A Mechanized Proof of Higman's Lemma by Open Induction

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Abstract I present a short, mechanically checked Isabelle/HOL formalization of Higman's lemma by open induction.

1 Introduction

In the winter of 2016 a mixed group of scientists met for a week in Dagstuhl, Germany, to discuss the present and future of Well Quasi-Orders in Computer Science.[1] *Having worked a little on mechanizing results from well-quasi-order theory with the proof assistant Isabelle in the past, I was for a time thinking hard about any new results I could present. Then I remembered a clingy item on my mental to-do list: applying a previous Isabelle formalization of open induction to obtain an alternative mechanization of Higman's lemma. The following exposition is supposed to give an accessible account of my formalization.*

The study of *well-quasi-orders* dates back at least to the early 1940s. (And already in the 1970s, a tendency to duplicate work prompted Kruskal to give an introductory overview of well-quasi-orders, including their history, present, and future [9].)

While initially, the goal was mostly to show that certain structures (like pairs, finite words [7], and finite trees [8]) are indeed well-quasi-ordered, later on a significant amount of work was invested into obtaining shorter/simpler/more elegant proofs of known results. A prime example of this kind of work is Nash-Williams' *minimal bad sequence* argument [13] (which allowed him to shorten the previous, rather involved 7-page proof by Kruskal [8] down to a conceptually simple half-page proof).

Despite the elegance of Nash-Williams' proof, some consider its non-constructive nature a major drawback. Therefore, another line of work focuses on constructive proofs of results in well-quasi-order theory [12, 3, 17]. Also, in order to obtain

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insight into the *computational content* of a proof – in essence, the goal is to obtain an algorithm directly from a proof, a process that is also known as *program extraction*.

A more recent branch of research is dedicated to the mechanization of results from well-quasi-order theory with the help of proof assistants [11, 5, 2, 20, 10, 18]. Such machine-checked proofs, while often hard to establish – are highly trustworthy and have also other advantages, like the possibility to machine-generate verified programs (eliminating the more traditional but potentially error-prone approaches of either manually transforming an existing computer program into a machine-readable specification or manually writing a program adhering to an existing specification).

While in practice investigations of the computational content of a proof and its formalization often go hand in hand, I want to stress that neither is being constructive a prerequisite for a formalization, nor is having a formalization a prerequisite for investigating the computational content of a proof. Which is why I distinguish between these two goals above.

This work is part of an effort towards combining the above three strands of research by simplifying existing proofs, investigating their computational content, and providing corresponding mechanizations. My starting point was an existing formalization of well-quasi-order theory [19] in Isabelle/HOL (by myself), employing the minimal bad sequence argument, together with the idea (of others) that a classically equivalent but more constructive way of expressing the same kind of reasoning is via a proof method called *open induction* [16], whose computational interpretation was investigated by Berger [1]. My main result is a new mechanized proof of Higman's lemma by open induction.

Below, we repeat the (semi-)formal statement of Higman's well-known result [7, Theorem 4.3] (where A^* denotes the set of finite words over an alphabet A).

Higman's Lemma. *If A is well-quasi-ordered, then so is A^*.*

As another new result I provide a mechanized equivalence proof between the classical definition of almost-full relations and a more recent inductive definition due to Vytiniotis, Coquand, and Wahlstedt [24].

Isabelle is a generic interactive proof assistant. Its most popular incarnation is Isabelle/HOL [14] (for higher-order logic) which is a classical logic based on Church's simply typed lambda calculus and with Hilbert's choice operator built in. Every mechanized proof is ultimately broken down to the handful of basic axioms of HOL – where every single step of this reduction is machine checked – yielding a very high degree of reliability.

Related Work. A similar proof, but without an accompanying formalization and yet more involved, was presented in an unpublished manuscript by Geser [6].

Overview. I start, in Section 2, by recalling the fundamental notions of almost-full relations and well-quasi-orders and also give some basic results. This includes a new proof of the fact that almost-full relations admit a (classically) equivalent inductive definition. Then, in Section 3, it is shown how the proof principle of open induction almost naturally arises when searching for a constructive counterpart to proofs by minimal counterexample. Afterwards, in Section 4, the stage is set by discharging the prerequisites of open induction one at a time. My main result, a proof of Higman's lemma by open induction, is presented in Section 5. Finally, I conclude in Section 6.

2 Preliminaries

Let us start by recalling the formal definition of well-quasi-orders, based on the, probably less well known, notion of almost-full relations. (The notion of almost-full relations was introduced by Veldman and Bezem [23]; another very accessible exposition is given by Vytiniotis, Coquand, and Wahlstedt [24].)

Definition 1 (Almost-Full Relations and Well-Quasi-Orders). Let $\sqsubseteq$ be a binary relation with domain A. An infinite sequence $a_1, a_2, a_3, \ldots$ of elements in A (or infinite A-sequence for short) is *($\sqsubseteq$-)good* if it contains an "increasing pair," that is, $a_i \sqsubseteq a_j$ for some $i < j$. Infinite A-sequences that do not satisfy this condition are called *($\sqsubseteq$-)bad*. A relation $\sqsubseteq$ is *almost-full* (on A) if all infinite A-sequences are good. If in addition $\sqsubseteq$ is a quasi-order (on A),[2] it is called a *well-quasi-order* (on A).

A nice property of almost-full relations is that in combination with transitivity, we obtain well-foundedness for free. Therefore almost-full relations and well-quasi-orders are of special interest for proving termination (of programs, term rewrite systems, etc.; which is also my angle on the subject).

Lemma 1. *Every transitive extension $\preceq$ of an almost-full relation $\sqsubseteq$ is well-founded.*

Proof. Assume to the contrary that there is an infinite descending sequence $a_1 \succ a_2 \succ a_3 \succ \ldots$ (where $x \succ y$ iff $x \succeq y$ and $x \not\preceq y$). By transitivity, we obtain $a_i \succ a_j$ for all $i < j$. But then also $a_i \not\sqsubseteq a_j$ for all $i < j$, and thus the sequence above is $\sqsubseteq$-bad, contradicting the assumption that $\sqsubseteq$ is almost-full. □

It turns out that it is often easy to extend results about almost-full relations to well-quasi-orders (remember that the latter differ from the former only by transitivity), which is an indication that "being almost-full' somehow captures the essence of "being a well-quasi-order."

In my initial presentation of Higman's lemma above, the well-quasi-order on finite words was left implicit. Let us amend this omission with the following definition.

Definition 2 (Homeomorphic Embedding). Given a binary relation $\sqsubseteq$, the induced *(homeomorphic) embedding relation* on finite words is defined inductively by the following three clauses (where finite words are constructed from the empty word $[]$ together with the binary constructor $\cdot$ which puts a single letter in front of another finite word):

$$\frac{}{[] \sqsubseteq^* ys} \qquad \frac{xs \sqsubseteq^* ys}{xs \sqsubseteq^* y \cdot ys} \qquad \frac{x \sqsubseteq y \quad xs \sqsubseteq^* ys}{x \cdot xs \sqsubseteq^* y \cdot ys}$$

[2] In fact, demanding transitivity suffices, since reflexivity is immediate for almost-full relations.

Now, a more explicit version of Higman's lemma is as follows.

Theorem 1 (Higman's Lemma). *Given a well-quasi-order $\sqsubseteq$ on A, the induced embedding relation $\sqsubseteq^*$ is a well-quasi-order on A^*.*

Incidentally, the above statement is already true when replacing all occurrences of "*a well-quasi-order*" by "*an almost-full relation*." Which is to say that transitivity does not pose any additional difficulties.

I conclude this section by some (classically) equivalent definitions of almost-full relations. (For well-quasi-orders more equivalences are known.)

Lemma 2. *Given a relation $\sqsubseteq$ on A, the following statements are equivalent:*

(1) The relation $\sqsubseteq$ is almost-full on A.
(2) Every infinite A-sequence a admits a $\sqsubseteq$-homogeneous subsequence, that is, there is a strictly monotone mapping $\sigma : \mathbb{N} \to \mathbb{N}$ such that $a_{\sigma(i)} \sqsubseteq a_{\sigma(j)}$ for all $i < j$.
(3) The relation $\sqsubseteq$ satisfies the predicate $\mathsf{af}(\cdot)$ which is defined inductively by the two clauses:

$$\frac{\forall x, y \in A.\, x \sqsubseteq y}{\mathsf{af}(\sqsubseteq)} \qquad\qquad \frac{\forall x \in A.\, \mathsf{af}(\lambda y\, z.\, y \sqsubseteq z \vee x \sqsubseteq y)}{\mathsf{af}(\sqsubseteq)}$$

Property (3) above, is due to Vytiniotis, Coquand, and Wahlstedt [24] and gives a nice intuition why such relations are called "almost full": it is possible within a finite number of steps (since the definition is inductive) to turn them into full relations (which is the only base case).

Proof. Detailed Isabelle/HOL proofs of the above equivalences are available in the *Archive of Formal Proofs* [19, `Almost_Full.thy`]. Their basic outline follows.

For the implication from (1) to (2), consider the infinite 2-colored graph whose vertices are the natural numbers such that i and j are connected by an edge with color 0 if and only if a_i and a_j are related by $\sqsubseteq$ (in either direction) and by an edge with color 1, otherwise. An application of Ramsey's theorem yields an infinite homogeneously colored subgraph. Since an infinite 1-subgraph contradicts the fact that $\sqsubseteq$ is almost-full, an infinite 0-subgraph is obtained. Enumerating the corresponding indices in increasing order yields the desired homogeneous subsequence of a.

The implication from (2) to (3) only holds classically. For the sake of a contradiction, let us first assume that (3) does not hold, and then construct a counterexample to (2). To this end, let $\mathsf{NAF}_{\preceq}$ denote some $x \in A$ such that $\mathsf{af}(\lambda y\, z.\, y \preceq z \vee x \preceq y)$ does not hold (which is obtained using Hilbert's choice operator in Isabelle/HOL). Then construct an infinite sequence $c^{\preceq}$ such that $c^{\preceq}_1$ is $\mathsf{NAF}_{\preceq}$ and $c^{\preceq}_{i+1}$ is $c^{\preceq'}_i$ with $\preceq' = (\lambda y\, z.\, y \preceq z \vee \mathsf{NAF}_{\preceq} \preceq y)$ for all $i \geq 1$. In the following c abbreviates $c^{\sqsubseteq}$. Now, from the assumption $\neg\,\mathsf{af}(\sqsubseteq)$, it is shown by induction on n that

$$\mathsf{af}\left(\lambda y\, z.\, y \sqsubseteq z \vee \bigvee_{i \leq n} c_i \sqsubseteq y \vee \bigvee_{1 \leq i < j \leq n} c_i \sqsubseteq c_j\right)$$

does not hold for any n, contradicting the fact that c admits an infinite homogeneous subsequence.

The proof of the implication from (3) to (1) proceeds by an easy rule-induction according to the definition of $\mathsf{af}(\cdot)$. □

3 From Minimal Counterexamples to Open Induction

Before I give the necessary prerequisites for proving Higman's lemma, let us discuss how *open induction* enters the picture. Since, ideally we want to have a simple, constructive, and formalized proof, I thought it a good idea to start from Nash-Williams proof (which is way simpler than any other proof I am aware of). His proof proceeds by contradiction: assuming that there is a bad sequence, he then argues that there is a minimal one, and finally constructs an even smaller bad sequence. This corresponds to a proof by contradiction assuming a minimal counterexample (which I will call proof "by minimal counterexample" in the remainder)

$$((\exists m.\, \neg P(m) \wedge (\forall x < m.\, P(x))) \rightarrow \bot) \rightarrow \forall x.\, P(x)$$

where P is the property we want to prove, $>$ is an "appropriate" order, and m denotes a minimal counterexample.

If $>$ is well-founded, then the above formula is (classically) equivalent to a proof by well-founded induction

$$(\forall x.\, (\forall y < x.\, P(y)) \rightarrow P(x)) \rightarrow \forall x.\, P(x)$$

which is a desirable alternative, since the outermost proof structure is now constructive, while the inner proof structure stays the same.

So at least superficially it seems that it should be possible to prove whatever we can prove by minimal counterexample, also by well-founded induction. The problem, however, is that the order on infinite sequences that Nash-Williams used is not well-founded. Indeed no suitable well-founded order on infinite sequences immediately suggests itself.

Raoult [16] introduced a viable alternative to well-founded induction in the form of *open induction*, a variation of well-founded induction that exchanges well-foundedness of the order by two other prerequisites. To begin with, the order has to be *downward complete*.

Definition 3 (Chains and Downward Completeness). Let $\sqsubseteq$ be a relation with domain A. A $\sqsubseteq$-*chain* C is a totally ordered subset of A, that is, for all $c, d \in C$ we have either $c \sqsubseteq d$ or $d \sqsubseteq c$. The relation $\sqsubseteq$ is called *downward complete* if every non-empty $\sqsubseteq$-chain has a greatest lower bound in A.

Moreover, open induction is only valid for proving *open* properties.

Definition 4 (Open Properties). A property P is ($\sqsubseteq$-)open if for every non-empty $\sqsubseteq$-chain C it holds that whenever some greatest lower bound g of C satisfies $P(g)$, then $P(x)$ also holds for some $x \in C$.

Theorem 2 (Open Induction). *Let $\sqsubseteq$ be a downward complete quasi-order on A and P be an $\sqsubseteq$-open property, then the principle of* open induction *reads as follows*

$$(\forall x \in A. (\forall y \in A. y \sqsubset x \to P(y)) \to P(x)) \to \forall x \in A. P(x)$$

where $x \sqsubset y$ abbreviates $x \sqsubseteq y \land y \not\sqsubseteq x$. □

Here, I state open induction as a theorem, since its correctness has been formalized by Mizuhito Ogawa and myself in Isabelle/HOL (basically by an appeal to Zorn's lemma; the development is available in the *Archive of Formal Proofs* [15]).

At this point, three ingredients are still missing before we can actually apply open induction to prove Higman's lemma. First, we need to fix the property of infinite sequences we want to prove (which must of course be a property which implies that $\sqsubseteq^*$ is almost-full). Second, we need to provide an appropriate (that is, downward complete) order on infinite sequences. And third, we have to make sure that the chosen property is open with respect to the chosen order.

4 Setting the Stage: An open property and an appropriate order

The idea to apply open induction to well-quasi-order theory dates back to Raoult [16]. I am not aware of any actual execution of this idea until the work of Geser [6], who chose a rather complicated order on infinite sequences after arguing that the much simpler lexicographic extension of the proper suffix relation on finite words would make it impossible to use the induction hypothesis (moreover, he tried to prove a variation on Lemma 2(2), namely that every infinite sequence contains an infinite ordered subsequence, instead of Lemma 2(1)).

It turns out, that simply using the lexicographic extension of the proper suffix relation to infinite sequences yields a simpler proof than Geser's initial attempt.

Definition 5 (Lexicographic Extension to Infinite Sequences). Let $\prec$ be a relation with domain A. Then the *lexicographic extension* of $\prec$ to infinite A-sequences a and b is given by $a \prec_{\mathsf{lex}} b$ iff $a_k \prec b_k$ and $\forall i < k. a_i = b_i$ for some k.

The following construction will provide a greatest lower bound for each non-empty $\prec_{\mathsf{lex}}$-chain (and is actually the same one I also used to obtain *minimal bad sequences* in some Nash-Williams-style proofs I formalized [19] and could therefore be reused).

Definition 6 (Minimal Infinite Sequences). Let $\prec$ be a well-founded partial order with domain A, C be a non-empty set of infinite A-sequences, and a be an infinite A-sequence. Then the set E_k^a of sequences in C that are equal to a up to, but not including, position k is defined by $E_k^a = \{b \in C. \forall i < k. a_i = b_i\}$. Now, a *lexicographically $\prec$-minimal sequence* is constructed inductively as follows:

$$\mu_i = \min_\prec \{a_i \mid a \in E_i^\mu\}$$

That is to say that the ith element of μ is a $\prec$-minimal element of the ith "column" of sequences in E_i^μ. This construction is well-defined, since obtaining the ith element of m only requires access to elements of m whose positions are strictly smaller.

Lemma 3. *Given a well-founded partial order $\prec$, the infinite sequence μ is a greatest lower bound of any non-empty $\prec_{\mathsf{lex}}$-chain C.*

Proof. Let C be a non-empty $\prec_{\mathsf{lex}}$-chain. Let us first establish that μ is a lower bound of C. To this end, let a be an arbitrary infinite sequence in C. If $\mu = a$ we are done. Otherwise, $a \neq \mu$ and thus there is some position k at which a and μ differ for the first time, that is, $a_i = \mu_i$ for all $i < k$. Then $a \in E_k^\mu$ and hence $a_k \in \{b_k \mid b \in E_k^\mu\}$. But then $\mu_k \prec a_k$ since we have $a_k \neq \mu_k$, $a_k \prec \mu_k$ is impossible by construction of μ, and C is a $\prec_{\mathsf{lex}}$-chain.

It remains to be shown that μ is greater than or equal to any other lower bound $\ell \neq \mu$. Again, take the least k such that $\ell_k \neq \mu_k$ (thus $\ell_i = \mu_i$ for all $i < k$). Now, obtain an infinite sequence $a \in E_{k+1}^\mu$, that is, $a_i = \mu_i$ for all $i \leq k$. Then, $\ell_k \prec a_k$ (since ℓ is a lower bound) and thus $\ell \prec_{\mathsf{lex}} \mu$. □

Corollary 1. *The lexicographic extension $\prec_{\mathsf{lex}}$ is downward complete for every well-founded partial order $\prec$.*

Below, I will use the *(proper) suffix relation* $\lhd$ as base order, which is a well-founded partial order given by $xs \lhd ys$ iff ys is obtained by taking some non-empty finite word zs and appending xs (or in words: *xs is a proper suffix of ys*).

Now that we have an appropriate order on infinite sequences we still have to fix a property and show that it is open. The property of infinite sequences I will use in my proof of Higman's lemma below, is "being good" (thus, in contrast to Geser, I am using Lemma 2(1) as the defining property of almost-full relations).

Lemma 4. *For any well-founded partial order $\prec$, being good is an $\prec_{\mathsf{lex}}$-open property for arbitrary relations.*

Proof. Assume that C is a non-empty $\prec_{\mathsf{lex}}$-chain with $\sqsubseteq$-good greatest lower bound g for some arbitrary but fixed relation $\sqsubseteq$. Then also μ is $\sqsubseteq$-good, since for antisymmetric relations greatest lower bounds are unique and thus $g = m$. This means that $\mu_i \sqsubseteq \mu_j$ for some $i < j$. Moreover, take some $a \in E_{j+1}^\mu$, which has to exist since C is non-empty. But then, $a_i = \mu_i$ and $a_j = \mu_j$ and thus also $a_i \sqsubseteq a_j$, showing that $a \in C$ is good. □

5 The Proof via Open Induction

Finally, we are ready for proving Higman's lemma by open induction (the Isabelle/HOL formalization of the proof below is available in the *Archive of Formal Proofs* [19, `Higman_OI.thy`]; it might be interesting to note that the formalization is about the same size).

Proof (of Theorem 1). By assumption $\sqsubseteq$ is almost-full on A. Since the suffix relation $\lhd$ is a well-founded partial order, its lexicographic extension $\lhd_{\mathsf{lex}}$ is downward complete by Lemma 1. Together with the fact that being $\sqsubseteq^*$-good is an open property (Lemma 4), this means – according to Theorem 2 – that we can apply open induction in order to prove that every infinite A-sequence a is $\sqsubseteq^*$-good (which is to say that $\sqsubseteq^*$ is almost-full on A^*).

By induction hypothesis (IH) any infinite A-sequence $b \lhd_{\mathsf{lex}} a$ is $\sqsubseteq^*$-good.

If a contains the empty word then it is trivially good. Thus we concentrate on the case where for each $i \geq 1$ we have $a_i = h_i \cdot t_i$, that is, a_i consists of a head (letter) $h_i \in A$ and a tail (word) $t_i \in A^*$. Since $\sqsubseteq$ is almost-full on A, we obtain an infinite increasing subsequence of h by Lemma 2(2): $h_{\sigma(1)} \sqsubseteq h_{\sigma(2)} \sqsubseteq h_{\sigma(3)} \sqsubseteq \cdots$. We form a new infinite A-sequence a' by extending the finite initial segment $a_1, a_2, a_3, \ldots, a_{\sigma(1)-1}$ of a by the infinite A-sequence $t_{\sigma(1)}, t_{\sigma(2)}, t_{\sigma(3)}, \ldots$. Then, by construction of a' we have $a' \lhd_{\mathsf{lex}} a$ and thus we obtain an increasing pair $a'_i \sqsubseteq^* a'_j$ for some $i < j$ by IH. We conclude by an exhaustive case analysis on the positioning of i and j within a':

- If $j < \sigma(1)$, then $a_i = a'_i \sqsubseteq a'_j = a_j$ and thus a is good.
- If $i < \sigma(1) \leq j$, then $a_i = a'_i \sqsubseteq^* a'_j = t_{\sigma(j-\sigma(1)+1)} \sqsubseteq^* a_{\sigma(j-\sigma(1)+1)}$. Moreover, $i < \sigma(j - \sigma(1) + 1)$ and thus a is good.
- If $\sigma(1) \leq i$ then $t_{\sigma(i-\sigma(1)+1)} = a'_i \sqsubseteq^* a'_j = t_{\sigma(j-\sigma(1)+1)}$. Which trivially implies $a_{\sigma(i-\sigma(1)+1)} \sqsubseteq^* a_{\sigma(j-\sigma(1)+1)}$. Moreover $\sigma(i - \sigma(1) + 1) < \sigma(j - \sigma(1) + 1)$ and thus a is good. □

6 Conclusions and Future Work

I have given a short, constructive, and mechanically checked proof of Higman's lemma by open induction.

My Isabelle/HOL mechanization of Higman's lemma has already been used by others (although for these applications it is not important *how* the result is obtained): Felgenhauer employs Higman's lemma to obtain well-foundedness of a complex induction order [4], while Wu et al. [25] use it to obtain a formalization of the fact that: *For every language A, the languages of sub- and superstrings of A are regular.*

My initial goal was to apply mechanized results from well-quasi-order theory to simplify well-foundedness proofs in IsaFoR [22] (an Isabelle Formalization of Rewriting), for example, to obtain well-foundedness of the Knuth-Bendix order (KBO) "for free." However, at the moment it is not clear whether the generalized variant of KBO of IsaFoR [21] which is required to certify generated proofs by several different automated termination provers is a simplification order at all (and frankly, I doubt it).

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