Loops under Strategies ... Continued

René Thiemann (rene.thiemann@uibk.ac.at), University of Innsbruck, Austria
Jürgen Giesl* (giesl@informatik.rwth-aachen.de), RWTH Aachen University, Germany
Peter Schneider-Kamp (petersk@imada.sdu.dk), University of Southern Denmark, Denmark
Christian Sternagel† (christian.sternagel@uibk.ac.at), University of Innsbruck, Austria

1 Introduction

Termination is an important property of term rewrite systems (TRSs). Therefore, much effort has been spent on developing and automating techniques for showing termination of TRSs. But in order to detect bugs, it is at least as important to prove *non-termination* of TRSs. Note that for rewriting under strategies, one cannot ignore the strategy, since a non-terminating TRS may still be terminating due to the strategy. Thus, it is important to develop automated techniques to disprove termination of TRSs under strategies.

Most of the techniques for showing non-termination detect loops, i.e., derivations of the form $t \to_{\mathcal{R}}^{+} C[t\mu]$ for some context C and some substitution μ. To prove non-termination of a TRS $\mathcal{R}$ under a strategy $\mathcal{S}$, we may use a complete transformation $T_{\mathcal{S}}$ (e.g., [2, 4, 6]) where $\mathcal{R}$ terminates under the strategy $\mathcal{S}$ iff the TRS $T_{\mathcal{S}}(\mathcal{R})$ terminates without considering any strategy. Then one could try to find a loop of the transformed system $T_{\mathcal{S}}(\mathcal{R})$. There are three main drawbacks using these transformations. The first problem is an increased search space, as loops of $\mathcal{R}$ are often transformed into much longer loops in $T_{\mathcal{S}}(\mathcal{R})$. Second, the complete transformations in [2, 4, 6] translate a loop $t \to_{\mathcal{R}}^{+} C[t\mu]$ into a non-looping infinite derivation in $T_{\mathcal{S}}(\mathcal{R})$ whenever $C \neq \Box$. To solve these two problems, in [7, 8], decision procedures were presented. Given a loop, they decide whether the loop is also a loop under the respective strategy. Here, [7] treats the innermost strategy and [8] deals with the context-sensitive [3] and the outermost strategies. The third problem is the availability of complete transformations. As a TRS is terminating for the leftmost-innermost, parallel-innermost, or max-parallel-innermost strategy iff it is innermost terminating [5], one can use the decision procedure for innermost loops [7] to disprove termination for all these innermost strategies. However, we are not aware of any transformations for the strategies leftmost-outermost, parallel-outermost, and max-parallel-outermost. Therefore, in this paper we built upon the direct methods of [7, 8] and give decision procedures for all these strategies.[1] Note that our decision procedures can also be extended to the context-sensitive case, e.g., to leftmost-innermost context-sensitive.

Example 1. *Consider the following TRS $\mathcal{R}$ to compute factorial numbers which is a variant of [7, Ex. 1].*

$$
\begin{aligned}
\mathsf{factorial}(y) &\to \mathsf{fact}(\mathsf{0},y) && (1) & \mathsf{0}\cdot y &\to \mathsf{0} && (7)\\
\mathsf{fact}(x,y) &\to \mathsf{if}(x == y, \mathsf{s}(\mathsf{0}), \mathsf{fact}(\mathsf{s}(x),y)\cdot \mathsf{s}(x)) && (2) & \mathsf{s}(x)\cdot y &\to y + (x\cdot y) && (8)\\
\mathsf{if}(\mathsf{true},x,y) &\to x && (3) & x == y &\to \mathsf{eq}(\mathsf{chk}(x),\mathsf{chk}(y)) && (9)\\
\mathsf{if}(\mathsf{false},x,y) &\to y && (4) & \mathsf{eq}(x,x) &\to \mathsf{true} && (10)\\
\mathsf{0}+y &\to y && (5) & \mathsf{chk}(x) &\to \mathsf{false} && (11)\\
\mathsf{s}(x)+y &\to \mathsf{s}(x+y) && (6) & \mathsf{eq}(\mathsf{false},y) &\to \mathsf{false} && (12)
\end{aligned}
$$

Here, the intended strategy is leftmost-outermost: otherwise, rule (2) would directly cause non-termination. Moreover, this strategy is needed for the equality-test encoded by rules (9)–(12) that needs at most 3 reductions. Nevertheless, we obtain the following looping reduction:

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[1] Note that by [5] an innermost loop implies leftmost-innermost non-termination. However, this does not imply leftmost-innermost-loopingness: As an example, consider $\{\mathsf{a} \to \mathsf{f}(\mathsf{nloop},\mathsf{a})\} \cup \mathcal{R}$, where nloop is a non-terminating, but non-looping term w.r.t. the TRS $\mathcal{R}$. Therefore, in this paper we also develop decision procedures for the various innermost strategies.

$$\begin{aligned} t = \mathsf{fact}(x,y) &\to_{\mathcal{R}} \mathsf{if}(x == y, \mathsf{s}(0), \mathsf{fact}(\mathsf{s}(x),y) \cdot \mathsf{s}(x)) \\ &\to_{\mathcal{R}} \mathsf{if}(\mathsf{eq}(\mathsf{chk}(x),\mathsf{chk}(y)), \mathsf{s}(0), \mathsf{fact}(\mathsf{s}(x),y) \cdot \mathsf{s}(x)) \\ &\to_{\mathcal{R}} \mathsf{if}(\mathsf{eq}(\mathsf{false},\mathsf{chk}(y)), \mathsf{s}(0), \mathsf{fact}(\mathsf{s}(x),y) \cdot \mathsf{s}(x)) \\ &\to_{\mathcal{R}} \mathsf{if}(\mathsf{false}, \mathsf{s}(0), \mathsf{fact}(\mathsf{s}(x),y) \cdot \mathsf{s}(x)) \qquad \to_{\mathcal{R}} \mathsf{fact}(\mathsf{s}(x),y) \cdot \mathsf{s}(x) = C[t\mu] \end{aligned}$$

where $\mu = \{x/\mathsf{s}(x)\}$ *and* $C = \Box \cdot \mathsf{s}(x)$. *Applying our algorithm will show that the above loop indeed is a leftmost-outermost loop, and hence,* $\mathcal{R}$ *does not terminate under the leftmost-outermost strategy.*

2 Loops under Strategies

We only regard finite signatures and TRSs and refer to [1] for the basics of rewriting. We use ℓ, r, s, t, u for terms, f, g for function symbols, x, y for variables, μ, σ for substitutions, i, j, k, n, m for natural numbers, p, q for positions and C, D for contexts. Here, contexts are terms which contain exactly one hole $\Box$. The set of variables is denoted by $\mathcal{V}$. Throughout this paper we assume a fixed TRS $\mathcal{R}$ and we write $t \to_p s$ if one can reduce t to s at position p with $\mathcal{R}$, i.e., $t = C[\ell\sigma]$ and $s = C[r\sigma]$ for some $\ell \to r \in \mathcal{R}$, substitution σ, and context C with $C|_p = \Box$. In this case, the term $\ell\sigma$ is called a redex at position p. The reduction is leftmost / innermost / outermost, written $t \xrightarrow{\mathsf{l}}_p / \xrightarrow{\mathsf{i}}_p / \xrightarrow{\mathsf{o}}_p s$, iff p is a leftmost / innermost / outermost position of t such that $t|_p$ is a redex. The leftmost-innermost reduction is defined as $\xrightarrow{\mathsf{li}}_p = \xrightarrow{\mathsf{l}}_p \cap \xrightarrow{\mathsf{i}}_p$. Similarly, the leftmost-outermost reduction is $\xrightarrow{\mathsf{lo}}_p = \xrightarrow{\mathsf{l}}_p \cap \xrightarrow{\mathsf{o}}_p$. If the position is irrelevant we just write $\to$, $\xrightarrow{\mathsf{l}}$, $\xrightarrow{\mathsf{i}}$, $\xrightarrow{\mathsf{o}}$, $\xrightarrow{\mathsf{li}}$, or $\xrightarrow{\mathsf{lo}}$.

We also consider parallel reductions. Here, $t \xrightarrow{\mathsf{p}}_{p_1 \dots p_k} s$ is a parallel reduction iff $k > 0$, the p_i's are pairwise parallel positions, and $t \to_{p_1} \dots \to_{p_k} s$. The max-parallel reduction relation is defined as $t \xrightarrow{\mathsf{m}}_{p_1 \dots p_k} s$ iff $t \xrightarrow{\mathsf{p}}_{p_1 \dots p_k} s$ and t has no further redex at a position that is parallel to all positions $p_1 \dots p_k$. The (max-)parallel-innermost reduction is defined as $t \xrightarrow{\mathsf{mi}} / \xrightarrow{\mathsf{pi}}_{p_1 \dots p_k} s$ iff $t \xrightarrow{\mathsf{m}} / \xrightarrow{\mathsf{p}}_{p_1 \dots p_k} s$ and all redexes $t|_{p_i}$ are innermost redexes. The (max-)parallel-outermost reductions $\xrightarrow{\mathsf{mo}}$ and $\xrightarrow{\mathsf{po}}$ are defined analogously.

To shortly illustrate the difference between the strategies, observe that $0 == 0 \xrightarrow{\mathsf{i}}{}^* / \xrightarrow{\mathsf{li}}{}^* / \xrightarrow{\mathsf{mi}}{}^* / \xrightarrow{\mathsf{o}}{}^* / \xrightarrow{\mathsf{mo}}{}^* \mathsf{false}$ whereas $0 == 0 \xrightarrow{\mathsf{lo}}{}^* \mathsf{false}$ does not hold for the TRS $\mathcal{R}$ of Ex. 1.

A TRS $\mathcal{R}$ is non-terminating iff there is an infinite derivation $t_1 \to t_2 \to \dots$. It is leftmost-innermost / leftmost-outermost / parallel-innermost / parallel-outermost / max-parallel-innermost / max-parallel-outermost non-terminating iff there is such an infinite derivation using $\xrightarrow{\mathsf{li}} / \xrightarrow{\mathsf{lo}} / \xrightarrow{\mathsf{pi}} / \xrightarrow{\mathsf{po}} / \xrightarrow{\mathsf{mi}} / \xrightarrow{\mathsf{mo}}$ instead of $\to$. To describe the infinite derivation that is induced by a loop, we use context-substitutions.

Definition 2 (Context-substitutions [8])**.** *A* context-substitution *is a pair* (C,μ) *consisting of a context* C *and a substitution* μ. *The n-fold application of* (C,μ) *to a term* t*, written* $t(C,\mu)^n$ *is defined as follows.*

$$t(C,\mu)^0 = t \qquad\qquad t(C,\mu)^{n+1} = C[t(C,\mu)^n\mu]$$

From the definition it is obvious that in $t(C,\mu)^n$, the context C is added n-times above t and t is instantiated by μ^n. Note that also the added contexts are instantiated by μ. For the term $t(C,\mu)^3$ this is illustrated in Fig. 1. Context-substitutions have similar properties to both contexts and substitutions.

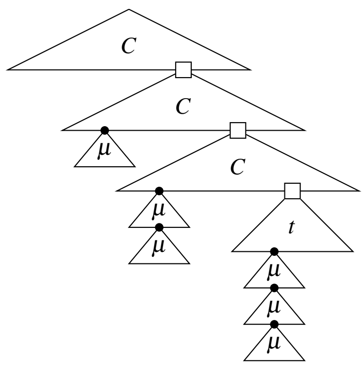


Figure 1: The term $t(C,\mu)^3$

Lemma 3 (Properties of context-substitutions [8])**.**

(i) $t(C,\mu)^n\mu = t\mu(C\mu,\mu)^n$.

(ii) $t(C,\mu)^m(C,\mu)^n = t(C,\mu)^{m+n}$.

(iii) Whenever $t \to_q s$ *and* $C|_p = \Box$ *then* $t(C,\mu)^n \to_{p^n q} s(C,\mu)^n$.

Here, property (i) is similar to the fact that $C[t]\mu = C\mu[t\mu]$, and (ii) shows that context-substitutions can be combined just like substitutions where $\mu^m\mu^n = \mu^{m+n}$. Finally stability and monotonicity of rewriting are used to show in (iii) that rewriting is closed under context-substitutions. Using context-substitutions we can now concisely present the infinite derivation of a loop $t \to^+ C[t\mu] = t(C,\mu)$.

$$t(C,\mu)^0 \to^+ t(C,\mu)(C,\mu)^0 = t(C,\mu)^1 \to^+ \ldots \to^+ t(C,\mu)^n \to^+ \ldots$$

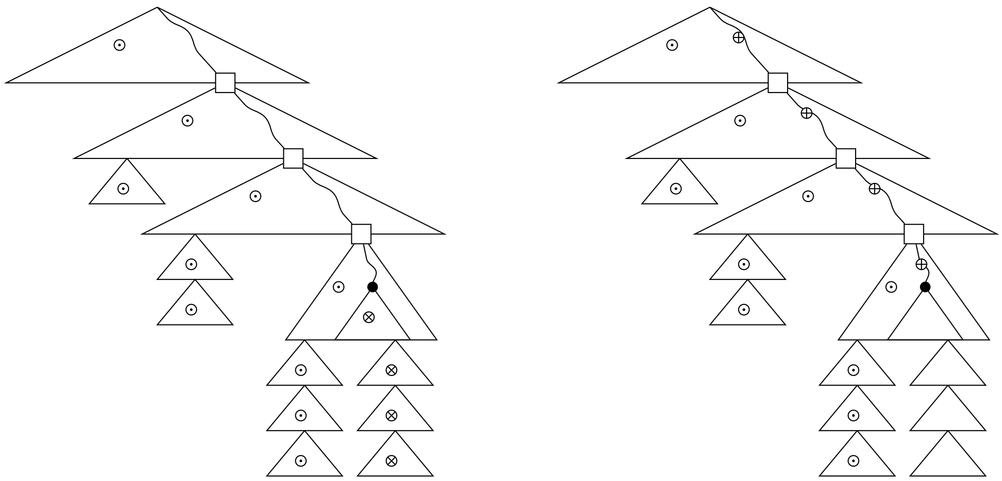
Hence, for every n the positions of the reductions in the loop are prefixed by an additional p^n where p is the position of the hole in C, cf. Lemma 3 (iii).

Definition 4 ($\mathcal{S}$-loop, [8]). *Let $\mathcal{S}$ be a strategy. A loop $t_1 \to_{q_1} t_2 \to_{q_2} \ldots t_n \to_{q_n} t_{n+1} = t_1(C,\mu)$ with $C|_p = \Box$ is an $\mathcal{S}$-loop iff the reduction $t_i(C,\mu)^m \to_{p^m q_i} t_{i+1}(C,\mu)^m$ respects the strategy $\mathcal{S}$ for all i and m.*

As a direct consequence of Def. 4, one can conclude that every $\mathcal{S}$-loop of a rewrite system $\mathcal{R}$ proves non-termination of $\mathcal{R}$ under strategy $\mathcal{S}$.

3 Leftmost-Innermost and Leftmost-Outermost Loops

Recall the definition of $\xrightarrow{\mathsf{li}}$ and $\xrightarrow{\mathsf{lo}}$. A leftmost-innermost / -outermost reduction of all terms $t(C,\mu)^n$ at positions p^nq requires that for no n there is a redex at a position left or below / above of p^nq. This is illustrated in the figure below: The reduction of the subterm at the black position p^nq respects the leftmost-innermost strategy iff p^nq is leftmost and innermost. This is the case whenever there are no redexes at positions $\odot$ and $\otimes$. We obtain a leftmost-outermost reduction iff p^nq is leftmost and outermost. This is the case whenever there are no redexes at positions $\odot$ and $\oplus$.



leftmost-innermost leftmost-outermost

In [7] a decision procedure for the existence of redexes at positions $\otimes$ was given, and [8] contains a decision procedure for the existence of redexes at positions $\oplus$. Hence, it remains to give a decision procedure for the existence of redexes at positions $\odot$, i.e., we have to be able to decide whether all p^nq point to leftmost redexes.

In total, there are four possibilities why p^nq might not point to a leftmost redex in the term $t(C,\mu)^n$:

(i) There might be a redex within $t\mu^n$ at position $q' \in \mathcal{P}\mathsf{os}(t)$ which is left of q. Hence, we have to consider all finitely many subterms $u = t|_{q'}$ where q' is left of q and guarantee that $u\mu^n$ is no redex.

(ii) There might be a redex within $t\mu^n$ at position $q' \in \mathcal{P}\mathsf{os}(t\mu^n) \setminus \mathcal{P}\mathsf{os}(t)$ which is left of q. Hence, this redex is of the form $v\mu^k$ for some $k \leq n$ and some subterm $v \trianglelefteq x\mu$ where x is a variable that occurs within some of u, $u\mu$, $u\mu^2$, ... for some subterm u of t that is listed in (i). Note that there are only finitely many such variables x and hence, again we obtain a finite set of terms where for each of these terms v and each n we have to guarantee that $v\mu^n$ is not a redex.

(iii) There might be a redex where the root is within C and left of the path p. Here, we have to consider all finitely many subterms $u = C|_{p'}$ where p' is left of q and guarantee that $u\mu^n$ is not a redex.

(iv) In analogy to (ii) we also have to consider redexes within μ where now the variables x are taken from the subterms u that are listed in (iii).

To summarize, we generate a finite set U of terms u such that (a) and (b) are equivalent:

(a) For every n, the reduction $t(C,\mu)^n \to_{p^n q} t'(C,\mu)^n$ is leftmost.

(b) There is no $u \in U$ and no number n such that $u\mu^n$ is a redex.

Note that the question whether $u\mu^n$ is a redex for some n is essentially a question of matching: does there exist a number n, a left-hand side ℓ, and a substitution σ such that $u\mu^n = \ell\sigma$? This question gives rise to the following definition of matching problems.

Definition 5 (Matching problems [7]). *A* matching problem *is a pair* $(u \gtrdot \ell, \mu)$. *It is* solvable *iff there are* n *and* σ *such that* $u\mu^n = \ell\sigma$.

Following the possibilities (i) - (iv) above, now we can formally define a set of matching problems to analyze leftmost reductions.

Definition 6 (Leftmost matching problems). *The set of* leftmost matching problems *for a reduction* $t \to_q t'$ *and a context-substitution* (C,μ) *with* $C|_p = \Box$ *is defined as the set consisting of:*

$(u \gtrdot \ell, \mu)$ *for each* $\ell \to r \in \mathcal{R}$ *and* $q' \in \mathcal{P}\mathsf{os}(t)$ *where* q' *is left of* q*, and* $u = t|_{q'}$

$(u \gtrdot \ell, \mu)$ *for each* $\ell \to r \in \mathcal{R}$ *and* $q' \in \mathcal{P}\mathsf{os}(t)$ *where* q' *is left of* q*,* $x \in \bigcup_{i \in \mathbb{N}} \mathcal{V}(t|_{q'}\mu^i)$*, and* $u \trianglelefteq x\mu$

$(u \gtrdot \ell, \mu)$ *for each* $\ell \to r \in \mathcal{R}$ *and* $p' \in \mathcal{P}\mathsf{os}(C)$ *where* p' *is left of* p*, and* $u = C|_{p'}$

$(u \gtrdot \ell, \mu)$ *for each* $\ell \to r \in \mathcal{R}$ *and* $p' \in \mathcal{P}\mathsf{os}(C)$ *where* p' *is left of* p*,* $x \in \bigcup_{i \in \mathbb{N}} \mathcal{V}(C|_{p'}\mu^i)$*, and* $u \trianglelefteq x\mu$

Note that the sets of variables in the second and fourth case are finite and can easily be computed.

Theorem 7 (Soundness of leftmost matching problems). *Let* $t \to_q t'$ *and* (C,μ) *such that* $C|_p = \Box$*. All reductions* $t(C,\mu)^n \to_{p^n q} t'(C,\mu)^n$ *are leftmost iff none of the leftmost matching problems is solvable.*

As solvability of matching problems is decidable [7], one can combine Thm. 7 with the decision procedures for innermost or outermost loops of [7,8] to construct a decision procedure which determines whether a given loop is a leftmost-innermost loop or a leftmost-outermost loop: for each loop construct the leftmost matching problems, ensure that all these matching problems are not satisfiable (then leftmost reductions are guaranteed), and moreover use the decision procedures of [7, 8] to further ensure that the loop is an innermost or outermost loop.

Corollary 8 (Leftmost-innermost and leftmost-outermost loops are decidable). *Let there be a loop* $t_1 \to_{q_1} t_2 \to_{q_2} \ldots t_n \to_{q_n} t_{n+1} = t_1(C,\mu)$ *with* $C|_p = \Box$*. Then the following two questions are decidable.*

- *Is the loop a leftmost-innermost loop?*
- *Is the loop a leftmost-outermost loop?*

Using Cor. 8 we can decide that the loop given in Ex. 1 is a leftmost-outermost loop, but not a leftmost-innermost loop. Nevertheless, there is also a reduction $\mathsf{fact}(x,y) \xrightarrow{\mathsf{li}}{}^{+} \mathsf{if}(\mathsf{false},\mathsf{s}(0),\mathsf{fact}(\mathsf{s}(x),y) \cdot \mathsf{s}(x))$ which is a leftmost-innermost loop.

4 (Max-)Parallel-Innermost and (Max-)Parallel-Outermost Loops

For the parallel innermost / outermost strategies it suffices to use the decision procedures for innermost- and outermost loops. The reason is that $t(C,\mu)^n \xrightarrow{\mathsf{p}}_{p^n q_1 \ldots p^n q_k} t'(C,\mu)^n$ is a $\xrightarrow{\mathsf{pi}}$ / $\xrightarrow{\mathsf{po}}$-reduction iff for every $1 \leq i \leq k$ there is some s_i such that $t(C,\mu)^n \rightarrow_{p^n q_i} s_i$ is an innermost / outermost reduction.

Hence, for the rest of the paper we consider the max-parallel strategies $\xrightarrow{\mathsf{mi}}$ and $\xrightarrow{\mathsf{mo}}$. Again, the innermost or outermost aspect can be decided using the respective decision procedures. It remains to consider the max-parallel aspect, i.e., we have to decide whether $t(C,\mu)^n \xrightarrow{\mathsf{m}}_{p^n q_1 \ldots p^n q_k} t'(C,\mu)^n$ for all n.

Here, we essentially proceed as in the leftmost case, where we replace the condition that some position is left of p or q by the condition that it is parallel to p or to each q_i.

Definition 9 (Max-parallel matching problems)**.** *The set of* max-parallel matching problems *for a reduction* $t \xrightarrow{\mathsf{p}}_{q_1 \ldots q_k} t'$ *and a context-substitution* (C,μ) *with* $C|_p = \Box$ *is defined as the set consisting of:*

$(u \gg \ell, \mu)$ *for each* $\ell \rightarrow r \in \mathcal{R}$ *and* $q' \in \mathcal{P}\mathsf{os}(t)$ *where* $q' \parallel q_i$ *for all* i, *and* $u = t|_{q'}$

$(u \gg \ell, \mu)$ *for each* $\ell \rightarrow r \in \mathcal{R}$ *and* $q' \in \mathcal{P}\mathsf{os}(t)$ *where* $q' \parallel q_i$ *for all* i, $x \in \bigcup_{i \in \mathbb{N}} \mathcal{V}(t|_{q'}\mu^i)$, *and* $u \trianglelefteq x\mu$

$(u \gg \ell, \mu)$ *for each* $\ell \rightarrow r \in \mathcal{R}$ *and* $p' \in \mathcal{P}\mathsf{os}(C)$ *where* $p' \parallel p$, *and* $u = C|_{p'}$

$(u \gg \ell, \mu)$ *for each* $\ell \rightarrow r \in \mathcal{R}$ *and* $p' \in \mathcal{P}\mathsf{os}(C)$ *where* $p' \parallel p$, $x \in \bigcup_{i \in \mathbb{N}} \mathcal{V}(C|_{p'}\mu^i)$, *and* $u \trianglelefteq x\mu$

Using this finite set of matching problems we again obtain a decision procedure.

Theorem 10 (Soundness of max-parallel matching problems)**.** *Let* $t \xrightarrow{\mathsf{p}}_{q_1 \ldots q_k} t'$ *and* (C,μ) *be given such that* $C|_p = \Box$. *All reductions* $t(C,\mu)^n \xrightarrow{\mathsf{p}}_{p^n q_1 \ldots p^n q_k} t'(C,\mu)^n$ *are max-parallel iff none of the max-parallel matching problems is solvable.*

Corollary 11 ((Max-)parallel-innermost and (max-)parallel-outermost loops are decidable)**.** *Let there be a loop* $t_1 \xrightarrow{\mathsf{p}}_{q_1^1 \ldots q_{k_1}^1} t_2 \xrightarrow{\mathsf{p}}_{q_1^2 \ldots q_{k_2}^2} \ldots t_n \xrightarrow{\mathsf{p}}_{q_1^n \ldots q_{k_n}^n} t_{n+1} = t_1(C,\mu)$ *with* $C|_p = \Box$. *Then the following questions are decidable.*

- *Is the loop a parallel-innermost loop? Is it a max-parallel-innermost loop?*
- *Is the loop a parallel-outermost loop? Is it a max-parallel-outermost loop?*

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