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Parallel Reductions in λ-Calculus*

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The notion of parallel reduction is extracted from the simple proof of the Church–Rosser theorem by Tait and Martin-Löf. Intuitively, this means to reduce a number of redexes (existing in a λ-term) simultaneously. Thus in the case of β-reduction the effect of a parallel reduction is same as that of a "complete development" which is defined by using "residuals" of β-redexes. A nice feature of parallel reduction, however, is that it can be defined directly by induction on the structure of λ-terms (without referring to residuals or other auxiliary notions), and the inductive definition provides us exactly what we need in proving the theorem inductively. Moreover, the notion can be easily extended to other reduction systems such as Girard's second-order system **F** and Gödel's system **T**. In this paper, after reevaluating the significance of the notion of parallel reduction in Tait-and-Martin-Löf type proofs of the Church–Rosser theorems, we show that the notion of parallel reduction is also useful in giving short and direct proofs of some other fundamental theorems in reduction theory of λ-calculus; among others, we give such simple proofs of the standardization theorem for β-reduction (a special case of which is known as the leftmost reduction theorem for β-reduction), the quasi-leftmost reduction theorem for β-reduction, the postponement theorem of η-reduction (in $\beta\eta$-reduction), and the leftmost reduction theorem for $\beta\eta$-reduction.

1. INTRODUCTION

Parallel β-reduction is the key notion of the Tait-and-Martin-Löf proof of the Church–Rosser theorem for β-reduction, which intuitively means to reduce a number of β-redexes (existing in a λ-term) simultaneously.

First, we define the notion inductively, and recall the Tait-and-Martin-Löf proof of the Church–Rosser theorem. We refer to Barendregt (1984) as the standard text, and unless otherwise stated we follow the notations there. In particular, we use capital letters $M, N, P, Q, R, \ldots$ for arbitrary (type-free) λ-terms, and $x, y, z, \ldots$ for arbitrary variables.

DEFINITION 1.1. The parallel β-reduction, which we denote by $\Rightarrow_\beta$, is defined inductively, as follows.

(β1) $x \Rightarrow_\beta x$.

(β2) $\lambda x.M \Rightarrow_\beta \lambda x.M'$ if $M \Rightarrow_\beta M'$.

(β3) $MN \Rightarrow_\beta M'N'$ if $M \Rightarrow_\beta M'$ and $N \Rightarrow_\beta N'$.

(β4) $(\lambda x.M)\,N \Rightarrow_\beta M'[x := N']$ if $M \Rightarrow_\beta M'$ and $N \Rightarrow_\beta N'$.

The rules (β1)$\sim$(β3) mean that the relation $\Rightarrow_\beta$ includes the identity on λ-terms, i.e., $M \Rightarrow_\beta M$ holds for each M. In order to get $M \Rightarrow_\beta M'$, however, we may apply at any β-redexes in M the rule (β4) rather than (β3). Thus $M \Rightarrow_\beta M'$ means intuitively that M' is obtained from M by simultanuous contraction of some β-redexes possibly overlapping each other.

Based on the inductive definition of $\Rightarrow_\beta$, we can easily verify the following properties.

$$M \to_\beta M' \;\Rightarrow\; M \Rightarrow_\beta M'. \tag{1}$$

$$M \Rightarrow_\beta M' \;\Rightarrow\; M \twoheadrightarrow_\beta M'. \tag{2}$$

$$M \Rightarrow_\beta M',\, N \Rightarrow_\beta N' \;\Rightarrow\; M[y := N] \Rightarrow_\beta M'[y := N']. \tag{3}$$

(Property (1) can be verified by induction on the context of the redex, while (2) and (3) are by induction on M.) From (1) and (2), we know $\twoheadrightarrow_\beta$ is the reflexive, transitive closure of $\Rightarrow_\beta$. Therefore, to prove the Church–Rosser theorem for β-reduction,

$$N_1 \twoheadleftarrow_\beta M \twoheadrightarrow_\beta N_2 \;\Rightarrow\; N_1 \twoheadrightarrow_\beta M' \twoheadleftarrow_\beta N_2 \quad \text{for some } M',$$

it suffices to show the "diamond property" of $\Rightarrow_\beta$,

$$N_1 \Leftarrow_\beta M \Rightarrow_\beta N_2 \;\Rightarrow\; N_1 \Rightarrow_\beta M' \Leftarrow_\beta N_2 \quad \text{for some } M'. \tag{4}$$

But we can prove the following stronger statement more easily:

$$M \Rightarrow_\beta N \;\Rightarrow\; N \Rightarrow_\beta M^*. \tag{5}$$

* A previous version of this paper appeared in the *Journal of Symbolic Computation* 7 (1989), 113–123. In this revised version, I added a new introduction (Section 1) explaining the significance of the simplest proof à la Tait–Martin-Löf of the Church–Rosser theorem and the importance of the notion of parallel reduction therein. Some new material is also included (in particular, 2.7, 2.9, 2.10, 2.11 in Section 2). Besides, the whole paper is restructured, while some of the old proofs are improved (Lemma 2.2, Theorem 4.3, etc.) I express my sincere gratitude to Albert Meyer for his encouragement and valuable advice for this revision. Thanks are also due to Yohji Akama, Henk Barendregt, Roger Hindley, Kenichi Noguchi, Hirofumi Yokouchi, and the anonymous referees for their helpful comments and suggestions, which led to the final form of the paper.

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Here M^* is a term determined by M (but independent from N). Intuitively, the property (5) is satisfied by the term M^* which is obtained from M by contracting all the redexes existing in M simultanuously. The intuition can easily be verified by induction.

First we define M^* by induction on the λ-term M.

($\beta 1^*$) $x^* \equiv x$.

($\beta 2^*$) $(\lambda x.M)^* \equiv \lambda x.M^*$.

($\beta 3^*$) $(M_1 M_2)^* \equiv M_1^* M_2^*$ if $M_1 M_2$ is not a β-redex.

($\beta 4^*$) $((\lambda x.M_1)\, M_2)^* \equiv M_1^*[x := M_2^*]$.

Then property (5) can be verified by induction on M, as follows.

Case 1. If $M \equiv x \Rightarrow_\beta N$, then $N \equiv x \Rightarrow_\beta x \equiv M^*$.

Case 2. If $M \equiv \lambda x.M_1 \Rightarrow_\beta N$, then $N \equiv \lambda x.N_1$ for some N_1 with $M_1 \Rightarrow_\beta N_1$. Since M_1 is a subterm of M, by the induction hypothesis we get $N_1 \Rightarrow_\beta M_1^*$. This implies $\lambda x.N_1 \Rightarrow_\beta \lambda x.M_1^* \equiv M^*$.

Case 3. If $M \equiv M_1 M_2 \Rightarrow_\beta N$ and M is not a β-redex, then $N \equiv N_1 N_2$ for some N_i with $M_i \Rightarrow_\beta N_i$ $(i = 1, 2)$. Then we have $N_1 N_2 \Rightarrow_\beta M_1^* M_2^* \equiv M^*$.

Case 4. If $M \equiv (\lambda x.M_1)\, M_2 \Rightarrow_\beta N$, then either $N \equiv (\lambda x.N_1)\, N_2$ or $N \equiv N_1[x := N_2]$ both for some N_i with $M_i \Rightarrow_\beta N_i$ $(i = 1, 2)$. Here we have $N_i \Rightarrow_\beta M_i^*$ $(i = 1, 2)$ by the induction hypothesis.

Subcase 4.1. If $N \equiv (\lambda x.N_1)\, N_2$, then $N \Rightarrow_\beta M_1^*[x := M_2^*] \equiv M^*$.

Subcase 4.2. If $N \equiv N_1[x := N_2]$, we also have $N \Rightarrow_\beta M_1^*[x := M_2^*] \equiv M^*$ by property (3) above.

This completes the proof of (5), and hence that of the Church–Rosser theorem. The proof is rigorous, direct, and perhaps the shortest among all the known proofs of the theorem. In addition, to the author's view the proof is mathematically clear and easy to understand because the notion of parallel reduction captures precisely what we need to express our intuitive idea for the proof. Moreover the notion has a nice internal structure, which makes simple inductive argument on the structure of terms suffice for verifying the statement.

In the literature, variations of the proof are also known as the Tait-and-Martin-Löf proof. In those proofs, the diamond property (4) of $\Rightarrow_\beta$ rather than (5) is often proved (see, e.g., Barendregt, 1984; Hindley and Seldin, 1986; Rosser, 1982; Stenlund, 1972). In such cases the term M' in (4) is specified depending on M, N_1, and N_2 rather than M alone. In this way, one might get a term M' which is the "closest" to N_1 and N_2, but the case analysis needs a little more lines than the proof above.

The idea of parallel reduction also applies to the Church–Rosser theorems for other reduction systems, such as the extensional λ-calculus $\beta\eta$, the systems $\beta\Omega$ and $\beta\eta\Omega$ (cf. Barendregt, 1984), $\beta\eta^{-1}$ (with β-reduction and η-expansion), Girard's second-order system **F** (with and without η-reduction), etc.

For example, in the case of a non-extensional system **F** (see, e.g., Girard *et al.*, 1989), we define the parallel reduction $\Rightarrow_{\mathbf{F}}$ inductively on well-typed terms as

(F1) $x \Rightarrow_{\mathbf{F}} x$,

(F2) $\lambda x{:}\, U.t \Rightarrow_{\mathbf{F}} \lambda x{:}\, U.t'$,

(F3) $ts \Rightarrow_{\mathbf{F}} t's'$,

(F4) $\Lambda X.t \Rightarrow_{\mathbf{F}} \Lambda X.t'$,

(F5) $tU \Rightarrow_{\mathbf{F}} t'U$,

(F6) $(\lambda x{:}\, U.t)\, s \Rightarrow_{\mathbf{F}} t'[x := s']$,

(F7) $(\Lambda X.t)\, U \Rightarrow_{\mathbf{F}} t'[X := U]$,

assuming $t \Rightarrow_{\mathbf{F}} t'$ and $s \Rightarrow_{\mathbf{F}} s'$. (Here we write x for term variables, t, t', s, s' for terms of appropriate types, X for type variables, and U for types.) Then as before by induction we can easily verify the properties

$$t \Rightarrow_{\mathbf{F}} t',\ s \Rightarrow_{\mathbf{F}} s' \quad\Rightarrow\quad t[x := s] \Rightarrow_{\mathbf{F}} t'[x := s'], \tag{6}$$

$$t \Rightarrow_{\mathbf{F}} t' \quad\Rightarrow\quad t[X := U] \Rightarrow_{\mathbf{F}} t'[X := U], \tag{7}$$

$$t \Rightarrow_{\mathbf{F}} t' \quad\Rightarrow\quad t' \Rightarrow_{\mathbf{F}} t^*, \tag{8}$$

where for each term t of **F** we define t^* inductively as

(F1*) $x^* \equiv x$.

(F2*) $(\lambda x{:}\, U.t)^* \equiv \lambda x{:}\, U.t^*$.

(F3*) $(ts)^* \equiv t^* s^*$ if $t \not\equiv \lambda x{:}\, U.t'$ for any x, U and t'.

(F4*) $(\Lambda X.t)^* \equiv \Lambda X.t^*$.

(F5*) $(tU)^* \equiv t^* U$ if $t \not\equiv \Lambda X.t'$ for any X and t'.

(F6*) $((\lambda x{:}\, U.t)\, s)^* \equiv t^*[x := s^*]$.

(F7*) $((\Lambda X.t)\, U)^* \equiv t^*[X := U]$.

Property (8) then immediately yields the Church–Rosser theorem for **F** as before.

In the case of an extensional Girard's system, we add

(F8) $\lambda z{:}\, U.tz \Rightarrow_{\mathbf{F}} t'$ if $t \Rightarrow_{\mathbf{F}} t'$ and $z \notin \mathrm{FV}(t)$

to the inductive definition of $\Rightarrow_{\mathbf{F}}$, add

(F8*) $(\lambda z{:}\, U.tz)^* \equiv t^*$ if $z \notin \mathrm{FV}(t)$

to the inductive definition of t^*, and let (F2*) be valid only when $\lambda x{:}\, U.t$ is not an η-redex. With these modifications, again we can prove statements (6) ~ (8) by induction on the structure of the term t. For example, to verify (8) where t is an η-redex, we observe that in this case $t \Rightarrow_{\mathbf{F}} t'$ means either

- $t \equiv \lambda z{:}\, U.sz \Rightarrow_{\mathbf{F}} s' \equiv t'$,
- $t \equiv \lambda z{:}\, U.sz \Rightarrow_{\mathbf{F}} \lambda z{:}\, U.s'z \equiv t'$, or
- $t \equiv \lambda z{:}\, U.(\lambda y{:}\, U.s)\, z \Rightarrow_{\mathbf{F}} \lambda z{:}\, U.s'[y := z] \equiv t'$,

for some s, s', z, y such that $s \Rightarrow_F s'$ and $z \notin FV(s)$. In the first two cases, clearly $t' \Rightarrow_F s^* \equiv t^*$ since by the induction hypothesis $s' \Rightarrow_F s^*$. In the last case, we have $t' \equiv \lambda y{:}\ U.s' \Rightarrow_F (\lambda y{:}\ U.s)^* \equiv t^*$ by the induction hypothesis applied to $\lambda y{:}\ U.s \Rightarrow_F \lambda y{:}\ U.s'$. Other cases are straightforward.

This proof technique is also applicable to systems with certain operators and additional reduction rules. As an example, consider Gödel's system **T** for primitive recursive functionals of finite type. Here we take the version with pairing, recursor, and conditional operators (cf. Girard *et al.*, 1989). For this system, we define the parallel reduction $\Rightarrow_T$ inductively on well-typed terms as

(T1) $x \Rightarrow_T x$, $0 \Rightarrow_T 0$, $\mathsf{T} \Rightarrow_T \mathsf{T}$, $\mathsf{F} \Rightarrow_T \mathsf{F}$,

(T2) $\lambda x{:}\ U.t \Rightarrow_T \lambda x{:}\ U.t'$,

(T3) $ts \Rightarrow_T t's'$,

(T4) $\langle t, s\rangle \Rightarrow_T \langle t', s'\rangle$,

(T5) $\pi^1 t \Rightarrow_T \pi^1 t'$, $\pi^2 t \Rightarrow_T \pi^2 t'$,

(T6) $\mathsf{R}tsr \Rightarrow_T \mathsf{R}t's'r'$,

(T7) $\mathsf{D}tsr \Rightarrow_T \mathsf{D}t's'r'$,

(T8) $(\lambda x{:}\ U.t)\, s \Rightarrow_T t'[x := s']$,

(T9) $\pi^1\langle t, s\rangle \Rightarrow_T t'$, $\pi^2\langle t, s\rangle \Rightarrow_T s'$,

(T10) $\mathsf{R}ts0 \Rightarrow_T t'$, $\mathsf{R}ts(\mathsf{S}r) \Rightarrow_T s'(\mathsf{R}t's'r')\, r'$,

(T11) $\mathsf{D}ts\mathsf{T} \Rightarrow_T t'$, $\mathsf{D}ts\mathsf{F} \Rightarrow_T s'$,

assuming $t \Rightarrow_T t'$, $s \Rightarrow_T s'$, $r \Rightarrow_T r'$. (Here x stands for term variables, t, t', s, s', r, r' for terms of appropriate types, and U for types.) Then by defining t^* for each term t of **T** in the same spirit as before, we can readily verify

$$t \underset{T}{\Rightarrow} t' \quad \Rightarrow \quad t' \underset{T}{\Rightarrow} t^*,$$

which shows the Church–Rosser theorem for **T**.

See Sato (1991) and Takahashi (1993) for other applications of the proof technique to systems with more operators and reduction rules.

From these examples, one might get the impression that the proof technique always works successfully in proving the confluence of reduction systems. But this is not the case; there are certain systems, such as the restriction of $\beta\eta^{-1}$ discussed in Mints (1979), for which the proof technique does not work (because of the context-sensitiveness of the reduction rules) (cf. Akama, 1993).

As for the origin of the Tait-and-Martin-Löf proof, it was first presented in a lecture by William W. Tait for combinatory logic, and Per Martin-Löf adapted it in 1971 for his unpublished work (Martin-Löf, 1993). In published form, to my knowledge, Lévy (1975, 1976) is the only literature that contains it until late 1980s, when a preliminary version of the present work appeared (see the footnote in the first page), while a number of textbooks and articles carry its variations as mentioned earlier.

The aim of this paper is to show that the notion of parallel reduction is useful for proving not only Church–Rosser theorems but also some other fundamental theorems in the reduction theory of λ-calculus. It is useful in the sense that the notion provides excellent means to express fundamental ideas of the proofs, and at the same time, what we so expressed can be verified by induction without difficulty. In fact, it often happens as in the case of Church–Rosser theorems that once the idea is stated properly in terms of the parallel reduction, the essential part of the proof is almost over, because the inductive verification of the statement is so easy, even mechanical.

In Section 2, we apply the notion of parallel β-reduction to the proof of the standardization theorem for β. The key statement there on $\Rightarrow_\beta$ (cf. Lemma 2.2) may seem to be less intuitive at first sight than the case of Church–Rosser theorems. But when one wants it to be verified by induction, the idea behind the statement would be understood with ease. The statement is also shown to yield simple proofs of some other fundamental theorems such as the head normalization theorem, the quasi-head reduction theorem and the quasi-leftmost reduction theorem for β-reduction.

In Section 3, we define the notion of parallel η-reduction $\Rightarrow_\eta$, and use it to give a short proof of the postponement theorem of η-reduction (in $\beta\eta$-reduction). The last section is devoted to a simple inductive proof of the leftmost reduction theorem for the extensional λ-calculus $\beta\eta$, based on the results in previous sections.

In the literature, most of the theorems above mentioned have been proved by way of "residuals" (based on the theorem of "finiteness of developments"). This paper exhibits as a whole that, as far as these theorems are concerned, the role played by "residuals" in conventional proofs can be fulfilled by the notion of parallel reduction in a succinct way. In passing, we note that recently type-theoretic proofs are found for the head normalization theorem, the (quasi-)leftmost reduction theorem for β, and some others (Krivine, 1990).

We also note that another application of the parallel reduction in the realm of typed λ-calculus is found in Yokouchi (1993).

2. STANDARDIZATION THEOREM FOR β-REDUCTION

In this section, we first establish a fundamental property of $\Rightarrow_\beta$ (cf. Lemma 2.2), and apply it to prove the standardization theorem and some other fundamental theorems in the non-extensional λ-calculus β.

We will write $\xrightarrow{h}_\beta$ ($\xrightarrow{i}_\beta$, $\xrightarrow{l}_\beta$, respectively) for one-step head (internal, leftmost) β-reduction, and $\xrightarrow{h}\!\!\!\rightarrow_\beta$ ($\xrightarrow{i}\!\!\!\rightarrow_\beta$, $\xrightarrow{l}\!\!\!\rightarrow_\beta$) for the reflexive transitive closure. The following lemma is immediate from the definition of $\xrightarrow{h}_\beta$.

LEMMA 2.1. (1) *$M \xrightarrow{h}_\beta N$ implies $\lambda x.M \xrightarrow{h}_\beta \lambda x.N$.*

(2) *$M \xrightarrow{h}_\beta N$ implies $M[x := P] \xrightarrow{h}_\beta N[x := P]$.*

(3) *$M \xrightarrow{h}_\beta N$ implies $MP \xrightarrow{h}_\beta NP$, unless M is an abstraction.*

We write $P \overset{i}{\Longrightarrow}_\beta Q$ when $P \Longrightarrow_\beta Q$ and $P \overset{i}{\twoheadrightarrow}_\beta Q$. In case P is in head normal form, say $P \equiv \lambda \vec{y}.xP_1P_2 \cdots P_n$, $P \overset{i}{\Longrightarrow}_\beta Q$ means simply $P \Longrightarrow_\beta Q$; in this case, of course $Q \equiv \lambda \vec{y}.xQ_1Q_2 \cdots Q_n$ with $P_j \Longrightarrow_\beta Q_j$ ($j = 1, 2, ..., n$) for some $Q_1, Q_2, ..., Q_n$. On the other hand, if P has a head redex, say $P \equiv \lambda \vec{y}.(\lambda x.P_0)P_1P_2 \cdots P_n$ with $n \geq 1$, then $P \overset{i}{\Longrightarrow}_\beta Q$ means $Q \equiv \lambda \vec{y}.(\lambda x.Q_0)Q_1Q_2 \cdots Q_n$ with $P_j \Longrightarrow_\beta Q_j$ ($j = 0, 1, ..., n$) for some $Q_0, Q_1, ..., Q_n$.

The key lemma in our proof of the standardization theorem is the following.

LEMMA 2.2 (Main Lemma). *$M \Longrightarrow_\beta N$ implies $M \overset{h}{\twoheadrightarrow}_\beta P \overset{i}{\Longrightarrow}_\beta N$ for some P.*

Proof. By induction we will verify a stronger statement:

$$\begin{aligned} &M \Longrightarrow_\beta N \text{ implies } M \equiv M_0 \xrightarrow{h}_\beta M_1 \xrightarrow{h}_\beta \\ &\quad M_2 \xrightarrow{h}_\beta \cdots \xrightarrow{h}_\beta M_m \overset{i}{\Longrightarrow}_\beta N \\ &\quad \text{for some } m \geq 0 \text{ and } M_j \text{ such that} \\ &\quad M_j \Longrightarrow_\beta N \; (j = 0, 1, ..., m). \end{aligned} \qquad (*)$$

For simplicity we call the reduction sequence $(*)$ from M to N (with the associated condition for M_j's) a $*$-sequence, and when there is a $*$-sequence from M to N we will write $M * N$.

Now we prove the statement

$$M \underset{\beta}{\Longrightarrow} N \quad \Longrightarrow \quad M * N \qquad (**)$$

by induction on the structure of M.

Assume $M \Longrightarrow_\beta N$. Then we are in one of the following cases:

(β1) $M \equiv x \equiv N$,

(β2) $M \equiv \lambda x.M' \Longrightarrow_\beta \lambda x.N' \equiv N$,

(β3) $M \equiv M'M'' \Longrightarrow_\beta N'N'' \equiv N$,

(β4) $M \equiv (\lambda x.M')M'' \Longrightarrow_\beta N'[x := N''] \equiv N$.

Here $M' \Longrightarrow_\beta N'$ and $M'' \Longrightarrow_\beta N''$. In case ($\beta$1), clearly we have $M * N$. To study the other cases, we use the following properties of $*$-sequences:

$$M * N \quad \Longrightarrow \quad \lambda x.M * \lambda x.N \qquad (1)$$

$$M * N, P \underset{\beta}{\Longrightarrow} Q \quad \Longrightarrow \quad MP * NQ \qquad (2)$$

$$M * N, P * Q \quad \Longrightarrow \quad P[x := M] * Q[x := N]. \qquad (3)$$

Property (1) is clear from the definition; (2) and (3) will be proved in Lemmas 2.3 and 2.4 below. Once we get these properties, it is easy to verify $(**)$ in cases (β2) ~ (β4) by induction on M. Indeed, in case (β2), by the induction hypothesis we know $M' * N'$ and therefore $M \equiv \lambda x.M' * \lambda x.N' \equiv N$ by property (1). Similarly in case (β3), from the induction hypothesis $M' * N'$ and property (2), we get $M \equiv M'M'' * N'N'' \equiv N$. In case ($\beta$4), since $M' * N'$ and $M'' * N''$ by the induction hypothesis, we get

$$M \equiv (\lambda x.M')M'' \xrightarrow{h}_\beta M'[x := M''] * N'[x := N''] \equiv N,$$

using the property (3). This is surely a $*$-sequence, and hence $M * N$. ∎

LEMMA 2.3. *$M * N, P \Longrightarrow_\beta Q \Longrightarrow MP * NQ$.*

Proof. Suppose

$$M_0 \xrightarrow{h}_\beta M_1 \xrightarrow{h}_\beta M_2 \xrightarrow{h}_\beta \cdots \xrightarrow{h}_\beta M_m \overset{i}{\Longrightarrow}_\beta N$$

is a $*$-sequence from M to N. If there is an abstraction among $M_0, M_1, ..., M_m$, let M_k be the first one. Then

$$M_0P \xrightarrow{h}_\beta M_1P \xrightarrow{h}_\beta M_2P \xrightarrow{h}_\beta \cdots \xrightarrow{h}_\beta M_kP \overset{i}{\Longrightarrow}_\beta NQ$$

is the required $*$-sequence from MP to NQ. (Recall Lemma 2.1(3) and the fact that $M_kP \equiv (\lambda x.M')P \overset{i}{\Longrightarrow}_\beta (\lambda x.N')Q \equiv NQ$ for some x, M', N' such that $M' \Longrightarrow_\beta N'$ since $M_k \Longrightarrow_\beta N$.) On the other hand, if there is no abstraction in $M_0, M_1, ..., M_m$, the same holds for $k = m$. ∎

LEMMA 2.4. *$M * N, P * Q \Longrightarrow P[x := M] * Q[x := N]$.*

Proof. First, we consider the case where $P * Q$ is just $P \overset{i}{\Longrightarrow}_\beta Q$. Then we have either

(i) $P \equiv \lambda \vec{y}.yP_1P_2 \cdots P_n$ and $Q \equiv \lambda \vec{y}.yQ_1Q_2 \cdots Q_n$ with $n \geq 0$ and $P_j \Longrightarrow_\beta Q_j$ ($j = 1, 2, ..., n$), or

(ii) $P \equiv \lambda \vec{y}.(\lambda y.P_0)P_1P_2 \cdots P_n$ and $Q \equiv \lambda \vec{y}.(\lambda y.Q_0) \times Q_1Q_2 \cdots Q_n$ with $n \geq 1$ and $P_j \Longrightarrow_\beta Q_j$ ($j = 0, 1, ..., n$)

for a sequence $\vec{y}$ of variables, a variable y, and λ-terms P_j and Q_j. (In (i), x and y may be identical.) In both cases we may assume that $\vec{y}$ is empty, since the nonempty case immediately follows from the empty case and the property (1) of $*$-sequences.

For each j, let $P'_j \equiv P_j[x := M]$ and $Q'_j \equiv Q_j[x := N]$. Then we have $P'_j \Longrightarrow_\beta Q'_j$ since $P_j \Longrightarrow_\beta Q_j$ and $M \Longrightarrow_\beta N$. In case (i), if $y \equiv x$ then by applying Lemma 2.3 repeatedly we get

$$P[x := M] \equiv MP'_1P'_2 \cdots P'_n * NQ'_1Q'_2 \cdots Q'_n \equiv Q[x := N].$$

If $y \not\equiv x$ in case (i), clearly

$$\begin{aligned} P[x := M] &\equiv yP'_1P'_2 \cdots P'_n \overset{i}{\Longrightarrow}_\beta yQ'_1Q'_2 \cdots Q'_n \\ &\equiv Q[x := N]. \end{aligned}$$

In case (ii), it is also clear that

$$P[x:=M] \equiv (\lambda y.P'_0)\, P'_1 P'_2 \cdots P'_n \xrightarrow{i}_\beta (\lambda y.Q'_0)\, Q'_1 Q'_2 \cdots Q'_n \equiv Q[x:=N].$$

Thus the lemma holds when $P \stackrel{i}{\Rightarrow}_\beta Q$.

Now suppose $P * Q$, and

$$P^{(0)} \xrightarrow{h}_\beta P^{(1)} \xrightarrow{h}_\beta P^{(2)} \xrightarrow{h}_\beta \cdots \xrightarrow{h}_\beta P^{(p)} \stackrel{i}{\Rightarrow}_\beta Q$$

is a $*$-sequence from P to Q. Then by applying Lemma 2.1(2) and the fact we have just shown, we get

$$P^{(0)}[x:=M] \xrightarrow{h}_\beta P^{(1)}[x:=M] \xrightarrow{h}_\beta \cdots \xrightarrow{h}_\beta P^{(p)}[x:=M] * Q[x:=N],$$

which is a $*$-sequence from $P[x:=M]$ to $Q[x:=N]$. ▮

The proof of Lemma 2.2 is now completed. It says that $M \Rightarrow_\beta N$ implies $M \xrightarrow{h}\!\!\!\rightarrow_\beta P \xrightarrow{i}\!\!\!\rightarrow_\beta N$ for some P. The same holds true under the weaker condition $M \twoheadrightarrow_\beta N$. To see this we need an additional observation.

LEMMA 2.5. *$M \stackrel{i}{\Rightarrow}_\beta P \xrightarrow{h}_\beta N$ implies $M \xrightarrow{h}\!\!\!\rightarrow_\beta Q \stackrel{i}{\Rightarrow}_\beta N$ for some Q.*

Proof. Since $P \xrightarrow{h}_\beta N$, we have $P \equiv \lambda\vec{y}.(\lambda x.P_0)P_1 P_2 \cdots P_n$ and $N \equiv \lambda\vec{y}.(P_0[x:=P_1])\, P_2 \cdots P_n$ for some $n \geqslant 1$ and $\vec{y}, x, P_0, P_1, ..., P_n$. Next, since $M \stackrel{i}{\Rightarrow}_\beta P \equiv \lambda\vec{y}.(\lambda x.P_0)P_1 P_2 \cdots P_n$, we know that $M \equiv \lambda\vec{y}.(\lambda x.M_0)\, M_1 M_2 \cdots M_n$ for some M_j with $M_j \Rightarrow_\beta P_j$ $(j=0, 1, ..., n)$. This implies $M \xrightarrow{h}_\beta \lambda\vec{y}.(M_0[x:=M_1])\, M_2 \cdots M_n \Rightarrow_\beta \lambda\vec{y}.(P_0[x:=P_1])\, P_2 \cdots P_n \equiv N$, which together with Lemma 2.2 shows the lemma. ▮

COROLLARY 2.6 (Mitschke, 1979; Barendregt, 1984, Lemma 11.4.6). *$M \twoheadrightarrow_\beta N$ implies $M \xrightarrow{h}\!\!\!\rightarrow_\beta P \xrightarrow{i}\!\!\!\rightarrow_\beta N$ for some P.*

Proof. Recall that $\twoheadrightarrow_\beta$ ($\xrightarrow{i}\!\!\!\rightarrow_\beta$, resp.) is the reflexive transitive closure of $\Rightarrow_\beta$ (of $\stackrel{i}{\Rightarrow}_\beta$), and apply Lemmas 2.2 and 2.5. ▮

COROLLARY 2.7. (Barendregt, 1984, Corollary 11.4.8, Head Normalization Theorem). *If M has a head normal form, then $M \xrightarrow{h}\!\!\!\rightarrow_\beta P$ for some P in head normal form.*

Proof. Immediate from Corollary 2.6, since if $P \xrightarrow{i}\!\!\!\rightarrow_\beta N$ and N is in head normal form, then so is P. ▮

From Corollary 2.6, we can also obtain the leftmost reduction theorem and the standardization theorem for $\twoheadrightarrow_\beta$, by using Mitschke's argument (Mitschke, 1979). To make the paper somewhat self-contained, we include the argument.

THEOREM 2.8 (Barendregt, 1984, Theorem 13.2.2, Leftmost Reduction Theorem for β). *If M has a β-normal form N, then $M \xrightarrow{l}\!\!\!\rightarrow_\beta N$.*

Proof. By induction on the structure of N. Suppose $M \xrightarrow{h}\!\!\!\rightarrow_\beta P \xrightarrow{i}\!\!\!\rightarrow_\beta N$ and $N \equiv \lambda\vec{y}.xN_1 N_2 \cdots N_n$ where $n \geqslant 0$ and $N_1, ..., N_n$ are in β-normal form. Then $P \equiv \lambda\vec{y}.xP_1 P_2 \cdots P_n$ for some $P_j \twoheadrightarrow_\beta N_j$ $(j=0, 1, ..., n)$. Now by induction hypothesis we have $P_j \xrightarrow{l}\!\!\!\rightarrow_\beta N_j$ $(j=0, 1, ..., n)$, and therefore $M \xrightarrow{h}\!\!\!\rightarrow_\beta P \equiv \lambda\vec{y}.xP_1 P_2 \cdots P_n \xrightarrow{l}\!\!\!\rightarrow_\beta \lambda\vec{y}.xN_1 P_2 \cdots P_n \xrightarrow{l}\!\!\!\rightarrow_\beta \lambda\vec{y}.xN_1 N_2 P_3 \cdots P_n \xrightarrow{l}\!\!\!\rightarrow_\beta \cdots \xrightarrow{l}\!\!\!\rightarrow_\beta \lambda\vec{y}.xN_1 N_2 N_3 \cdots N_n \equiv N$, which is indeed the leftmost reduction from M to N. ▮

A reduction sequence σ: $M_0 \rightarrow_\beta M_1 \rightarrow_\beta \cdots \rightarrow_\beta M_n$ is said to be standard if, roughly speaking, the sequence of "positions" of the β-redexes contracted in σ moves from left to right. More precisely, when $M \rightarrow_\beta M'$ and the β-redex contracted in this reduction step begins with the pth symbol from the left in M (in writing M in unabbreviated fully parenthesized form), we denote the number p by $p(M \rightarrow_\beta M')$. (Here by "symbols" we mean occurrences of variables, λ, point and parentheses.) Then we say a reduction sequence $M_0 \rightarrow_\beta M_1 \rightarrow_\beta \cdots \rightarrow_\beta M_n$ is standard if $p(M_0 \rightarrow_\beta M_1) \leqslant p(M_1 \rightarrow_\beta M_2) \leqslant \cdots \leqslant p(M_{n-1} \rightarrow_\beta M_n)$.

THEOREM 2.9 (Mitschke, 1979; Barendregt, 1984, Theorem 11.4.7, Standardization Theorem for β). *If $M \twoheadrightarrow_\beta N$, then we can obtain N from M by a standard reduction sequence.*

Proof. By induction on the structure of N. From the assumption and Corollary 2.6, we have $M \xrightarrow{h}\!\!\!\rightarrow_\beta P \xrightarrow{i}\!\!\!\rightarrow_\beta N$ for some P. When N is a single variable, the theorem is trivial since $M \xrightarrow{h}\!\!\!\rightarrow_\beta P \equiv N$ is certainly a standard reduction sequence. Otherwise, we can write $N \equiv \lambda\vec{y}.N_1 N_2 \cdots N_n$ where each N_j is a proper subterm of N. Then we have $P \equiv \lambda\vec{y}.P_1 P_2 \cdots P_n$ where $P_j \twoheadrightarrow_\beta N_j$ $(j=1, 2, ..., n)$. In this case, by the induction hypothesis there exists a standard reduction sequence from P_j to N_j, say σ_j. Let σ be the head reduction sequence $M \xrightarrow{h}\!\!\!\rightarrow_\beta P$ followed by $\sigma_1, \sigma_2, ..., \sigma_n$ (in this order) applied to the subterms $P_1, P_2, ..., P_n$, respectively. Then clearly σ is a standard reduction sequence from M to N. ▮

The proof technique of Theorem 2.8 can be extended to show the quasi-leftmost reduction theorem for β-reduction. Before proving the theorem, we present the quasi-head reduction theorem which is another consequence of Lemma 2.5.

An (infinite) β-reduction sequence is called quasi-leftmost, if it contains infinitely many leftmost reduction steps $\xrightarrow{l}_\beta$. It is also called a quasi-head reduction, if it contains infinitely many head reduction steps $\xrightarrow{h}_\beta$.

THEOREM 2.10 (Quasi-Head Reduction Theorem). *If M has a head normal form, then there is no (infinite) quasi-head reduction sequence from M.*

Proof. The proof of Lemma 2.5 actually shows that $M \xrightarrow{i}_\beta \cdot \xrightarrow{h}_\beta N$ implies $M \xrightarrow{h}_\beta \cdot \xrightarrow{h}\!\!\!\rightarrow_\beta \cdot \xrightarrow{i}_\beta N$. It means that if there is an infinite quasi-head reduction sequence from M, then for any $n \geqslant 0$ there is a head reduction sequence from M whose length is n. So in this case M has no head normal form by Corollary 2.7. This proves the contraposition of the theorem. ∎

THEOREM 2.11 (Barendregt, 1984, Theorem 13.2.6, Quasi-Leftmost Reduction Theorem). *If M has a β-normal form, then there is no (infinite) quasi-leftmost β-reduction sequence from M.*

Proof. Suppose $N \equiv \lambda \vec{y}.xN_1 \cdots N_n$ is the β-normal form of M, and

$$M \twoheadrightarrow_\beta M_0 \xrightarrow{l}_\beta M_1 \twoheadrightarrow_\beta M_2 \xrightarrow{l}_\beta M_3 \twoheadrightarrow_\beta \cdots$$
$$\twoheadrightarrow_\beta M_{2j} \xrightarrow{l}_\beta M_{2j+1} \twoheadrightarrow_\beta \cdots \qquad (\dagger)$$

is a quasi-leftmost β-reduction sequence from M. Then by the previous theorem, only finitely many of $M_{2j} \xrightarrow{l}_\beta M_{2j+1}$ $(j = 0, 1, 2, ...)$ are head reduction steps. Suppose $M_{2k} \xrightarrow{l}_\beta M_{2k+1}$ is not a head reduction step. Then M_{2k} is in head normal form, and hence by the Church–Rosser theorem we can write $M_{2k} \equiv \lambda \vec{y}.xP_1 P_2 \cdots P_n$ where $P_i \twoheadrightarrow_\beta N_i$ $(i = 1, ..., n)$. Here, by induction hypothesis (on the structure of N), each P_i has no infinite quasi-leftmost reduction sequence. Then it implies that the tail segment of (†) starting from M_{2k} is finite, which is a contradiction. ∎

3. POSTPONEMENT THEOREM

In this section, we define parallel η-reduction, and based on certain properties of the notion we give a short proof of the postponement theorem: $M \twoheadrightarrow_{\beta\eta} N$ implies $M \twoheadrightarrow_\beta P \twoheadrightarrow_\eta N$ for some P.

DEFINITION 3.1. The parallel η-reduction $\Rightarrow_\eta$ is defined inductively, as follows:

(η1) $x \Rightarrow_\eta x$,

(η2) $\lambda x.M \Rightarrow_\eta \lambda x.M'$ if $M \Rightarrow_\eta M'$,

(η3) $MN \Rightarrow_\eta M'N'$ if $M \Rightarrow_\eta M'$ and $N \Rightarrow_\eta N'$,

(η4) $\lambda z.Mz \Rightarrow_\eta M'$ if $M \Rightarrow_\eta M'$ and $z \notin \mathrm{FV}(M)$.

As before, $M \Rightarrow_\eta M'$ intuitively means that M' is obtained from M by simultaneous contraction of η-redexes existing in M. It is easy to verify that $\twoheadrightarrow_\eta$ is the transitive closure of $\Rightarrow_\eta$, and the notion $\Rightarrow_\eta$ is substitution closed; i.e.,

$$M \Rightarrow_\eta M',\ N \Rightarrow_\eta N' \quad \Rightarrow \quad M[y := N] \Rightarrow_\eta M'[y := N'].$$

We will denote the k-fold η-expansion of M by $(M)_k$; more precisely,

$$(M)_k \equiv \lambda z_1.(\lambda z_2.(\cdots(\lambda z_k.Mz_k)\cdots)z_2)z_1,$$

where M is a λ-term, $k \geqslant 0$, and $z_1, z_2, ..., z_k \notin \mathrm{FV}(M)$. In particular, $(M)_0 \equiv M$.

LEMMA 3.2. (1) $M \Rightarrow_\eta x \Leftrightarrow M \equiv (x)_k$ *for some* $k \geqslant 0$.

(2) $M \Rightarrow_\eta N_1 N_2 \Leftrightarrow M \equiv (M_1 M_2)_k$ *for some* $k \geqslant 0$ *and* M_i *such that* $M_i \Rightarrow_\eta N_i$ $(i = 1, 2)$.

(3) $M \Rightarrow_\eta \lambda x.N \Leftrightarrow M \equiv (\lambda x.M')_k$ *for some* $k \geqslant 0$ *and* M' *such that* $M' \Rightarrow_\eta N$.

Proof. (1) By the definition of $\Rightarrow_\eta$, $M \Rightarrow_\eta x$ means either $M \equiv x$ or $M \equiv \lambda z.M_1 z$ with $M_1 \Rightarrow_\eta x$ and $z \notin \mathrm{FV}(M_1)$. Applying the same argument to M_1, we eventually get $M \equiv (x)_k$ for some k. The other direction is trivial. The proofs of (2) and (3) are similar. ∎

To establish the fundamental relation between $\Rightarrow_\beta$ and $\Rightarrow_\eta$ (Lemma 3.4), we first observe the following properties of $\Rightarrow_\beta$ in relation to η-expansion $(M)_k$.

LEMMA 3.3. *Suppose* $M \Rightarrow_\beta M'$, $N \Rightarrow_\beta N'$, *and* $k \geqslant 0$. *Then*

(1) $(\lambda x.M)_k \Rightarrow_\beta \lambda x.M'$,

(2) $(\lambda x.M)_k N \Rightarrow_\beta M'[x := N']$,

(3) $(M)_k N \Rightarrow_\beta M'N'$,

(4) $(M)_{k+1} \Rightarrow_\beta (M')_1$.

Proof. When $k = 0$, the statements are trivial. So we assume $k > 0$.

(1) $(\lambda x.M)_k \equiv \lambda z_1.(\lambda z_2.(\cdots(\lambda z_k.(\lambda x.M)z_k)\cdots)z_2)z_1 \Rightarrow_\beta \lambda z_1.M'[x := z_k][z_k := z_{k-1}]\cdots[z_2 := z_1] \equiv \lambda x.M'$.

(2) $(\lambda x.M)_k N \equiv (\lambda z_1.(\lambda z_2.(\cdots(\lambda z_k.(\lambda x.M)z_k)\cdots)z_2)z_1)\, N \Rightarrow_\beta M'[x := N']$.

(3) $(M)_k N \equiv (\lambda z.Mz)_{k-1} N \Rightarrow_\beta (M'z)[z := N'] \equiv M'N'$ by (2).

(4) $(M)_{k+1} \equiv (\lambda z.Mz)_k \Rightarrow_\beta \lambda z.M'z \equiv (M')_1$ by (1). (In (3) and (4), we assume $z \notin \mathrm{FV}(MM')$.) ∎

The following is the key lemma in our proof of the postponement theorem.

LEMMA 3.4. $M \Rightarrow_\eta P \Rightarrow_\beta N$ *implies* $M \Rightarrow_\beta P' \Rightarrow_\eta N$ *for some* P'.

Proof. By induction on the structure of P. According to the definition of $P \Rightarrow_\beta N$, we consider four cases:

(β1) $P \equiv x \equiv N$.

(β2) $P \equiv \lambda x.P_1 \Rightarrow_\beta \lambda x.N_1 \equiv N$ with $P_1 \Rightarrow_\beta N_1$.

(β3) $P \equiv P_1 P_2 \Rightarrow_\beta N_1 N_2 \equiv N$ with $P_i \Rightarrow_\beta N_i$ $(i = 1, 2)$.

(β4) $P \equiv (\lambda x.P_1)\, P_2 \Rightarrow_\beta N_1[x := N_2] \equiv N$ with $P_i \Rightarrow_\beta N_i$ $(i = 1, 2)$.

Except for ($\beta 4$), the proof is straightforward. In case ($\beta 4$), by Lemma 3.2 we can write $M \equiv ((\lambda x.M_1)_l M_2)_k$ for some $k, l \geqslant 0$ and M_i such that $M_i \Rightarrow_\eta P_i$ ($i = 1, 2$). Then by the induction hypothesis we have $M_i \Rightarrow_\beta P'_i \Rightarrow_\eta N_i$ for some P'_i ($i = 1, 2$). This together with Lemma 3.3 implies $M \equiv ((\lambda x.M_1)_l M_2)_k \Rightarrow_\beta (P'_1[x := P'_2])_k \Rightarrow_\eta N_1[x := N_2] \equiv N$. ∎

THEOREM 3.5 (Barendregt, 1984, Corollary 15.1.6, Postponement Theorem of η-Reduction). *$M \twoheadrightarrow_{\beta\eta} N$ implies $M \twoheadrightarrow_\beta P \twoheadrightarrow_\eta N$ for some P.*

Proof. Immediate from Lemma 3.4, since $\twoheadrightarrow_\beta$ ($\twoheadrightarrow_\eta$, resp.) is the transitive closure of $\Rightarrow_\beta$ (of $\Rightarrow_\eta$). ∎

In passing, we note that when the notion of parallel $\beta\eta$-reduction $\Rightarrow_{\beta\eta}$ is naturally defined (i.e., inductively with five clauses corresponding to (F1) ~ (F3), (F6), and (F8) in Section 1), the equivalence

$$M \underset{\beta\eta}{\Rightarrow} N \iff M \underset{\eta}{\Rightarrow} P \underset{\beta}{\Rightarrow} N \quad \text{for some } P$$

can be verified as in Lemma 3.4. The converse of Lemma 3.4 however does not hold. Indeed, $\lambda x.(\lambda y.yx)\, z \Rightarrow_\beta \lambda x.zx \Rightarrow_\eta z$, but not $\lambda x.(\lambda y.yx)\, z \Rightarrow_{\beta\eta} z$.

We list some other consequences of previous lemmas.

LEMMA 3.6. *If $P \Rightarrow_\eta Q$ and Q has a β-normal form, then P has a β-normal form.*

Proof. By virtue of Lemma 3.4, it suffices to show the lemma in case Q is in β-normal form. Suppose $Q \equiv \lambda x_1 x_2 \cdots x_m.xQ_1Q_2 \cdots Q_n$ where $Q_1, ..., Q_n$ are in β-normal form. Then by Lemma 3.2,

$$P \equiv (\lambda x_1.(\lambda x_2.\cdots(\lambda x_m.P')_{l_m}\cdots)_{l_2})_{l_1}$$

with

$$P' \equiv ((\cdots(((x)_{k_0} P_1)_{k_1} P_2)_{k_2} \cdots P_{n-1})_{k_{n-1}} P_n)_{k_n}$$

for some $P_i \Rightarrow_\eta Q_i$ ($i = 1, 2, ..., n$), and $k_0, ..., k_n, l_1, ..., l_m \geqslant 0$. By the induction hypothesis (on the structure of Q), each P_i has a β-normal form, say N_i; then P' has one, since by Lemma 3.3 $P' \Rightarrow_\beta (xP_1P_2 \cdots P_n)_k \twoheadrightarrow_\beta (xN_1N_2 \cdots N_n)_k$ for some $k \leqslant 1$. We also have $P \Rightarrow_\beta \lambda x_1 x_2 \cdots x_m.P'$ by the same lemma. Therefore P has a β-normal form. ∎

LEMMA 3.7. *If $P \Rightarrow_\eta Q$ and P is in β-normal form, then so is Q.*

Proof. If $P \equiv \lambda x_1 x_2 \cdots x_m.xP_1P_2 \cdots P_n \Rightarrow_\eta Q$, then either $Q \equiv \lambda x_1 x_2 \cdots x_m.xQ_1Q_2 \cdots Q_n$, or $Q \equiv \lambda x_1 x_2 \cdots x_{m-1}.xQ_1Q_2 \cdots Q_{n-1}$ where in both cases $P_i \Rightarrow_\eta Q_i$ for each i. Here by the induction hypothesis (on the structure of P) each Q_i is in β-normal form, and so is Q. ∎

From these two lemmas, we immediately know that M has a β-normal form if and only if M has a $\beta\eta$-normal form (Barendregt, 1984, Corollary 15.1.5).

4. LEFTMOST REDUCTION THEOREM FOR βη-REDUCTION

Based on the results in previous sections, we give a simple proof of the leftmost reduction (or normalization) theorem for $\beta\eta$-reduction.

The one-step leftmost $\beta\eta$-reduction $\xrightarrow{l}_{\beta\eta}$ is defined as follows.

($l\beta\eta 1$) If $z \notin \mathrm{FV}(P)$, then $M \equiv \lambda\vec{y}.(\lambda z.Pz) \xrightarrow{l}_{\beta\eta} \lambda\vec{y}.P$.

($l\beta\eta 2$) If M is not of the form above but has a head redex and $M \xrightarrow{h}_\beta N$, then $M \xrightarrow{l}_{\beta\eta} N$.

($l\beta\eta 3$) If M is not of the forms above and $M \equiv \lambda\vec{y}.xM_1M_2 \cdots M_n$ where, for some i ($1 \leqslant i \leqslant n$), $M_1, ..., M_{i-1}$ are in $\beta\eta$-normal form and $M_i \xrightarrow{l}_{\beta\eta} M'_i$, then $M \xrightarrow{l}_{\beta\eta} \lambda\vec{y}.xM_1M_2 \cdots M_{i-1}M'_iM_{i+1} \cdots M_n$.

The leftmost $\beta\eta$-redex is defined naturally; in case ($l\beta\eta 1$) the η-redex $\lambda z.Pz$ is the leftmost $\beta\eta$-redex of M, while in case ($l\beta\eta 2$) so is the head redex of M. In case ($l\beta\eta 3$) the leftmost $\beta\eta$-redex of M_i is that of M.

The reflexive transitive closure of $\xrightarrow{l}_{\beta\eta}$ is called the leftmost $\beta\eta$-reduction, and is denoted by $\overset{l}{\twoheadrightarrow}_{\beta\eta}$.

LEMMA 4.1. *If $P \xleftarrow{l}_{\beta\eta} Q \xrightarrow{l}_\beta R$, then either $P \equiv R$ or $P \xrightarrow{l}_\beta Q' \xleftarrow{l}_{\beta\eta} R$ for some Q'.*

Proof. We may assume that the leftmost $\beta\eta$-redex of Q is an η-redex, say $\lambda z.Mz$ with $z \notin \mathrm{FV}(M)$. Then $Q \equiv \cdots(\lambda z.Mz)\cdots \xrightarrow{l}_{\beta\eta} \cdots M \cdots \equiv P$, and the leftmost β-redex of Q is either contained in the η-redex $\lambda z.Mz$ or thereafter.

Case 1. If the leftmost β-redex is Mz, that is, $M \equiv \lambda z.M_1$ for some M_1, then $Q \equiv \cdots(\lambda z.(\lambda z.M_1)\, z)\cdots \xrightarrow{l}_\beta \cdots(\lambda z.M_1)\cdots \equiv R$. In this case $P \equiv R$.

Case 2. If the leftmost β-redex is in M, suppose $M \xrightarrow{l}_\beta M'$. Then we have $Q \equiv \cdots(\lambda z.Mz)\cdots \xrightarrow{l}_\beta R \equiv \cdots (\lambda z.M'z)\cdots \xrightarrow{l}_{\beta\eta} \cdots M' \cdots$ and $P \equiv \cdots M \cdots \xrightarrow{l}_\beta \cdots M' \cdots$.

Case 3. If the leftmost β-redex is after the η-redex, suppose $Q \equiv \cdots(\lambda z.Mz)\cdots \xrightarrow{l}_\beta R \equiv \cdots(\lambda z.Mz)\circ\circ\circ$. Then $R \xrightarrow{l}_{\beta\eta} \cdots M\circ\circ\circ$ and $P \equiv \cdots M \cdots \xrightarrow{l}_\beta \cdots M\circ\circ\circ$. ∎

COROLLARLY 4.2. *If $P \xleftarrow{l}_{\beta\eta} Q \overset{l}{\twoheadrightarrow}_\beta R$, then either*

(1) *$P \overset{l}{\twoheadrightarrow}_\beta R$ with $|P \overset{l}{\twoheadrightarrow}_\beta R| < |Q \overset{l}{\twoheadrightarrow}_\beta R|$, or*

(2) *there exists Q' such that $P \overset{l}{\twoheadrightarrow}_\beta Q' \xleftarrow{l}_{\beta\eta} R$ with $|P \overset{l}{\twoheadrightarrow}_\beta Q'| = |Q \overset{l}{\twoheadrightarrow}_\beta R|$.*

Here by $|\sigma|$ we mean the length of the leftmost β-reduction σ.

Proof. By induction on $|Q \overset{l}{\twoheadrightarrow}_\beta R|$, using Lemma 4.1. ∎

THEOREM 4.3 (Klop, 1980, Theorem 5.8). *If M has a $\beta\eta$-normal form N, then $M \xrightarrow{l}\!\!\twoheadrightarrow_{\beta\eta} N$.*

Proof. By Theorem 3.5, we have $M \twoheadrightarrow_\beta P \twoheadrightarrow_\eta N$ for some P. Here we may assume that P is in β-normal form, because by Lemma 3.6, P has a β-normal form, say P', and $P' \twoheadrightarrow_\eta N$ (by the Church–Rosser theorem for $\beta\eta$ and Theorem 3.5). Then by the leftmost reduction theorem for β (Theorem 2.8), we can write $M \equiv M_0 \xrightarrow{l}_\beta M_1 \xrightarrow{l}_\beta M_2 \xrightarrow{l}_\beta \cdots \xrightarrow{l}_\beta M_m \equiv P$ for some $m \geqslant 0$. On the other hand, clearly one can obtain N from P by a leftmost η-reduction, say $P \equiv P_0 \xrightarrow{l}_\eta P_1 \xrightarrow{l}_\eta P_2 \xrightarrow{l}_\eta \cdots \xrightarrow{l}_\eta P_p \equiv N$ (where $\xrightarrow{l}_\eta$ stands for a one-step leftmost η-reduction). Note that the leftmost η-reduction from P to N is also a leftmost $\beta\eta$-reduction, since each P_j is in β-normal form.

We now prove the theorem by induction on $m+p$, the length $|M \xrightarrow{l}\!\!\twoheadrightarrow_\beta P \xrightarrow{l}\!\!\twoheadrightarrow_\eta N|$ of the reduction sequence $M \xrightarrow{l}\!\!\twoheadrightarrow_\beta P \xrightarrow{l}\!\!\twoheadrightarrow_\eta N$. If $m+p=0$, there is nothing to prove. So assume otherwise, and let $M \xrightarrow{l}_{\beta\eta} M'$. Then by applying the previous corollary to $M' \xleftarrow{l}_{\beta\eta} M \xrightarrow{l}\!\!\twoheadrightarrow_\beta P$, we get either

- $M' \xrightarrow{l}\!\!\twoheadrightarrow_\beta P$ with $|M' \xrightarrow{l}\!\!\twoheadrightarrow_\beta P| < |M \xrightarrow{l}\!\!\twoheadrightarrow_\beta P| = m$, or
- $M' \xrightarrow{l}\!\!\twoheadrightarrow_\beta P_1 (\xleftarrow{l}_{\beta\eta} P)$ with $|M' \xrightarrow{l}\!\!\twoheadrightarrow_\beta P_1| = m$.

In the first case we have

$$M \xrightarrow{l}_{\beta\eta} M' \xrightarrow{l}\!\!\twoheadrightarrow_\beta P \xrightarrow{l}\!\!\twoheadrightarrow_\eta N \quad \text{with} \quad |M' \xrightarrow{l}\!\!\twoheadrightarrow_\beta P \xrightarrow{l}\!\!\twoheadrightarrow_\eta N| < m+p,$$

while in the second case

$$M \xrightarrow{l}_{\beta\eta} M' \xrightarrow{l}\!\!\twoheadrightarrow_\beta P_1 \xrightarrow{l}\!\!\twoheadrightarrow_\eta N \quad \text{with} \quad |M' \xrightarrow{l}\!\!\twoheadrightarrow_\beta P_1 \xrightarrow{l}\!\!\twoheadrightarrow_\eta N| < m+p.$$

In both cases, by the induction hypothesis we have $M' \xrightarrow{l}\!\!\twoheadrightarrow_{\beta\eta} N$, and hence $M \xrightarrow{l}\!\!\twoheadrightarrow_{\beta\eta} N$. ∎

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