# Termination and Complexity in Higher-Order Term Rewriting

## Part 2. Termination: non-termination

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Download handout and slides from:
https://www.cs.ru.nl/~cynthiakop/2024_isr/

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Example:

$$\begin{array}{lcl} \mathtt{a} & \rightarrow & \mathtt{a} \\ \mathtt{a} & \rightarrow & \mathtt{b} \end{array}$$

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$$\begin{array}{rcl} \mathtt{f}(\mathtt{s}(\mathtt{0}), F) & \to & \mathtt{f}(\mathtt{0}, \lambda y.\mathtt{s}(F \cdot y)) \\ \mathtt{f}(\mathtt{0}, F) & \to & \mathtt{f}(F \cdot \mathtt{s}(\mathtt{0}), F) \end{array}$$

# Finding self-loops

Question: how to automatically detect a self-loop?

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The primary higher-order difficulty is extending semi-unification techniques.

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$$\mathtt{@}(\omega, \omega) \rightarrow_{\mathcal{R}} (\lambda x.\mathtt{@}(x, x)) \cdot \omega \rightarrow_\beta \mathtt{@}(\omega, \omega)$$

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# Finding $\omega\omega$ elsewhere

A different example:

$$\begin{aligned} \texttt{f} &:: (A \Rightarrow B \Rightarrow C) \Rightarrow A \\ \texttt{g} &:: A \Rightarrow B \Rightarrow A \Rightarrow C \\ \texttt{h} &:: C \Rightarrow C \end{aligned}$$

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$$\texttt{g}(\omega, z, \omega) \to^*_{\mathcal{R}} \texttt{h}(\texttt{g}(\omega, z, \omega))$$

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- Reduction: $C[D[F], x] \to^* E[F \cdot s_1 \cdots x \cdots s_k]$

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- Then let $\omega := D[\lambda x_1 \ldots x_k . C[x_i, x_i]]$

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- Then let $\omega := D[\lambda x_1 \ldots x_k . C[x_i, x_i]]$
- We have:
  $C[\omega, \omega] \rightarrow^* E[\ (\lambda x_1 \ldots x_k . C[x_i, x_i]) \cdot \omega\ ] \rightarrow^*_\beta E[\ C[\omega, \omega]\ ]$

# Exercises

Construct a (general) self-loop for the following HTRSs:

$$\begin{array}{rcl} \mathtt{f} & :: & \mathsf{o} \Rightarrow \mathsf{o} \Rightarrow \mathsf{o} \\ \mathtt{g} & :: & \mathsf{o} \Rightarrow \mathsf{o} \\ \mathtt{h} & :: & (\mathsf{o} \Rightarrow \mathsf{o}) \Rightarrow \mathsf{o} \end{array}$$

$$\begin{array}{rcl} \mathtt{f}(y, \mathtt{h}(F)) & \rightarrow & F \cdot \mathtt{g}(y) \\ \mathtt{g}(x) & \rightarrow & x \end{array}$$

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$$\begin{array}{rcl} \mathtt{f} & :: & \mathsf{c} \Rightarrow \mathsf{a} \\ \mathtt{g} & :: & \mathsf{a} \Rightarrow \mathsf{c} \\ \mathtt{h} & :: & (\mathsf{a} \Rightarrow \mathsf{b}) \Rightarrow \mathsf{c} \\ \mathtt{k} & :: & \mathsf{a} \Rightarrow \mathsf{c} \Rightarrow \mathsf{b} \end{array}$$

$$\mathtt{k}(\mathtt{f}(\mathtt{h}(F)), \mathtt{g}(y)) \rightarrow F \cdot y$$

# Bonus exercises

Construct a (general) self-loop for the following HTRSs:

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$$\underline{\quad \mathtt{f}(\mathtt{g}(x)) \quad \to \quad x \quad}$$

$$\begin{array}{lcl} \mathtt{f} & :: & (\mathsf{b} \Rightarrow \mathsf{a} \Rightarrow \mathsf{b} \Rightarrow \mathsf{a}) \Rightarrow \mathsf{c} \\ \mathtt{g} & :: & \mathsf{b} \Rightarrow \mathsf{c} \\ \mathtt{h} & :: & \mathsf{c} \Rightarrow \mathsf{b} \\ \mathtt{k} & :: & \mathsf{c} \Rightarrow \mathsf{b} \Rightarrow \mathsf{b} \Rightarrow \mathsf{a} \Rightarrow \mathsf{a} \end{array}$$

$$\mathtt{k}(\mathtt{g}(x), y, \mathtt{h}(\mathtt{f}(F)), z) \to F \cdot \mathtt{h}(\mathtt{g}(y)) \cdot z \cdot x$$

# Nasty example

$$\begin{array}{rcl} \mathtt{map}(F, []) & \rightarrow & [] \\ \mathtt{map}(F, \mathtt{cons}(x, y)) & \rightarrow & \mathtt{cons}(F \cdot x, \mathtt{map}(F, y)) \end{array}$$

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Not terminating if:

$$\begin{array}{rcl} [] & :: & \mathsf{o} \\ \mathtt{cons} & :: & (\mathsf{o} \Rightarrow \mathsf{o}) \Rightarrow \mathsf{o} \Rightarrow \mathsf{o} \\ \mathtt{map} & :: & ((\mathsf{o} \Rightarrow \mathsf{o}) \Rightarrow \mathsf{o} \Rightarrow \mathsf{o}) \Rightarrow \mathsf{o} \Rightarrow \mathsf{o} \end{array}$$

Proof: choose $\omega := \mathtt{cons}(\lambda x_{\mathsf{o}}.\mathtt{map}(\lambda y_{\mathsf{o}\Rightarrow\mathsf{o}}.\lambda z_{\mathsf{o}}.y_{\mathsf{o}\Rightarrow\mathsf{o}} \cdot x_{\mathsf{o}}, x_{\mathsf{o}}))$.
Then:

# Nasty example

$$\begin{array}{rcl} \mathtt{map}(F, []) & \to & [] \\ \mathtt{map}(F, \mathtt{cons}(x, y)) & \to & \mathtt{cons}(F \cdot x, \mathtt{map}(F, y)) \end{array}$$

Not terminating if:

$$\begin{array}{rcl} [] & :: & \mathsf{o} \\ \mathtt{cons} & :: & (\mathsf{o} \Rightarrow \mathsf{o}) \Rightarrow \mathsf{o} \Rightarrow \mathsf{o} \\ \mathtt{map} & :: & ((\mathsf{o} \Rightarrow \mathsf{o}) \Rightarrow \mathsf{o} \Rightarrow \mathsf{o}) \Rightarrow \mathsf{o} \Rightarrow \mathsf{o} \end{array}$$

Proof: choose $\omega := \mathtt{cons}(\lambda x_{\mathsf{o}}.\mathtt{map}(\lambda y_{\mathsf{o}\Rightarrow\mathsf{o}}.\lambda z_{\mathsf{o}}.y_{\mathsf{o}\Rightarrow\mathsf{o}} \cdot x_{\mathsf{o}}, x_{\mathsf{o}}))$.
Then:

$$\begin{array}{rl} & \mathtt{map}(\lambda y_{\mathsf{o}\Rightarrow\mathsf{o}}.\lambda z_{\mathsf{o}}.y_{\mathsf{o}\Rightarrow\mathsf{o}} \cdot \omega, \omega) \\ \to & \mathtt{cons}(\ (\lambda y.\lambda z.y \cdot \omega)\ \langle \lambda x.\mathtt{map}(\lambda y.\lambda z.y \cdot x, x)\rangle\ ,\ \mathtt{map}(\dots)) \\ = & \mathtt{cons}(\ \lambda z.\langle \lambda x.\mathtt{map}(\lambda y.\lambda z'.y \cdot x, x)\rangle \cdot \omega, \mathtt{map}(\dots)) \\ \to_\beta & \mathtt{cons}(\ \lambda z.\underline{\mathtt{map}(\lambda y.\lambda z'.y \cdot \omega, \omega)}\ , \mathtt{map}(\dots)) \end{array}$$

# Nasty example

$$\begin{array}{rcl} \mathtt{map}(F, []) & \to & [] \\ \mathtt{map}(F, \mathtt{cons}(x, y)) & \to & \mathtt{cons}(F \cdot x, \mathtt{map}(F, y)) \end{array}$$

Not terminating if:

$$\begin{array}{rcl} [] & :: & \mathsf{o} \\ \mathtt{cons} & :: & (\mathsf{o} \Rightarrow \mathsf{o}) \Rightarrow \mathsf{o} \Rightarrow \mathsf{o} \\ \mathtt{map} & :: & ((\mathsf{o} \Rightarrow \mathsf{o}) \Rightarrow \mathsf{o} \Rightarrow \mathsf{o}) \Rightarrow \mathsf{o} \Rightarrow \mathsf{o} \end{array}$$

Proof: choose $\omega := \mathtt{cons}(\lambda x_{\mathsf{o}}.\mathtt{map}(\lambda y_{\mathsf{o}\Rightarrow\mathsf{o}}.\lambda z_{\mathsf{o}}.y_{\mathsf{o}\Rightarrow\mathsf{o}} \cdot x_{\mathsf{o}}, x_{\mathsf{o}}))$.
Then:

(But *is* terminating if $\mathtt{cons} :: (\mathsf{a} \Rightarrow \mathsf{a}) \Rightarrow \mathsf{o} \Rightarrow \mathsf{o}$.)