

Termination and Complexity in Higher-Order Term Rewriting

Part 5. Complexity:
tuple interpretations

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Hence: $\llbracket \text{add}(s^n(0), s^m(0)) \rrbracket = 1 + m + 2 * n$: linear!

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Higher-order monotonic algebras: definition

Given: a set \mathcal{A} with a well-founded ordering $>$ (for example: \mathbb{N})

Choose: a function $[\mathfrak{f}]$ from \mathcal{A}^k to \mathcal{A} for every \mathfrak{f} of arity k

Define: for a given α mapping variables to \mathcal{A} :

- $[[x]] = \alpha(x)$
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Prove: $[[\ell]] > [[r]]$ for all rules $\ell \rightarrow r$, all α

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Given: a set \mathcal{A} with a well-founded ordering $>$ (for example: \mathbb{N})

Choose: a function $[\varepsilon]$ in (σ) for every ε of type σ

Define: for a given α mapping variables to \mathcal{A} :

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- $[[s \cdot t]] = [[s]]([[t]])$

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Example:

$[]$:: list
 cons :: $\text{nat} \Rightarrow \text{list} \Rightarrow \text{list}$
 map :: $(\text{nat} \Rightarrow \text{nat}) \Rightarrow \text{list} \Rightarrow \text{list}$

$\text{map}(F, []) \rightarrow []$
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Example:

`[]` :: list
`cons` :: nat \Rightarrow list \Rightarrow list
`map` :: (nat \Rightarrow nat) \Rightarrow list \Rightarrow list

`map(F, [])` \rightarrow []
`map(F, cons(x, l))` \rightarrow `cons(F · x, map(F, l))`

Choose: $\mathcal{A}_\iota = \mathbb{N}$ for all ι

`[]` = 0
`[cons](x, y)` = $x + y + 1$
`[map](F, x)` = $(x + 1) * F(x)$

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Monotonicity: holds.

Example

$$\begin{aligned} \llbracket [] \rrbracket &= 0 \\ \llbracket \text{cons} \rrbracket(x, y) &= x + y + 1 \\ \llbracket \text{map} \rrbracket(F, x) &= (x + 1) * F(x) + 1 \end{aligned}$$

Goal 1:

$$\llbracket \text{map}(F, [] \rrbracket \rrbracket > \llbracket [] \rrbracket$$

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$$(0 + 1) * F(0) + 1 > 0$$

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Goal 2:

$$\llbracket \text{map}(F, \text{cons}(x, l)) \rrbracket > \llbracket \text{cons}(F \cdot x, \text{map}(F, l)) \rrbracket$$

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$$\begin{aligned} &((x + l + 1) + 1) * F(x + l + 1) + 1 > \\ &F(x) + ((l + 1) * F(l) + 1) + 1 \end{aligned}$$

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 \end{array}$$

Exercise

Given:

`[]` :: list
`cons` :: nat \Rightarrow list \Rightarrow list
`filter` :: (nat \Rightarrow bool) \Rightarrow list \Rightarrow list
`helper` :: bool \Rightarrow nat \Rightarrow list \Rightarrow list

`filter`(F , `[]`) \rightarrow `[]`
`filter`(F , `cons`(x , l)) \rightarrow `helper`($F \cdot x$, x , `filter`(F , l))
`helper`(`true`, x , l) \rightarrow `cons`(x , l)
`helper`(`false`, x , l) \rightarrow l

Task: show that the following interpretation suffices:

<code>[]</code>	=	0	<code>[true]</code>	=	1
<code>[cons]</code> (x , y)	=	$x + y + 1$	<code>[false]</code>	=	0
<code>[helper]</code> (b , x , y)	=	$b + x + y + 1$			
<code>[filter]</code> (F , x)	=	$(x + 1) * (F(x) + 1)$			

Bonus exercise

Given:

$[]$:: list

cons :: nat \Rightarrow list \Rightarrow list

zip :: (nat \Rightarrow nat) \Rightarrow list \Rightarrow list

zip(F , $[]$, l) = l

zip(F , l , $[]$) = l

zip(F , cons(x , l), cons(y , q)) = cons($F \cdot x \cdot y$, zip(F , l , q))

Task: find an interpretation that orients these rules!

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Example: (for $\sigma, \tau = \text{nat}$ and $\mathcal{A}_{\text{nat}} = \mathbb{N}$):

- if F is constant, then $\text{makeSm}_{\sigma, \tau}(F) = x \mapsto F(x) + x + 1$
- otherwise $\text{makeSm}_{\sigma, \tau}(F) = x \mapsto F(x) + 1$

An observation

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Mechanism: map to \mathbb{N}^2 instead of \mathbb{N} .

We let $\langle x, y \rangle > \langle x', y' \rangle$ if $x > x'$ and $y \geq y'$.

Separating cost and size

$$\begin{aligned} \text{add}(0, y) &\rightarrow y \\ \text{add}(s(x), y) &\rightarrow s(\text{add}(x, y)) \end{aligned}$$

Let:

	cost	,	size
$\llbracket 0 \rrbracket$	0	,	0
$\llbracket s(x) \rrbracket$	x_{cost}	,	$x_{size} + 1$
$\llbracket \text{add}(x, y) \rrbracket$	$x_{cost} + y_{cost} + x_{size}$,	$x_{size} + y_{size}$

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	cost	,	size	}
$\llbracket 0 \rrbracket$	0		0	
$\llbracket s(x) \rrbracket$	x_{cost}		$x_{size} + 1$	
$\llbracket \text{add}(x, y) \rrbracket$	$x_{cost} + y_{cost} + x_{size}$		$x_{size} + y_{size}$	

Then:

$$\begin{aligned} \llbracket \text{add}(0, y) \rrbracket &= \langle 1 + y_1, y_2 \rangle \\ &> \langle y_1, y_2 \rangle = \llbracket y \rrbracket \\ \llbracket \text{add}(s(x), y) \rrbracket &= \langle 2 + x_1 + y_1 + x_2, 1 + x_2 + y_2 \rangle \\ &> \langle 1 + x_1 + y_1 + x_2, 1 + x_2 + y_2 \rangle = \llbracket s(\text{add}(x, y)) \rrbracket \end{aligned}$$

Separating cost and size

$$\begin{aligned} \text{add}(0, y) &\rightarrow y \\ \text{add}(s(x), y) &\rightarrow s(\text{add}(x, y)) \end{aligned}$$

Let:

	cost	,	size
$\llbracket 0 \rrbracket$	0	,	0
$\llbracket s(x) \rrbracket$	x_{cost}	,	$x_{size} + 1$
$\llbracket \text{add}(x, y) \rrbracket$	$x_{cost} + y_{cost} + x_{size}$,	$x_{size} + y_{size}$

Then:

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Hence: $\llbracket \text{add}(s^n(0), s^m(0)) \rrbracket = \langle 1 + n, n + m \rangle$: precise!

When interpretations to \mathbb{N} are Not Great

$$a(b(x)) \rightarrow b(a(x))$$

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Hence: $\llbracket a^n(b^m(\epsilon)) \rrbracket = 2^n * m$: exponential!

Separating cost and size

$$a(b(x)) \rightarrow b(a(x))$$

Let:

	cost		size	
$\llbracket a(x) \rrbracket$	$=$	\langle	$x_{cost} + x_{size}$	\rangle
$\llbracket b(x) \rrbracket$	$=$	\langle	x_{cost}	\rangle
$\llbracket \epsilon \rrbracket$	$=$	\langle	0	\rangle

Separating cost and size

$$a(b(x)) \rightarrow b(a(x))$$

Let:

	cost	,	size	
$\llbracket a(x) \rrbracket$	$x_{cost} + x_{size}$		x_{size}	\rangle
$\llbracket b(x) \rrbracket$	x_{cost}		$x_{size} + 1$	\rangle
$\llbracket \epsilon \rrbracket$	0		0	\rangle

Then:

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Separating cost and size

$$a(b(x)) \rightarrow b(a(x))$$

Let:

	cost		size
$\llbracket a(x) \rrbracket$	$= \langle$	$x_{cost} + x_{size}$	$, x_{size} \rangle$
$\llbracket b(x) \rrbracket$	$= \langle$	x_{cost}	$, x_{size} + 1 \rangle$
$\llbracket \epsilon \rrbracket$	$= \langle$	0	$, 0 \rangle$

Then:

$$\llbracket a(b(x)) \rrbracket = \langle x_1 + x_2 + 1, x_2 + 1 \rangle > \langle x_1 + x_2, x_2 + 1 \rangle = \llbracket b(a(x)) \rrbracket$$

Hence: $\llbracket a^n(b^m(\epsilon)) \rrbracket = (n * m, m)$: precise!

Tuple interpretations

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- $\{\mathbf{N}\} = \mathbb{N}^2$ (cost, size of normal form)
- $\{\text{list}\} = \mathbb{N}^3$ (cost, list length, size of greatest element)
- $\{\text{bool}\} = \mathbb{N}^1$ (cost)

Example: interpreting list functions

`append([], l)` → `l`
`append(cons(x, l), q)` → `cons(x, append(l, q))`
`sum([])` → `0`
`sum(cons(x, l))` → `add(x, sum(l))`

Example: interpreting list functions

$$\begin{aligned} \text{append}([], l) &\rightarrow l \\ \text{append}(\text{cons}(x, l), q) &\rightarrow \text{cons}(x, \text{append}(l, q)) \\ \text{sum}([]) &\rightarrow 0 \\ \text{sum}(\text{cons}(x, l)) &\rightarrow \text{add}(x, \text{sum}(l)) \end{aligned}$$

Interpretations:

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 - size = $l_{\text{len}} * l_{\text{max}}$
 - cost = $l_{\text{cost}} + 2 * l_{\text{len}} + l_{\text{len}} * l_{\text{max}} + 1$

Higher-order tuple interpretations: an example

$$\begin{aligned} [] &:: \text{list} \\ \text{cons} &:: \mathbf{N} \Rightarrow \text{list} \Rightarrow \text{list} \\ \text{map} &:: (\mathbf{N} \Rightarrow \mathbf{N}) \Rightarrow \text{list} \Rightarrow \text{list} \\ \text{map}(F, []) &\rightarrow [] \\ \text{map}(F, \text{cons}(x, l)) &\rightarrow \text{cons}(F \cdot x, \text{map}(F, l)) \end{aligned}$$

Higher-order tuple interpretations: an example

$[] :: \text{list}$
 $\text{cons} :: \mathbf{N} \Rightarrow \text{list} \Rightarrow \text{list}$
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Let:

- $\llbracket [] \rrbracket = \langle 0, 0, 0 \rangle$
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 - length:
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Higher-order tuple interpretations: an example

$[]$:: list
 cons :: $\mathbf{N} \Rightarrow \text{list} \Rightarrow \text{list}$
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 - cost:

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 - maximum: $F(\langle l_{\text{cost}}, l_{\text{max}} \rangle)_s$
 - cost: $(l_{\text{len}} + 1) * (F(\langle l_{\text{cost}}, l_{\text{max}} \rangle)_{\text{cost}} + 1)$

Higher-order tuple interpretations: an example

Goal: $\llbracket \text{map}(F, \text{cons}(x, l)) \rrbracket > \llbracket \text{cons}(F \cdot x, \text{map}(F, l)) \rrbracket$

Higher-order tuple interpretations: an example

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$$\begin{aligned}
 & \langle (q + 2) * (F(\langle x_1 + l, \max(x_2, l_3) \rangle))_1 + 1, \\
 & \quad q + 1 \\
 & \quad F(\langle x_1 + l, \max(x_2, l_3) \rangle)_2 \\
 & \rangle > \\
 & \langle F(x)_1 + (q + 1) * (F(\langle l, l_3 \rangle))_1 + 1, \\
 & \quad q + 1 \\
 & \quad \max(F(x)_2, F(\langle l, l_3 \rangle)_2) \\
 & \rangle
 \end{aligned}$$

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$$(q + 2) * (F(\langle x_1 + l, \max(x_2, l_3) \rangle)_1 + 1) > \\ F(x)_1 + (q + 1) * (F(\langle l, l_3 \rangle)_1 + 1)$$

$$q + 1 \geq \\ q + 1$$

$$F(\langle x_1 + l, \max(x_2, l_3) \rangle)_2 \geq \\ \max(F(x)_2, F(\langle l, l_3 \rangle)_2)$$

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$$F(\langle x_1 + l, \max(x_2, l_3) \rangle)_2 \geq \\ F(x)_2$$

$$F(\langle x_1 + l, \max(x_2, l_3) \rangle)_2 \geq \\ F(\langle l, l_3 \rangle)_2$$

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Higher-order tuple interpretations: an example

Goal: $\llbracket \text{map}(F, \text{cons}(x, l)) \rrbracket > \llbracket \text{cons}(F \cdot x, \text{map}(F, l)) \rrbracket$

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 \end{array} \geq$$

Exercise

1. Find an interpretation, with $(\text{nat}) = \mathbb{N}^2$, for the following system:

$$\begin{aligned} \text{minus}(x, 0) &\rightarrow x \\ \text{minus}(s(x), s(y)) &\rightarrow \text{minus}(x, y) \\ \text{quot}(0, s(y)) &\rightarrow 0 \\ \text{quot}(s(x), s(y)) &\rightarrow s(\text{quot}(\text{minus}(x, y), s(y))) \end{aligned}$$

Warning: do not take $x_{\text{size}} - y_{\text{size}}$ for the size of $\text{minus}(x, y)$!

2. Find an interpretation for the following HTRS, where $\text{zip} :: (\text{nat} \Rightarrow \text{nat}) \Rightarrow \text{list} \Rightarrow \text{list}$.

$$\begin{aligned} \text{zip}(F, [], l) &= l \\ \text{zip}(F, l, []) &= l \\ \text{zip}(F, \text{cons}(x, l), \text{cons}(y, q)) &= \text{cons}(F \cdot x \cdot y, \text{zip}(F, l, q)) \end{aligned}$$

A more challenging higher-order tuple interpretation

$$\begin{aligned}
 \text{fold}(F, x, []) &\rightarrow [] \\
 \text{fold}(F, x, \text{cons}(y, l)) &\rightarrow \text{fold}(F, (F \cdot x \cdot y), l)
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Interpretation:

$$\llbracket \text{fold}(F, x, l) \rrbracket = \langle \text{cost}, \text{size} \rangle$$

Where:

- $\text{cost} = 1 + l_{\text{cost}} + F(\langle 0, 0 \rangle)_{\text{cost}} + \text{Helper}[F, \langle l_{\text{cost}}, l_{\text{max}} \rangle]^{l_{\text{len}}(x)}_{\text{cost}}$
- $\text{size} = \text{Helper}[F, \langle l_{\text{cost}}, l_{\text{max}} \rangle]^{l_{\text{len}}(x)}_{\text{size}}$
- And $\text{Helper}[F, y] = x \mapsto \langle F(x, y)_{\text{cost}}, \max(x_{\text{size}}, F(x, y)_{\text{size}}) \rangle$.

A more challenging higher-order tuple interpretation.

`add(0, y)` → `y`
`add(s(x), y)` → `add(x, s(y))`
`fold(F, x, [])` → `[]`
`fold(F, x, cons(y, l))` → `fold(F, (F · x · y), l)`
`sum(l)` → `fold(λx.λy.add(x, y), 0, l)`

Method: Plug `[[λx.λy.add(x, y)]]` into the interpretation for `fold`.

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Interpreting λ : use $make_{sm, l, \sigma_1 \Rightarrow \dots \Rightarrow \sigma_m \Rightarrow \kappa} =$

$$\left\{ \begin{array}{l} (F, x, y_1, \dots, y_m) \mapsto (F(x, \vec{y})_1 + 1 + x_1, F(x, \vec{y})_2, \dots, F(x, \vec{y})_{K[\kappa]}) \\ \quad \text{if } F \text{ is constant} \\ (F, x, y_1, \dots, y_m) \mapsto (F(x, \vec{y})_1 + 1, F(x, \vec{y})_2, \dots, F(x, \vec{y})_{K[\kappa]}) \\ \quad \text{if } F \text{ is monotonic} \end{array} \right.$$

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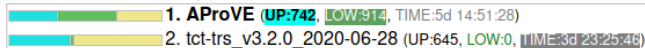
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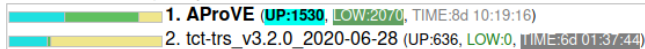
Termination (and complexity) competition

Complexity Analysis

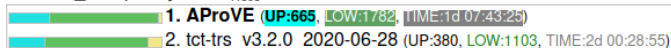
Derivational_Complexity: TRS ₄₁₄₉₉



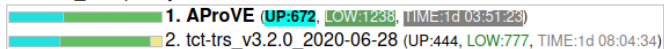
Derivational_Complexity: TRS Innermost ₄₁₅₀₀



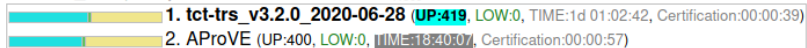
Runtime_Complexity: TRS ₄₁₅₀₈



Runtime_Complexity: TRS Innermost ₄₁₅₀₇



Runtime_Complexity: TRS Innermost Certified ₄₁₅₀₉



Complexity of higher-order term rewriting

Open question: do derivational and runtime complexity even make sense for higher-order rewriting?

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$$\begin{aligned} \text{fold}(F, x, []) &\rightarrow [] \\ \text{fold}(F, x, \text{cons}(y, l)) &\rightarrow \text{fold}(F, (F \cdot x \cdot y), l) \end{aligned}$$

Recall:

- What if: $F := \lambda x, y. \text{minimum}(x, y)$?
- What if: $F := \lambda x, y. \text{add}(x, y)$?
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Higher-order derivational complexity?

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Conclusion: exponential complexity at a minimum

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Choice: data must be a **first-order** term.

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Runtime complexity: $n \mapsto \mathcal{O}(n^2)$ (actually: length * max)

Exercises

1. Compute the runtime complexity of the following system.

$$\begin{aligned}\text{map}(F, []) &\rightarrow [] \\ \text{map}(F, \text{cons}(x, l)) &\rightarrow \text{cons}(F \cdot x, \text{map}(F, l)) \\ \text{doublemap}(l) &\rightarrow \text{map}(\text{double}, l) \\ \text{double}(0) &\rightarrow 0 \\ \text{double}(s(x)) &\rightarrow s(s(\text{double}(x)))\end{aligned}$$

2. Compute the runtime complexity of the following system.

$$\begin{aligned}\text{add}(x, 0) &\rightarrow x \\ \text{add}(x, s(y)) &\rightarrow s(\text{add}(x, y)) \\ \text{zip}(F, [], l) &= l \\ \text{zip}(F, l, []) &= l \\ \text{zip}(F, \text{cons}(x, l), \text{cons}(y, q)) &= \text{cons}(F \cdot x \cdot y, \text{zip}(F, l, q)) \\ \text{zipadd}(l, q) &\rightarrow \text{zip}(\lambda x. \lambda y. \text{add}(y, x), l, q)\end{aligned}$$

Higher-order complexity notion?

Idea:

- complexity of `map` is $\mathcal{O}(n * F(n))$?

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- complexity of `map` is $\mathcal{O}(n * F(n))$?
- complexity of `fold` is $\mathcal{O}(F^n(n))$?

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Relevance: determined by polynomial tuple interpretation!