Termination Tools

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Contents: Proving (Non-)Termination Automatically

TRS category in termCOMP 2024:

tool/method	YES	NO
AProVE	1030	282
NaTT	876	169
MU-TERM	717	135
NTI	293	275
autonon	—	233

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Termination by Linear Interpretations

Well-Founded Monotone Algebras

Definition

let $\mathcal{A} = (A, \{f_{\mathcal{A}}\})$ be algebra equipped with strict order $>$

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- $\mathcal{A}$ is well-founded if $>$ is well-founded
- $\mathcal{A}$ is (strictly) monotone if

$$a_i > b \implies f_\mathcal{A}(a_1, \dots, a_i, \dots, a_n) > f_\mathcal{A}(a_1, \dots, b, \dots, a_n)$$

for all $f^{(n)} \in \mathcal{F}$, $a_1, \dots, a_n, b \in A$, and $i \in \{1, \dots, n\}$

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- $s >_{\mathcal{A}} t$ if $[\alpha]_{\mathcal{A}}(s) > [\alpha]_{\mathcal{A}}(t)$ for all assignments $\alpha : \mathcal{V} \to A$

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- $s >_{\mathcal{A}} t$ if $[\alpha]_{\mathcal{A}}(s) > [\alpha]_{\mathcal{A}}(t)$ for all assignments $\alpha : \mathcal{V} \to A$

Theorem (Lankford 1979, Zantema 1994)

$>_{\mathcal{A}}$ *is reduction order if* $\mathcal{A}$ *is well-founded monotone algebra*

Termination Proof of Addition

consider TRS $\mathcal{R}$ and well-founded monotone algebra $\mathcal{A}$ on $\mathbb{N}$:

$$\mathcal{R} = \left\{ \begin{aligned} \mathsf{a}(x, \mathsf{o}) &\to x \\ \mathsf{a}(x, \mathsf{s}(y)) &\to \mathsf{s}(\mathsf{a}(x, y)) \end{aligned} \right\} \qquad \mathcal{A}\colon \left\{ \begin{aligned} \mathsf{o}_{\mathcal{A}} &= 1 \\ \mathsf{s}_{\mathcal{A}}(x) &= x + 1 \\ \mathsf{a}_{\mathcal{A}}(x, y) &= x + 2y \end{aligned} \right.$$

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$$\mathcal{R} \subseteq >_{\mathcal{A}} \iff \bigwedge \left\{ \begin{array}{lc} \forall x \in \mathbb{N}. & 2 + x > 0 + x \\ \forall x, y \in \mathbb{N}. & 2 + x + 2y > 1 + x + 2y \end{array} \right\}$$

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$$\iff \bigwedge \left\{ \begin{array}{lll} 2 > 0, & 1 \geqslant 1 & \\ 2 > 1, & 1 \geqslant 1, & 2 \geqslant 2 \end{array} \right\}$$

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Finding Interpretations Automatically

consider TRS $\mathcal{R}$ and algebra $\mathcal{A}$ on $\mathbb{N}$

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with $o_0, s_0, a_0 \in \mathbb{N}$ and $s_1, a_1, a_2 \in \mathbb{N}_+$ (why?):

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$$\begin{array}{ll} & \mathcal{R} \subseteq >_{\mathcal{A}} \\ \iff & \bigwedge \left\{ \begin{array}{ll} \forall x \in \mathbb{N}. & a_0 + a_1 x + a_2 o_0 > x \\ \forall x, y \in \mathbb{N}. & a_0 + a_1 x + a_2 s_0 + a_2 s_1 y > s_0 + s_1 a_0 + s_1 a_1 x + s_1 a_2 y \end{array} \right\} \end{array}$$

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Solving Constraints by SMT Solvers

```
(declare-const o0 Int)
(declare-const s0 Int)
(declare-const s1 Int)
(declare-const a0 Int)
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(assert (and
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```
(assert (and
  (>= o0 0)
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  (>= a0 0) (>= a1 1) (>= a2 1)
  (> (+ a0 (* a2 o0)) 0)
  (>= a1 1)
```

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(check-sat)
(get-value (o0 s0 s1 a0 a1 a2))
```

output:

```
sat
((o0 11) (s0 13) (s1 1) (a0 1) (a1 12) (a2 8))
```

Linear Interpretations

Definition

linear interpretations are of form

$$f_{\mathcal{A}}(x_1, \ldots, x_n) = f_0 + f_1 x_1 + \cdots + f_n x_n \quad (f_0, f_1, \ldots, f_n \in \mathbb{N})$$

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Theorem

solving polynomial equations over $\mathbb{Z}$ is undecidable

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Theorem (Mitterwallner et al. 2024)

termination by linear interpretations is undecidable

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Derivational Complexity

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- $\mathsf{dc}_{\mathcal{R}}(n) = \max\{\mathsf{dh}(t, \rightarrow_{\mathcal{R}}) \mid t \text{ is term with } |t| \leqslant n\}$

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- $\mathsf{dc}_{\mathcal{R}}(n) = \max\{\mathsf{dh}(t, \rightarrow_{\mathcal{R}}) \mid t \text{ is term with } |t| \leqslant n\}$

Theorem

$\mathsf{dc}_{\mathcal{R}}(n)$ *is bound from above by exponetial function if*

$\mathcal{R} \subseteq >_{\mathcal{A}}$ *for some linear interpretation* $\mathcal{A}$ *on* $\mathbb{N}$

Complexity Analysis of Bubble Sort Algorithm

TRS $\mathcal{R}$

$$\mathsf{b}(\mathsf{a}(x)) \to \mathsf{a}(\mathsf{b}(x))$$

take linear interpretation $\mathcal{A}$ on $\mathbb{N}$:

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thus, $\mathsf{dc}_{\mathcal{R}}(n)$ is boudned by exponetial function

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but bubble sort is quadratic-time algorithm, isn't it?

Idea: Counting a below b

TRS $\mathcal{R}$

$$\mathsf{b}(\mathsf{a}(x)) \to \mathsf{a}(\mathsf{b}(x))$$

rewriting

$$\begin{aligned}
& \underset{2}{\underline{\mathsf{b}}}(\mathsf{a}(\underset{1}{\underline{\mathsf{b}}}(\mathsf{a}(\underset{0}{\underline{\mathsf{b}}}(x))))) && 2+1+0=3 \\
\to_{\mathcal{R}}\ & \mathsf{a}(\underset{1}{\underline{\mathsf{b}}}(\underset{1}{\underline{\mathsf{b}}}(\mathsf{a}(\underset{0}{\underline{\mathsf{b}}}(x))))) && 1+1+0=2 \\
\to_{\mathcal{R}}\ & \mathsf{a}(\underset{1}{\underline{\mathsf{b}}}(\mathsf{a}(\underset{0}{\underline{\mathsf{b}}}(\underset{0}{\underline{\mathsf{b}}}(x))))) && 1+0+0=1 \\
\to_{\mathcal{R}}\ & \mathsf{a}(\mathsf{a}(\underset{0}{\underline{\mathsf{b}}}(\underset{0}{\underline{\mathsf{b}}}(\underset{0}{\underline{\mathsf{b}}}(x))))) && 0+0+0=0
\end{aligned}$$

Complexity Analysis by Matrix Interpretations

Definition

triangular matrix interpretation $\mathcal{A}$ on $\mathbb{N}^2$ is of form

$$f_{\mathcal{A}}(\mathbf{x_1}, \ldots, \mathbf{x_n}) = \mathbf{f}_0 + F_1\mathbf{x_1} + \cdots + F_n\mathbf{x_n} \quad \text{with } \mathbf{f}_0 \in \mathbb{N}^2 \text{ and } F_i = \begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix}$$

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$(x, y) > (x', y')$ if $x > x'$ and $y \geqslant y'$

Theorem (Moser, Schnabl, and Waldmann 2010)

$\mathsf{dc}_{\mathcal{R}}(n) \in O(n^2)$ *if* $\mathcal{R} \subseteq >_{\mathcal{A}}$ *for some triangular matrix interpretation* $\mathcal{A}$ *on* $\mathbb{N}^2$

Exercise: Find Triangular Matrix Interpretation

TRS $\mathcal{R}$

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TRS $\mathcal{R}$

$$\mathsf{b}(\mathsf{a}(x)) \to \mathsf{a}(\mathsf{b}(x))$$

take triangular matrix interpretation $\mathcal{A}$ on $\mathbb{N}^2$:

$$\mathsf{a}_{\mathcal{A}}(\mathbf{x}) = \mathbf{x} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \mathsf{b}_{\mathcal{A}}(\mathbf{x}) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \mathbf{x}$$

thus, $\mathsf{dc}_{\mathcal{R}}(n) \in O(n^2)$

Dependency Pair Method

Literature

- Arts and Giesl
  Termination of Term Rewriting Using Dependency Pairs
  TCS 236:133–178, 2000

- Hirokawa and Middeldorp
  Automating the Dependency Pair Method
  I&C 199(1,2), pp. 172–199, 2005

- Giesl, Thiemann, and Schneider-Kamp
  The Dependency Pair Framework: Combining Techniques for Automated Termination Proofs, LPAR-11, LNAI 3452, 301–331, 2005

Dependency Pair Method

Definition

- $t^\sharp = f^\sharp(t_1, \ldots, t_n)$ if $t = f(t_1, \ldots, t_n)$, where $f^\sharp$ is fresh symbol
- DP$(\mathcal{R}) = \{\ell^\sharp \to t^\sharp \mid \ell \to r \in \mathcal{R},\ r \trianglerighteq t,\ \text{root}(t) \in \mathcal{D}_\mathcal{R},\ \text{and}\ \ell \not\triangleright t\}$

Dependency Pair Method

Definition

- $t^\sharp = f^\sharp(t_1, \dots, t_n)$ if $t = f(t_1, \dots, t_n)$, where $f^\sharp$ is fresh symbol
- $\mathsf{DP}(\mathcal{R}) = \{\ell^\sharp \to t^\sharp \mid \ell \to r \in \mathcal{R},\ r \trianglerighteq t,\ \mathsf{root}(t) \in \mathcal{D}_\mathcal{R},\ \text{and } \ell \not\triangleright t\}$

Example

$$\mathcal{R} = \left\{ \begin{array}{r@{}l} \mathsf{ack}(0, y) &{} \to y \\ \mathsf{ack}(\mathsf{s}(x), 0) &{} \to \mathsf{ack}(x, \mathsf{s}(0)) \\ \mathsf{ack}(\mathsf{s}(x), \mathsf{s}(y)) &{} \to \mathsf{ack}(x, \mathsf{ack}(\mathsf{s}(x), y)) \end{array} \right\}$$

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Finiteness

Definition

$(\mathcal{P}, \mathcal{R})$ is finite if it has no infinite sequence

$$s_1 \xrightarrow{\epsilon}_{\mathcal{P}} t_1 \rightarrow^*_{\mathcal{R}} s_2 \xrightarrow{\epsilon}_{\mathcal{P}} t_2 \rightarrow^*_{\mathcal{R}} \cdots$$

that each t_i is $\mathcal{R}$-terminating

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$(\mathcal{P}, \mathcal{R})$ is finite if $\mathcal{P} = \varnothing$

how to prove finiteness when $\mathcal{P} \neq \varnothing$?

Subterm Criterion

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if $\pi(\mathsf{ack}^\sharp) = 1$ then $\mathsf{ack}^\sharp(\mathsf{s}(x), \mathsf{s}(y)) \triangleright^\pi \mathsf{ack}^\sharp(x, \mathsf{ack}(\mathsf{s}(x), y))$

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Theorem (Hirokawa and Middeldorp 2007)

if simple projection π *satisfies* $\mathcal{P} \subseteq \trianglerighteq^\pi$ *then*

$$(\mathcal{P}, \mathcal{R}) \text{ is finite} \iff (\mathcal{P} \setminus \triangleright^\pi, \mathcal{R}) \text{ is finite}$$

consider TRS $\mathcal{R}$

$$\begin{aligned} \mathsf{ack}(0, y) &\to y \\ \mathsf{ack}(\mathsf{s}(x), 0) &\to \mathsf{ack}(x, \mathsf{s}(0)) \\ \mathsf{ack}(\mathsf{s}(x), \mathsf{s}(y)) &\to \mathsf{ack}(x, \mathsf{ack}(\mathsf{s}(x), y)) \end{aligned}$$

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$\boxed{1}$ subterm criterion with $\pi(\mathsf{ack}^\sharp) = 1$ removes $\mathbf{1}$ and $\mathbf{2}$

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$\boxed{1}$ subterm criterion with $\pi(\mathsf{ack}^\sharp) = 1$ removes $\mathbf{1}$ and $\mathbf{2}$

$\boxed{2}$ subterm criterion with $\pi(\mathsf{ack}^\sharp) = 2$ removes $\mathbf{3}$

consider TRS $\mathcal{R}$

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$\boxed{2}$ subterm criterion with $\pi(\mathsf{ack}^\sharp) = 2$ removes $\mathbf{3}$

$\boxed{3}$ no dependency pair remains; hence $\mathcal{R}$ is terminating

Limitation of Subterm Criterion

we show termination of TRS $\mathcal{R}$

$$\begin{aligned}
x - \mathsf{0} &\to x \\
\mathsf{s}(x) - \mathsf{s}(y) &\to x - y \\
\mathsf{0} \div \mathsf{s}(y) &\to \mathsf{0} \\
\mathsf{s}(x) \div \mathsf{s}(y) &\to \mathsf{s}((x - y) \div \mathsf{s}(y))
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semantically decreasing!

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semantically decreasing! ☞ revisit interpretation method

Reduction Pairs and Removal of Dependency Pairs

Definition

$(\gtrsim, >)$ is reduction pair if

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Fact

$(\geqslant_{\mathcal{A}}, >_{\mathcal{A}})$ is reduction pair if $\mathcal{A}$ is well-founded weakly monotone algebras

Theorem

if $(\gtrsim, >)$ *is reduction pair and* $\mathcal{R} \subseteq \gtrsim$ *then*

$$(\mathcal{P}, \mathcal{R}) \text{ is finite } \iff (\mathcal{P} \setminus >, \mathcal{R}) \text{ is finite}$$

we show termination of TRS $\mathcal{R}$

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$\boxed{1}$ well-founded weakly monotone algebra $\mathcal{A}$ on $\mathbb{N}$ removes $\mathbf{1}$, $\mathbf{2}$, and $\mathbf{3}$

$$\mathsf{0}_{\mathcal{A}} = 0 \qquad \mathsf{s}_{\mathcal{A}}(x) = x + 1 \qquad x -_{\mathcal{A}} y = x -^\sharp_{\mathcal{A}} y = x \div_{\mathcal{A}} y = x \div^\sharp_{\mathcal{A}} y = x$$

we show termination of TRS $\mathcal{R}$

$$\begin{aligned} x - \mathsf{0} &\to x & \qquad\qquad x &\geqslant x \\ \mathsf{s}(x) - \mathsf{s}(y) &\to x - y & x + 1 &\geqslant x + 1 \\ \mathsf{0} \div \mathsf{s}(y) &\to \mathsf{0} & 0 &\geqslant 0 \\ \mathsf{s}(x) \div \mathsf{s}(y) &\to \mathsf{s}((x - y) \div \mathsf{s}(y)) & x + 1 &\geqslant x + 1 \end{aligned}$$

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Exercise: Prove Termination!

TRS $\mathcal{R}$

$$\begin{aligned} x + \mathsf{0} &\to x \\ x + \mathsf{s}(y) &\to \mathsf{s}(x + y) \\ x \times \mathsf{0} &\to \mathsf{0} \\ x \times \mathsf{s}(y) &\to (x \times y) + x \end{aligned}$$

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$x + \mathsf{0} \to x$	$x \geqslant x$
$x + \mathsf{s}(y) \to \mathsf{s}(x + y)$	$\max\{x, y\} \geqslant \max\{x, y\}$
$x \times \mathsf{0} \to \mathsf{0}$	$x \geqslant 0$
$x \times \mathsf{s}(y) \to (x \times y) + x$	$\max\{x, y\} \geqslant \max\{x, y\}$

$\mathsf{DP}(\mathcal{R})$ consists of

$x +^\sharp \mathsf{s}(y) \to x +^\sharp y$	$0 \geqslant 0$
$x \times^\sharp \mathsf{s}(y) \to (x \times y) +^\sharp x$	$1 > 0$
$x \times^\sharp \mathsf{s}(y) \to x \times^\sharp y$	$1 \geqslant 1$

Exercise: Prove Termination!

TRS $\mathcal{R}$

$$\begin{aligned} x + \mathsf{0} &\to x & x &\geqslant x \\ x + \mathsf{s}(y) &\to \mathsf{s}(x + y) & \max\{x, y\} &\geqslant \max\{x, y\} \\ x \times \mathsf{0} &\to \mathsf{0} & x &\geqslant 0 \\ x \times \mathsf{s}(y) &\to (x \times y) + x & \max\{x, y\} &\geqslant \max\{x, y\} \end{aligned}$$

DP$(\mathcal{R})$ consists of

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Recommendable Interpretations

- $0, 1$-coefficient linear interpretation $\mathcal{A}$ on $\mathbb{N}$:

$$f_{\mathcal{A}}(x_1, \ldots, x_n) = f_0 + f_1 x_1 + \cdots + f_n x_n$$

with $f_0 \in \mathbb{N}$ and $f_1, \ldots, f_n \in \{0, 1\}$

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- max/plus interpretation $\mathcal{A}$ on $\mathbb{N}$:

  $$f_{\mathcal{A}}(x_1, \ldots, x_n) = \max\{f_0, f'_1(f_1 + x_1), \ldots, f'_n(f_n + x_n)\}$$

  with $f_0 \in \mathbb{N}$, $f_1, \ldots, f_n \in \mathbb{Z}$, and $f'_1, \ldots, f'_n \in \{0, 1\}$

Contents: Proving (Non-)Termination Automatically

TRS category in termCOMP 2024:

	tool/method	YES	NO
	AProVE	1030	282
	NaTT	876	169
	MU-TERM	717	135
	NTI	293	275
	autonon	—	233
$\boxed{1}$	linear interpretations	145	—
$\boxed{2}$	**dependency pair method with linear and max/plus**	431	—
$\boxed{3}$	termination of imperative programs	—	—
$\boxed{4}$	loop detection	—	$\geqslant$ 178

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1	linear interpretations	145	—
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4	loop detection	—	$\geqslant 178$

Termination of Imperative Programs

From C to Conditional TRS (CTRS)

see e.g. Fuhs, Kop, and Nishida 2017

```
int sum(x)
{
  int y = 0;
  while (x > 0)
  {
    y = x + y;
    x = x - 1;
  }
  return y;
}
```

From C to Conditional TRS (CTRS)

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      int sum(x)
      {
sum:    int y = 0;
 f1:    while (x > 0)
        {
 f2:      y = x + y;
 f3:      x = x - 1;
        }
 f4:    return y;
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```

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```

$\mathsf{sum}(x) \to \mathsf{f}_1(x, \mathsf{0})$

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          }
f4 :      return y;
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```

$$\mathsf{sum}(x) \to \mathsf{f}_1(x, \mathsf{0})$$

$$\mathsf{f}_1(x, y) \to \mathsf{f}_2(x, y) \quad \text{if } x > 0 \to^* \mathsf{true}$$

From C to Conditional TRS (CTRS)

see e.g. Fuhs, Kop, and Nishida 2017

```
      int sum(x)
      {
sum:    int y = 0;
f1:     while (x > 0)
        {
f2:        y = x + y;
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                }
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```

$$
\begin{aligned}
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From C to Conditional TRS (CTRS)

see e.g. Fuhs, Kop, and Nishida 2017

	`int` $\mathsf{sum}(x)$	
	$\{$	
$\mathsf{sum}:$	`int` $y = 0;$	$\mathsf{sum}(x) \to \mathsf{f}_1(x, 0)$
$\mathsf{f}_1:$	`while` $(x > 0)$	$\mathsf{f}_1(x, y) \to \mathsf{f}_2(x, y) \quad \text{if } x > 0 \to^* \mathsf{true}$
	$\{$	$\mathsf{f}_1(x, y) \to \mathsf{f}_4(x, y) \quad \text{if } x > 0 \to^* \mathsf{false}$
$\mathsf{f}_2:$	$y = x + y;$	$\mathsf{f}_2(x, y) \to \mathsf{f}_3(x, x + y)$
$\mathsf{f}_3:$	$x = x - 1;$	$\mathsf{f}_3(x, y) \to \mathsf{f}_1(x - \mathsf{s}(0), y)$
	$\}$	
$\mathsf{f}_4:$	`return` $y;$	$\mathsf{f}_4(x, y) \to y$
	$\}$	

Unraveling CTRS into TRS

Example

$$\begin{aligned} x + 0 &\to x \\ x + \mathsf{s}(y) &\to \mathsf{s}(x + y) \\ x - 0 &\to x \\ \mathsf{s}(x) - \mathsf{s}(y) &\to x - y \\ 0 > x &\to \mathsf{false} \\ \mathsf{s}(x) > 0 &\to \mathsf{true} \\ \mathsf{s}(x) > \mathsf{s}(y) &\to x > y \end{aligned}$$

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Fact (Ohlebusch 2001)

deterministic 3-CTRS is operationally terminating if unraveled TRS is terminating

Can Termination Tools Prove Termination?

$$
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AProVE	NaTT	TTT2
YES	YES	MAYBE

Contents: Proving (Non-)Termination Automatically

TRS category in termCOMP 2024:

	tool/method	YES	NO
	AProVE	1030	282
	NaTT	876	169
	MU-TERM	717	135
	NTI	293	275
	autonon	—	233
1	linear interpretations	145	—
2	dependency pair method with linear and max/plus	431	—
3	termination of imperative programs	—	—
4	**loop detection**	—	$\geqslant 178$

Non-Termination

Looping

Fact

TRS $\mathcal{R}$ is non-terminating if it admits loop $t \to_{\mathcal{R}}^{+} C[t\sigma]$

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Example

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Example

$\boxed{1}$ $\mathcal{R} = \{\ \mathsf{from}(x) \to x : \mathsf{from}(\mathsf{s}(x))\ \}$

Looping

Fact

TRS $\mathcal{R}$ is non-terminating if it admits loop $t \to_{\mathcal{R}}^{+} C[t\sigma]$

Example

1. $\mathcal{R} = \{\ \mathsf{from}(x) \to x : \mathsf{from}(\mathsf{s}(x))\ \}$
2. $\mathcal{R} = \{\ \mathsf{f}(x, \mathsf{s}(y)) \to \mathsf{f}(\mathsf{s}(x), x)\ \}$

Looping

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TRS $\mathcal{R}$ is non-terminating if it admits loop $t \rightarrow_{\mathcal{R}}^{+} C[t\sigma]$

Example

$\boxed{1}$ $\mathcal{R} = \{\ \mathsf{from}(x) \rightarrow x : \mathsf{from}(\mathsf{s}(x))\ \}$

$\boxed{2}$ $\mathcal{R} = \{\ \mathsf{f}(x, \mathsf{s}(y)) \rightarrow \mathsf{f}(\mathsf{s}(x), x)\ \}$

$\boxed{3}$ $\mathcal{R} = \{\ \mathsf{ab} \rightarrow \mathsf{bbaa}\ \}$

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$$\boxed{4} \ \mathcal{R} = \left\{ \begin{aligned} \mathsf{f}(\mathsf{a}, \mathsf{b}, x) &\to \mathsf{f}(x, x, x) \\ \mathsf{g}(x, y) &\to x \\ \mathsf{g}(x, y) &\to y \end{aligned} \right\}$$

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$\mathsf{abb} \rightarrow_{\mathcal{R}} \mathsf{bbaab} \rightarrow_{\mathcal{R}} \mathsf{bbabbaa}$

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$\mathsf{f}(\mathsf{a}, \mathsf{b}, \mathsf{g}(\mathsf{a}, \mathsf{b})) \rightarrow^{3}_{\mathcal{R}} \mathsf{f}(\mathsf{a}, \mathsf{b}, \mathsf{g}(\mathsf{a}, \mathsf{b}))$

Example beyond Self-Reducible Rule

TRS $\mathcal{R}$

$$\mathsf{f}(x, \mathsf{s}(y)) \to \mathsf{f}(\mathsf{s}(x), x)$$

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how to find such loop?

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how to find such loop? ☞ semi-unification

Loop Detection by Semi-Unifiability

Definition

(s, t) is semi-unifiable if $s\sigma\tau = t\sigma$ for some σ and τ

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TRS $\mathcal{R}$ is non-terminating if $s \to_{\mathcal{R}}^{+} C[t]$ for some semi-unifiable pair (s, t)

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Proof.

if $s \to_{\mathcal{R}}^{+} C[t]$ and $s\sigma\tau = t\sigma$ then

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Proof.

if $s \rightarrow_{\mathcal{R}}^{+} C[t]$ and $s\sigma\tau = t\sigma$ then $s\sigma \rightarrow_{\mathcal{R}}^{+} C\sigma[t\sigma]$ □

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if $s \to_{\mathcal{R}}^{+} C[t]$ and $s\sigma\tau = t\sigma$ then $s\sigma \to_{\mathcal{R}}^{+} C\sigma[t\sigma] = C\sigma[s\sigma\tau]$ ☐

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Note

semi-unification algorithm exists see Aoto and Iwami 2013

Example for Semi-Unification

rule $\mathsf{f}(x, \mathsf{s}(y)) \to \mathsf{f}(\mathsf{s}(x), x)$ is semi-unifiable:

$$\mathsf{f}(x\sigma\tau, \mathsf{s}(y\sigma\tau)) = \mathsf{f}(\mathsf{s}(x\sigma), x\sigma)$$

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\iff & x\sigma\tau = \mathsf{s}(x\sigma) & \wedge & \mathsf{s}(y\sigma\tau) = x\sigma
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\end{array}
$$

therefore, $\sigma = \{x \mapsto \mathsf{s}(z)\}$ and $\tau = \{y \mapsto z,\ y \mapsto \mathsf{s}(z)\}$,

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 & \multicolumn{3}{c}{\mathsf{f}(x\sigma\tau, \mathsf{s}(y\sigma\tau)) = \mathsf{f}(\mathsf{s}(x\sigma), x\sigma)} \\
\Longleftrightarrow & x\sigma\tau = \mathsf{s}(x\sigma) & \wedge & \mathsf{s}(y\sigma\tau) = x\sigma \\
\Longleftrightarrow & \mathsf{s}(y\sigma\tau\tau) = \mathsf{s}(\mathsf{s}(y\sigma\tau)) & \wedge & x\sigma = \mathsf{s}(y\sigma\tau) \\
\Longleftrightarrow & \underset{\mathsf{s}(z)}{\underline{\underset{z}{\underline{\underset{y}{\underline{y\sigma}}\tau}}\tau}} = \underset{\mathsf{s}(z)}{\underline{\mathsf{s}(\underset{z}{\underline{\underset{y}{\underline{y\sigma}}\tau}})}} & \wedge & \underset{\mathsf{s}(z)}{\underline{x\sigma}} = \underset{\mathsf{s}(z)}{\underline{\mathsf{s}(\underset{z}{\underline{\underset{y}{\underline{y\sigma}}\tau}})}}
\end{array}$$

therefore, $\sigma = \{x \mapsto \mathsf{s}(z)\}$ and $\tau = \{y \mapsto z,\ y \mapsto \mathsf{s}(z)\}$, which yield loop:

$$\mathsf{f}(\mathsf{s}(z), \mathsf{s}(y)) \to_{\mathcal{R}} \mathsf{f}(\mathsf{s}(\mathsf{s}(z)), \mathsf{s}(z))$$

Loop Consisting of Many-Steps

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how to find such loop? ☞ narrowing

Forward Narrowing

Definition

$\ell_1\sigma \to r_1[r_2]\sigma$ is narrowed rule of $\mathcal{R}$ if

- $\ell_1 \to r_1$ and $\ell_2 \to r_2$ are variants of $\mathcal{R}$-rules
- $p \in \mathcal{P}\mathsf{os}(r_1)$, and $\sigma = \mathrm{mgu}(r_1|_p, \ell_2)$

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Proof.

immediate from $\ell \to_{\mathcal{R}} r$ or $\ell \to_{\mathcal{R}}^2 r$ ☐

Example for Narrowing

consider TRS

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since it admits loop, TRS is non-terminating

Non-termination Proof of Toyama's Example

TRS $\mathcal{R}$

$$\mathsf{f}(\mathsf{a}, \mathsf{b}, x) \to \mathsf{f}(\underbrace{x}_{\mathsf{g}(x,y)\to x}, x, x) \qquad \mathsf{g}(x, y) \to x \qquad \mathsf{g}(x, y) \to y$$

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thus, $\mathcal{R}$ is non-terminating due to loop $\mathsf{f}(\mathsf{a},\mathsf{b},\mathsf{g}(\mathsf{a},\mathsf{b})) \to_{\mathcal{R}}^{+} \mathsf{f}(\mathsf{a},\mathsf{b},\mathsf{g}(\mathsf{a},\mathsf{b}))$

Automation of Non-Termination Analysis

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1. perform narrowing twice
2. rewrite right-hand sides of rules (e.g.) 5 times
3. check if rule admits loop by using semi-unification

Contents: Proving (Non-)Termination Automatically

TRS category in termCOMP 2024:

	tool/method	YES	NO
	AProVE	1030	282
	NaTT	876	169
	MU-TERM	717	135
	NTI	293	275
	autonon	—	233
1	linear interpretations	145	—
2	dependency pair method with linear and max/plus	431	—
3	termination of imperative programs	—	—
4	**loop detection**	—	$\geqslant$ 178

Summary

- linear interpretations
- dependency pairs
- termination of imperative programs (still challenging)
- loop detection

thanks for your attention!