Confluence Tools

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https://www.jaist.ac.jp/~hirokawa/24isr/

Confluence

Definition

confluence

Confluence

Definition

$$
\begin{tikzcd}
 & \cdot \arrow[dl, "*"'] \arrow[dr, "*"] & \\
\cdot & \textbf{confluence} & \cdot
\end{tikzcd}
$$

Confluence

Definition

$$\begin{tikzcd}
 & \cdot \arrow[dl, "*"'] \arrow[dr, "*"] & \\
\cdot \arrow[dr, dashed, "*"'] & \textcolor{red}{\textbf{confluence}} & \cdot \arrow[dl, dashed, "*"] \\
 & \cdot &
\end{tikzcd}$$

Confluence

Definition

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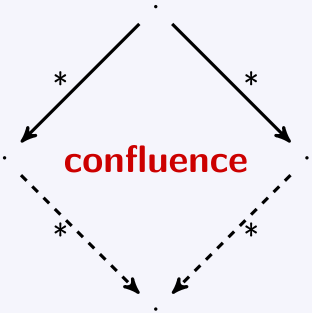
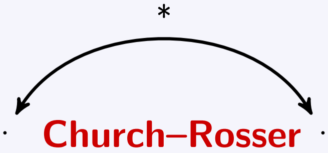
 & \cdot \arrow[dl, "*"'] \arrow[dr, "*"] & \\
\cdot \arrow[dr, dashed, "*"'] & \textbf{confluence} & \cdot \arrow[dl, dashed, "*"] \\
 & \cdot &
\end{tikzcd}
$$

Church–Rosser

Confluence

Definition





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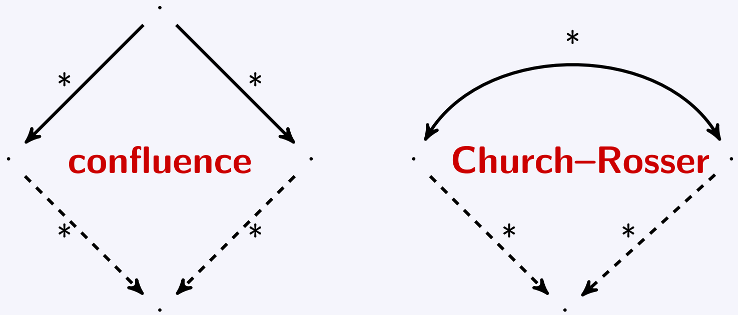

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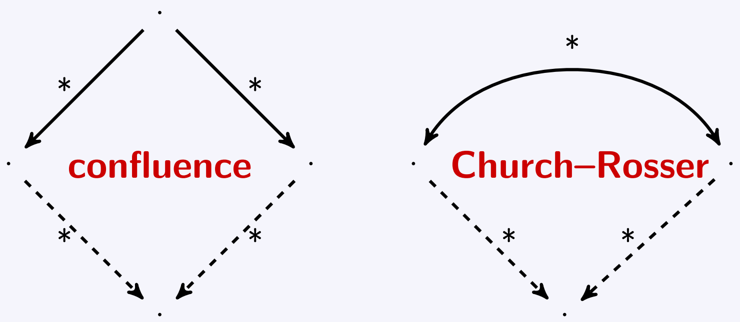

Fact

confluence and Church–Rosser property are equivalent

ARI Format

```
(format TRS)
(fun 0 0)
(fun s 1)
(fun + 2)
(rule (+ 0 x) x)
(rule (+ (s x) y) (s (+ x y)))
(rule (+ x y) (+ y x))
```

Contents: Proving (Non-)Confluence Automatically

full run of TRS category in CoCo 2024

tool/method	YES	NO
CSI	272	205
ACP	257	168
Hakusan	139	74
CONFident	108	142
FORT-h	34	89

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2	persistency decomposition (RL+KB)	$\geqslant$ 150	—

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$\boxed{3}$ generating convertible pairs	—	—

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Rule Labeling

Rule Labeling

given rule labeling function $\phi : \mathcal{R} \to \mathbb{N}$

- $s \to_k t$ if $s \to_\alpha t$ and $\phi(\ell \to r) \leqslant k$ for some $\ell \to r \in \mathcal{R}$
- $s \to_{\curlyvee km} t$ if $s \to_i t$ for some $i < \max\{k, m\}$

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Theorem (van Oostrom 2008)

linear TRS is confluent if every critical peak $t \leftarrow s \xrightarrow{\epsilon} u$ is decreasing wrt ϕ, i.e.:

$$
\begin{tikzcd}
& & s \arrow[dll, bend right, "k"'] \arrow[drrrr, bend left, "m"] & & & \\
t \arrow[r, leftrightarrow, dashed, "*", "\curlyvee k"'] & \cdot \arrow[r, dashed, "=", "m"'] & \cdot \arrow[r, leftrightarrow, dashed, "*", "\curlyvee km"'] & \cdot & \cdot \arrow[l, dashed, "=", "k"'] \arrow[r, leftrightarrow, dashed, "*", "\curlyvee m"'] & u
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$$

Proof.

immediate from decreasing diagram techniques (van Oostrom 1995) □

Example for Rule Labeling (COPS #20)

consider linear TRS and labeling for rules

$$\mathsf{hd}(x : y) \xrightarrow{\mathbf{0}} x \qquad \mathsf{inc}(x : y) \xrightarrow{\mathbf{0}} \mathsf{s}(x) : \mathsf{inc}(y) \qquad \mathsf{inc}(\mathsf{tl}(\mathsf{nats})) \xrightarrow{\mathbf{1}} \mathsf{tl}(\mathsf{inc}(\mathsf{nats}))$$

$$\mathsf{tl}(x : y) \xrightarrow{\mathbf{0}} y \qquad \mathsf{nats} \xrightarrow{\mathbf{0}} 0 : \mathsf{inc}(\mathsf{nats})$$

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confluence follows from decreasingness of critical peak:

$$\begin{tikzcd}
\mathsf{inc}(\mathsf{tl}(0 : \mathsf{inc}(\mathsf{nats}))) \arrow[d, dashed, "\mathbf{0}"'] & \mathsf{inc}(\mathsf{tl}(\mathsf{nats})) \arrow[l, "\mathbf{0}"'] \arrow[r, "\mathbf{1}"] & \mathsf{tl}(\mathsf{inc}(\mathsf{nats})) \arrow[d, dashed, "\mathbf{0}"'] \\
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\mathsf{inc}(\mathsf{inc}(\mathsf{nats})) & \mathsf{tl}(\mathsf{s}(0) : \mathsf{inc}(\mathsf{inc}(\mathsf{nats}))) \arrow[l, dashed, "0"'] & \mathsf{tl}(\mathsf{inc}(0 : \mathsf{inc}(\mathsf{nats}))) \arrow[l, dashed, "0"']
\end{tikzcd}$$

Note: given join sequences, suitable labeling can be found by SAT/SMT

Parallel Critical Pairs

Definition

$(\ell\sigma)[r_p\sigma]_{p\in P} \nleftrightarrow \ell\sigma \xrightarrow{\epsilon} r\sigma$ is parallel critical peak if

- P is non-empty set of parallel function positions in ℓ
- none of rules $\ell \to r$ and $\ell_p \to r_p$ for $p \in P$ shares variable with other rules
- σ is most general unifier of $\{\ell_p \approx (\ell|_p)\}_{p\in P}$, and
- if $P = \{\epsilon\}$ then $\ell_\epsilon \to r_\epsilon$ is not variant of $\ell \to r$

$\mathsf{PCP}(\mathcal{R}) = \{(t,u) \mid t \nleftrightarrow s \xrightarrow{\epsilon} u$ is parallel critical peak$\}$

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$\mathsf{PCP}(\mathcal{R}) = \{(t, u) \mid t \mathrel{\leftarrow\!\!\!\!\!\!\;+\!\!\!\!+} s \xrightarrow{\epsilon} u \text{ is parallel critical peak}\}$

Exercise

compute $\mathsf{PCP}(\mathcal{R})$ for $\mathcal{R} = \{\mathsf{f}(\mathsf{g}(x), \mathsf{g}(y)) \to x,\ \mathsf{g}(\mathsf{a}) \to \mathsf{a}\}$

Parallel-Critical-Pairs Closing Systems

Theorem (Shintani and Hirokawa 2024)

$\mathsf{CR}(\mathcal{S}) \implies \mathsf{CR}(\mathcal{R})$ *if* $\mathcal{R}$ *is left-linear,* $\mathcal{S} \subseteq \mathcal{R}$*, and* $\mathsf{PCP}(\mathcal{R}) \subseteq \leftrightarrow^*_{\mathcal{S}}$,

Parallel-Critical-Pairs Closing Systems

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Proof.

$\mathrel{+\!\!\!\!\!\!\twoheadrightarrow}_{\mathcal{R}} \cdot \rightarrow_{\mathcal{S}}^{*}$ has diamond property $\square$

we show confluence of left-linear TRS $\mathcal{R}$:

$$\mathsf{hd}(x : y) \xrightarrow{\mathbf{1}} x \qquad \mathsf{inc}(x : y) \xrightarrow{\mathbf{3}} \mathsf{s}(x) : \mathsf{inc}(y) \qquad \mathsf{inc}(\mathsf{tl}(\mathsf{nats})) \xrightarrow{\mathbf{5}} \mathsf{tl}(\mathsf{inc}(\mathsf{nats}))$$

$$\mathsf{tl}(x : y) \xrightarrow{\mathbf{2}} y \qquad \mathsf{nats} \xrightarrow{\mathbf{4}} 0 : \mathsf{inc}(\mathsf{nats})$$

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\mathsf{hd}(x : y) \xrightarrow{\mathbf{1}} x & \mathsf{inc}(x : y) \xrightarrow{\mathbf{3}} \mathsf{s}(x) : \mathsf{inc}(y) & \mathsf{inc}(\mathsf{tl}(\mathsf{nats})) \xrightarrow{\mathbf{5}} \mathsf{tl}(\mathsf{inc}(\mathsf{nats})) \\
\mathsf{tl}(x : y) \xrightarrow{\mathbf{2}} y & \mathsf{nats} \xrightarrow{\mathbf{4}} 0 : \mathsf{inc}(\mathsf{nats}) &
\end{array}$$

$\boxed{1}$ parallel critical peak is closed by $\mathcal{S} = \{2, 3, 5\}$:

$$\begin{tikzcd}
\mathsf{inc}(\mathsf{tl}(0 : \mathsf{inc}(\mathsf{nats}))) \arrow[d, "2"'] & \mathsf{inc}(\mathsf{tl}(\mathsf{nats})) \arrow[l, "\parallel" marking] \arrow[r] & \mathsf{tl}(\mathsf{inc}(\mathsf{nats})) \arrow[d, "5"'] \\
\mathsf{inc}(\mathsf{inc}(\mathsf{nats})) & \mathsf{tl}(\mathsf{s}(0) : \mathsf{inc}(\mathsf{inc}(\mathsf{nats}))) \arrow[l, "2"'] & \mathsf{tl}(\mathsf{inc}(0 : \mathsf{inc}(\mathsf{nats}))) \arrow[l, "3"']
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$\boxed{2}$ $\mathcal{S}$ is confluent as it is orthogonal

we show confluence of left-linear TRS $\mathcal{R}$:

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\end{tikzcd}$$

$\boxed{2}$ $\mathcal{S}$ is confluent as it is orthogonal

$\boxed{3}$ hence $\mathcal{R}$ is confluent

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Theorem (Zankl et al. 2015)

left-linear TRS $\mathcal{R}$ is confluent if every parallel critical peak is decreasing wrt ϕ

$$\begin{tikzcd}
 & & & s \arrow[dlll, bend right, "P"', "k"] \arrow[drr, bend left, "\epsilon", "m"'] & & \\
t \arrow[r, leftrightarrow, dashed, "*", "\curlyvee k"'] & \cdot \arrow[r, dashed, "m"'] & \cdot \arrow[r, leftrightarrow, dashed, "*", "\curlyvee km"'] & w & \cdot \arrow[l, dashed, "Q"', "k"] \arrow[r, leftrightarrow, dashed, "*", "\curlyvee m"'] & u
\end{tikzcd}$$

with $\mathcal{V}\mathrm{ar}(w, Q) \subseteq \mathcal{V}\mathrm{ar}(s, P)$

here $\to_k$ is rewrite step of $\{\ell \to r \in \mathcal{R} \mid \phi(\ell \to r) \leqslant k\}$

Confluence Proof by Rule Labeling

left-linear TRS

$$\mathsf{d}(x) \xrightarrow{\mathbf{0}} x + x \qquad \mathsf{s}(x) + y \xrightarrow{\mathbf{0}} \mathsf{s}(x + y) \qquad \infty \xrightarrow{\mathbf{0}} \mathsf{s}(\infty)$$

$$\mathsf{d}(\mathsf{s}(x)) \xrightarrow{\mathbf{1}} \mathsf{s}(\mathsf{s}(\mathsf{d}(x))) \qquad x + \mathsf{s}(y) \xrightarrow{\mathbf{0}} \mathsf{s}(x + y)$$

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all parallel critical peaks are decreasing; for instance,

$$\begin{tikzcd}
 & \mathsf{d}(\mathsf{s}(x)) \arrow[dl, bend right, "\{\epsilon\}"', "\mathbf{0}", "\parallel" marking] \arrow[dr, bend left, "\epsilon", "\mathbf{1}"'] & \\
\mathsf{s}(x) + \mathsf{s}(x) \arrow[d, "\mathbf{0}"'] & & \mathsf{s}(\mathsf{s}(\mathsf{d}(x))) \arrow[d, "\mathbf{0}"] \\
\mathsf{s}(x + \mathsf{s}(x)) \arrow[rr, bend right, "\mathbf{0}"'] & & \mathsf{s}(\mathsf{s}(x + x))
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \underbrace{\mathsf{s}(x) + \mathsf{s}(y)}_{\{x,y\}} \arrow[dl, bend right, "\{\epsilon\}"', "\mathbf{0}", "\parallel" marking] \arrow[dr, bend left, "\epsilon", "\mathbf{0}"'] & \\
\mathsf{s}(x + \mathsf{s}(y)) \arrow[dr, bend right, "\mathbf{0}"', "\parallel" marking] & & \mathsf{s}(\mathsf{s}(x) + y) \arrow[dl, bend left, "\mathbf{0}", "\parallel" marking] \\
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\end{tikzcd}$$

Confluence Proof by Rule Labeling

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\mathsf{s}(x + \mathsf{s}(y)) \arrow[dr, "\mathbf{0}"'] & & \mathsf{s}(\mathsf{s}(x) + y) \arrow[dl, "\mathbf{0}"] \\
 & \mathsf{s}(\underline{\mathsf{s}(x+y)}_{\{x,y\}}) &
\end{tikzcd}$$

hence TRS is confluent

Commutation

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\cdot \arrow[dr, blue, dashed, "*"'] & & \cdot \arrow[dl, red, dashed, "*"] \\
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Fact

confluence of $\mathcal{R}$ is equivalent to commutation of $\to_{\mathcal{R}}$ and $\to_{\mathcal{R}}$

given rule labeling functions $\phi, \psi : \mathcal{R} \to \mathbb{N}$

- $s \to_{\phi,k} t$ if $s \to_\alpha t$ and $k \geqslant \phi(\alpha)$ for some $\alpha \in \mathcal{R}$
- $s \leftrightarrow_{\curlyvee km} t$ if $s \to_{\psi,i} t$ or $s \;{}_{\phi,i}{\leftarrow}\; t$ for some $i < k$ or $i < m$

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Definition (parallel version of rule labeling)

parallel critical peak $t \;{}_{\phi,k}\!\overset{P}{\nparallel\!\leftarrow} s \xrightarrow{\epsilon}_{\psi,m} u$ is (ψ, ϕ)-decreasing if

$$
\begin{tikzcd}
 & & & s \arrow[dlll, bend right, "P"', "\phi,k"] \arrow[drr, bend left, "\epsilon", "\psi,m"'] & & \\
t \arrow[r, dashed, leftrightarrow, "*", "\curlyvee k"'] & \cdot \arrow[r, dashed, "\psi,m"'] & \cdot \arrow[r, dashed, leftrightarrow, "*", "\curlyvee km"'] & w & \cdot \arrow[l, dashed, "Q"', "\phi,k"] \arrow[r, dashed, leftrightarrow, "*", "\curlyvee m"'] & u
\end{tikzcd}
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& & & & & \\
& & & & & \\
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with $\mathcal{V}\mathrm{ar}(w, Q) \subseteq \mathcal{V}\mathrm{ar}(s, P)$

Theorem

left-linear TRS is confluent if

every parallel critical peak is (ϕ, ψ)- and (ψ, ϕ)-decreasing

Example for Rule Labeling with Two Labelings

TRS

$$x + y \xrightarrow{\mathbf{1},\mathbf{0}} y + x \qquad\qquad (x + y) + z \xrightarrow{\mathbf{1},\mathbf{0}} x + (y + z)$$

all parallel critical peaks $t\ {}_1{\leftarrow}\ s \rightarrow_0 u$ satisfies

$$t \rightarrow_0^* u$$

hence TRS is confluent

Remark on Related Criteria

rule labeling based on PCPs subsumes confluence criteria for left-linear TRSs:

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1. orthogonality (Rozen 1973)

Remark on Related Criteria

rule labeling based on PCPs subsumes confluence criteria for left-linear TRSs:

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full run of TRS category in CoCo 2024

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Persistency Decomposition

Example

TRS

$$x - 0 \to x \qquad \mathsf{s}(x) - \mathsf{s}(y) \to x - y \qquad x - x \to 0 \qquad \mathsf{c} \to \mathsf{f}(\mathsf{c})$$

is non-left-linear and non-terminating

Example

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how to prove confluence?

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is non-left-linear and non-terminating

how to prove confluence? ☞ split it into two subsystems

Modularity

Theorem (Toyama 1987)

$\mathsf{CR}(\mathcal{R} \cup \mathcal{S}) \iff \mathsf{CR}(\mathcal{R}) \wedge \mathsf{CR}(\mathcal{S})$ *if* $\mathcal{F}\mathsf{un}(\mathcal{R}) \cap \mathcal{F}\mathsf{un}(\mathcal{S}) = \varnothing$

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Proof.

difficult $\square$

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difficult (induction on rank) $\square$

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Example

non-left-linear and non-terminating TRS

$$x - \mathsf{0} \xrightarrow{\mathbf{1}} x \qquad \mathsf{s}(x) - \mathsf{s}(y) \xrightarrow{\mathbf{2}} x - y \qquad x - x \xrightarrow{\mathbf{3}} \mathsf{0} \qquad \mathsf{c} \xrightarrow{\mathbf{4}} \mathsf{f}(\mathsf{c})$$

is confluent because

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is confluent because $\underbrace{\{\mathbf{1},\mathbf{2},\mathbf{3}\}}_{\text{complete}}$ and $\underbrace{\{\mathbf{4}\}}_{\text{orthogonal}}$ are confluent

Limitation of Modularity

consider TRS

$$\mathsf{s}(x) + y \xrightarrow{\mathbf{1}} \mathsf{s}(x+y) \qquad \mathsf{eq}(x,x) \xrightarrow{\mathbf{2}} \mathsf{T} \qquad \mathsf{from}(x) \xrightarrow{\mathbf{4}} x : \mathsf{from}(\mathsf{s}(x))$$

$$\mathsf{eq}(\mathsf{0},\mathsf{s}(x)) \xrightarrow{\mathbf{3}} \mathsf{F}$$

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$$\mathsf{eq}(0, \mathsf{s}(x)) \xrightarrow{\mathbf{3}} \mathsf{F}$$

last modularity result is not applicable because

$$\mathsf{s} \in \mathcal{F}\mathsf{un}(\{\mathbf{1}, \mathbf{2}, \mathbf{3}\}) \cap \mathcal{F}\mathsf{un}(\{\mathbf{4}\})$$

Persistency

Theorem (Aoto 1997)

many-sorted TRS and its untyped version are equi-confluent

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very similar to proof of modularity $\square$

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Theorem (van de Pol 1995)

$\mathsf{CR}(\mathcal{R})$ *if and only if* $\mathsf{CR}(\mathcal{R}_\alpha)$ *for all sorts* α

Example of Persistency Decomposition

TRS

$$\mathsf{s}(x) + y \xrightarrow{\mathbf{1}} \mathsf{s}(x + y) \qquad \mathsf{eq}(x, x) \xrightarrow{\mathbf{2}} \mathsf{T} \qquad \mathsf{from}(x) \xrightarrow{\mathbf{4}} x : \mathsf{from}(\mathsf{s}(x))$$

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$$\mathsf{s}(x) + y \xrightarrow{1} \mathsf{s}(x + y) \qquad \mathsf{eq}(x, x) \xrightarrow{2} \mathsf{T} \qquad \mathsf{from}(x) \xrightarrow{4} x : \mathsf{from}(\mathsf{s}(x))$$
$$\mathsf{eq}(0, \mathsf{s}(x)) \xrightarrow{3} \mathsf{F}$$

admits many-sorted signature:

$$\begin{array}{llll} 0 : \mathsf{N} & \mathsf{T} : \mathsf{B} & (+) : \mathsf{N} \times \mathsf{N} \to \mathsf{N} & \mathsf{eq} : \mathsf{N} \times \mathsf{N} \to \mathsf{B} \\ \mathsf{s} : \mathsf{N} \to \mathsf{N} & \mathsf{F} : \mathsf{B} & (:) : \mathsf{N} \times \mathsf{L} \to \mathsf{L} & \mathsf{from} : \mathsf{N} \to \mathsf{L} \end{array}$$

Example of Persistency Decomposition

TRS

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$$\mathsf{eq}(0, \mathsf{s}(x)) \xrightarrow{\mathbf{3}} \mathsf{F}$$

admits many-sorted signature:

$$\begin{array}{llll} 0 : \mathsf{N} & \mathsf{T} : \mathsf{B} & (+) : \mathsf{N} \times \mathsf{N} \to \mathsf{N} & \mathsf{eq} : \mathsf{N} \times \mathsf{N} \to \mathsf{B} \\ \mathsf{s} : \mathsf{N} \to \mathsf{N} & \mathsf{F} : \mathsf{B} & (:) : \mathsf{N} \times \mathsf{L} \to \mathsf{L} & \mathsf{from} : \mathsf{N} \to \mathsf{L} \end{array}$$

$\mathcal{R}$ is confluent because ($\mathcal{R}_{\mathsf{N}} = \underbrace{\{\mathbf{1}\}}_{\text{orthogonal}}$,) $\mathcal{R}_{\mathsf{B}} = \underbrace{\{\mathbf{1}, \mathbf{2}, \mathbf{3}\}}_{\text{complete}}$, and $\mathcal{R}_{\mathsf{L}} = \underbrace{\{\mathbf{1}, \mathbf{4}\}}_{\text{orthogonal}}$

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Non-Confluence Analysis

Proving Non-Confluence Automatically

Fact

confluence $\iff$ $\forall t, u.$

$$\begin{tikzcd}
t \arrow[rr, leftrightarrow, bend left, "*"] \arrow[dr, dashed, "*"'] & \text{CR} & u \arrow[dl, dashed, "*"] \\
& \cdot &
\end{tikzcd}$$

Proving Non-Confluence Automatically

Fact

confluence $\iff$ $\forall t, u.$

$$\begin{tikzcd} t \arrow[rr, leftrightarrow, bend left, "*"] \arrow[dr, dashed, "*"'] & \arrow[d, phantom, "\textbf{CR}"] & u \arrow[dl, dashed, "*"] \\ & \cdot & \end{tikzcd}$$

How To Prove Non-Confluence

Proving **Non**-Confluence Automatically

Fact

$$\textbf{confluence} \iff \forall t, u.\ \begin{tikzcd} t \arrow[rr, leftrightarrow, bend left, "*"] \arrow[dr, dashed, "*"'] \arrow[rr, phantom, "\mathbf{CR}", near end, yshift=-3ex] & & u \arrow[dl, dashed, "*"] \\ & \cdot & \end{tikzcd}$$

How To Prove Non-Confluence

$\boxed{1}$ generate convertible pairs $t \leftrightarrow^* u$

Proving Non-Confluence Automatically

Fact

$$\textbf{confluence} \quad \iff \quad \forall t, u. \quad \begin{tikzcd} t \arrow[rr, leftrightarrow, bend left, "*"] \arrow[dr, dashed, "*"'] & \arrow[d, phantom, "\mathbf{CR}"] & u \arrow[dl, dashed, "*"] \\ & \cdot & \end{tikzcd}$$

How To Prove Non-Confluence

1. generate convertible pairs $t \leftrightarrow^* u$
2. test non-joinability $t \not\downarrow u$

Generating Convertible Pairs

Associativity — Critical Pairs

TRS

$$x + \mathsf{0} \to x \qquad\qquad (x + y) + z \to x + (y + z)$$

Associativity — Critical Pairs

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$$x + 0 \to x \qquad\qquad (x + y) + z \to x + (y + z)$$

admits non-joinable critical pair:

$$\begin{tikzcd} & (x + 0) + z \arrow[dl, bend right] \arrow[dr, bend left] & \\ \underset{\mathsf{NF}}{\underline{0 + z}} & & \underset{\mathsf{NF}}{\underline{x + (0 + z)}} \end{tikzcd}$$

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$$\begin{tikzcd}
 & (x+0)+z \arrow[dl, bend right] \arrow[dr, bend left] & \\
\underset{\mathsf{NF}}{\underline{0+z}} & & \underset{\mathsf{NF}}{\underline{x+(0+z)}}
\end{tikzcd}$$

hence $\mathcal{R}$ is not confluent

ARI-COPS #493 — Critical Pairs

TRS

$$\begin{array}{rlrlrl}
\mathsf{br}(0, y, z) \to y & & \mathsf{p}(0) \to 0 & & x + y \to \mathsf{br}(x, y, \mathsf{p}(x) + s(y)) \\
\mathsf{br}(\mathsf{s}(x), y, z) \to z & & \mathsf{p}(\mathsf{s}(x)) \to x & & x + y \to \mathsf{br}(y, x, \mathsf{s}(x) + p(y))
\end{array}$$

ARI-COPS #493 — Critical Pairs

TRS

$$
\begin{array}{rrr}
\mathsf{br}(0, y, z) \to y & \mathsf{p}(0) \to 0 & x + y \to \mathsf{br}(x, y, \mathsf{p}(x) + s(y)) \\
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\end{array}
$$

admits non-joinable critical pair:

$$
\begin{tikzcd}
 & x + y \arrow[dl, bend right] \arrow[dr, bend left] & \\
\underset{\text{stable}}{\underline{\mathsf{br}(x, y}}, \mathsf{p}(x) + \mathsf{s}(y)) & & \underset{\text{stable}}{\underline{\mathsf{br}(y, x}}, \mathsf{s}(x) + \mathsf{p}(y))
\end{tikzcd}
$$

ARI-COPS #493 — Critical Pairs

TRS

$$
\begin{array}{rclrclrcl}
\mathsf{br}(0, y, z) & \to & y & \mathsf{p}(0) & \to & 0 & x + y & \to & \mathsf{br}(x, y, \mathsf{p}(x) + s(y)) \\
\mathsf{br}(\mathsf{s}(x), y, z) & \to & z & \mathsf{p}(\mathsf{s}(x)) & \to & x & x + y & \to & \mathsf{br}(y, x, \mathsf{s}(x) + p(y))
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Kleene's Example — Critical Pairs, Normalized

TRS

$$\mathsf{b} \to \mathsf{a} \qquad \mathsf{b} \to \mathsf{c} \qquad \mathsf{c} \to \mathsf{b} \qquad \mathsf{c} \to \mathsf{d}$$

Kleene's Example — Critical Pairs, Normalized

TRS

$$b \to a \qquad b \to c \qquad c \to b \qquad c \to d$$

admits non-joinable pair originating from critical pair:

$$\begin{tikzcd} & b \arrow[dl, bend right] \arrow[dr, bend left] & \\ \underset{\mathsf{NF}}{\underline{a}} & & c \end{tikzcd}$$

Kleene's Example — Critical Pairs, **Normalized**

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hence $\mathcal{R}$ is not confluent

Huet's TRS — No Critical Pairs...

TRS $\mathcal{R}$

$$\mathsf{f}(x, x) \to \mathsf{a} \qquad \mathsf{f}(x, \mathsf{s}(x)) \to \mathsf{b} \qquad \mathsf{c} \to \mathsf{s}(\mathsf{c})$$

Huet's TRS — No Critical Pairs...

TRS $\mathcal{R}$

$$\mathsf{f}(x, x) \to \mathsf{a} \qquad\qquad \mathsf{f}(x, \mathsf{s}(x)) \to \mathsf{b} \qquad\qquad \mathsf{c} \to \mathsf{s}(\mathsf{c})$$

has admits non-joinable conversion:

$$\begin{tikzcd}
& \mathsf{f}(\mathsf{c}, \mathsf{c}) \arrow[dl, bend right] \arrow[dr, bend left] & \\
\underset{\text{NF}}{\underline{\mathsf{a}}} & & \mathsf{f}(\mathsf{c}, \mathsf{s}(\mathsf{c})) \arrow[d] \\
& & \underset{\text{NF}}{\underline{\mathsf{b}}}
\end{tikzcd}$$

Huet's TRS — No Critical Pairs...

TRS $\mathcal{R}$

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has admits non-joinable conversion:

$$\begin{array}{ccc} & \mathsf{f}(\mathsf{c},\mathsf{c}) & \\ \swarrow & & \searrow \\ \underset{\text{NF}}{\underline{\mathsf{a}}} & & \mathsf{f}(\mathsf{c},\mathsf{s}(\mathsf{c})) \\ & & \downarrow \\ & & \underset{\text{NF}}{\underline{\mathsf{b}}} \end{array}$$

how to find such peak automatically?

Huet's TRS — No Critical Pairs...

TRS $\mathcal{R}$

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 & & \underset{\text{NF}}{\underline{\mathsf{b}}}
\end{tikzcd}$$

how to find such peak automatically? ☞ peak between $\mathcal{R}$ and $\mathcal{R}^{-1}$

Huet's Example — Critical Pairs of $\mathcal{R} \cup \mathcal{R}^{-1}$

TRS $\mathcal{R}$

$$\mathsf{f}(x, x) \to \mathsf{a} \qquad \mathsf{f}(x, \mathsf{s}(x)) \to \mathsf{b} \qquad \mathsf{c} \to \mathsf{s}(\mathsf{c})$$

Huet's Example — Critical Pairs of $\mathcal{R} \cup \mathcal{R}^{-1}$

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has admits non-joinable critical peak between $\mathcal{R}$ and $\mathcal{R}^{-1}$:

$$\begin{tikzcd}
 & \mathsf{f}(\mathsf{c}, \underline{\mathsf{s}(\mathsf{c})}) \arrow[dl, bend right, "\mathcal{R}^{-1}"'] \arrow[dr, bend left, "\mathcal{R}"] & \\
\mathsf{f}(\mathsf{c}, \mathsf{c}) \arrow[d] & & \underset{\mathsf{NF}}{\underline{\mathsf{b}}} \\
\underset{\mathsf{NF}}{\underline{\mathsf{a}}} & &
\end{tikzcd}$$

Huet's Example — Critical Pairs of $\mathcal{R} \cup \mathcal{R}^{-1}$

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has admits non-joinable critical peak between $\mathcal{R}$ and $\mathcal{R}^{-1}$:

$$\begin{tikzcd} & \mathsf{f}(\mathsf{c}, \underline{\mathsf{s}(\mathsf{c})}) \arrow[dl, bend right, "\mathcal{R}^{-1}"'] \arrow[dr, bend left, "\mathcal{R}"] & \\ \mathsf{f}(\mathsf{c}, \mathsf{c}) \arrow[d] & & \underset{\mathsf{NF}}{\underline{\mathsf{b}}} \\ \underset{\mathsf{NF}}{\underline{\mathsf{a}}} & & \end{tikzcd}$$

hence $\mathcal{R}$ is not confluent

Klop's Example — Reducing Right-Hand Sides

Klop's TRS

$$\mathsf{f}(x,x) \to \mathsf{a} \qquad \mathsf{g}(x) \to \mathsf{f}(x,\mathsf{g}(x)) \qquad \mathsf{c} \to \underline{\mathsf{g}(\mathsf{c})}$$

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$$\mathsf{f}(x, x) \to \mathsf{a} \qquad \mathsf{g}(x) \to \mathsf{f}(x, \mathsf{g}(x)) \qquad \mathsf{c} \to \underline{\mathsf{g}(\mathsf{c})}$$

admits non-joinable conversion:

$$\underset{\mathsf{WN}}{\underline{\mathsf{g}(\,\underset{\mathsf{WN}}{\underline{\mathsf{c}}}\,)}}$$

Klop's Example — Reducing Right-Hand Sides

Klop's TRS

$$\mathsf{f}(x,x) \to \mathsf{a} \qquad\qquad \mathsf{g}(x) \to \mathsf{f}(x,\mathsf{g}(x)) \qquad\qquad \mathsf{c} \to \underline{\mathsf{g}(\mathsf{c})}$$

admits non-joinable conversion:

$$\begin{tikzcd}
 & \mathsf{g}(\underset{\color{red}\mathsf{WN}}{\underline{\mathsf{c}}}) \;\; \dfrac{}{\color{red}\mathsf{WN}} \arrow[dl, bend right] \\
\mathsf{f}(\mathsf{c},\mathsf{g}(\mathsf{c})) \arrow[d] & \\
\mathsf{f}(\mathsf{g}(\mathsf{c}),\mathsf{g}(\mathsf{c})) \arrow[d] & \\
\underset{\color{red}\mathsf{NF}}{\underline{\mathsf{a}}} &
\end{tikzcd}$$

Klop's Example — Reducing Right-Hand Sides

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admits non-joinable conversion:

$$\begin{tikzcd}
 & \underset{\mathsf{WN}}{\underline{\mathsf{g}(\underset{\mathsf{WN}}{\underline{\mathsf{c}}})}} \arrow[dl] \arrow[r] & \mathsf{g}(\mathsf{g}(\mathsf{c})) \arrow[dr] & \\
\mathsf{f}(\mathsf{c}, \mathsf{g}(\mathsf{c})) \arrow[d] & & & \mathsf{g}(\mathsf{f}(\mathsf{c}, \mathsf{g}(\mathsf{c}))) \arrow[d] \\
\mathsf{f}(\mathsf{g}(\mathsf{c}), \mathsf{g}(\mathsf{c})) \arrow[d] & & & \mathsf{g}(\mathsf{f}(\mathsf{g}(\mathsf{c}), \mathsf{g}(\mathsf{c}))) \arrow[d] \\
\underset{\mathsf{NF}}{\underline{\mathsf{a}}} & & & \underset{\text{not } \mathsf{WN}}{\underline{\mathsf{g}(\mathsf{a})}}
\end{tikzcd}$$

Testing Non-Joinability by tcap

Associativity — Distinct Normal Forms

TRS

$$x + \mathsf{0} \to x \qquad\qquad (x + y) + z \to x + (y + z)$$

Associativity — Distinct Normal Forms

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$$x + 0 \to x \qquad\qquad (x + y) + z \to x + (y + z)$$

admits non-joinable critical pair:

$$\begin{tikzcd}[column sep=small] & (x + 0) + z \arrow[dl, bend right] \arrow[dr, bend left] & \\ \underbrace{0 + z}_{\mathsf{NF}} & & \underbrace{x + (0 + z)}_{\mathsf{NF}} \end{tikzcd}$$

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hence $\mathcal{R}$ is not confluent

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$$\mathsf{b} \to \mathsf{a} \qquad \mathsf{b} \to \mathsf{c} \qquad \mathsf{c} \to \mathsf{b} \qquad \mathsf{c} \to \mathsf{d}$$

Kleene's Example — Distinct Normal Forms

TRS

$$\mathsf{b} \to \mathsf{a} \qquad \mathsf{b} \to \mathsf{c} \qquad \mathsf{c} \to \mathsf{b} \qquad \mathsf{c} \to \mathsf{d}$$

admits non-joinable pair originating from critical pair:

$$\begin{tikzcd} & \mathsf{b} \arrow[dl, bend right] \arrow[dr, bend left] & \\ \underset{\text{NF}}{\underline{\mathsf{a}}} & & \mathsf{c} \end{tikzcd}$$

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$$\begin{tikzcd}
& \mathsf{b} \arrow[dl, bend right] \arrow[dr, bend left] & \\
\underset{\textcolor{red}{\mathsf{NF}}}{\underline{\mathsf{a}}} & & \mathsf{c} \arrow[d] \\
& & \underset{\textcolor{red}{\mathsf{NF}}}{\underline{\mathsf{d}}}
\end{tikzcd}$$

Kleene's Example — Distinct Normal Forms

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$$\begin{tikzcd}
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\underset{\text{NF}}{\underline{\mathsf{a}}} & & \mathsf{c} \arrow[d] \\
 & & \underset{\text{NF}}{\underline{\mathsf{d}}}
\end{tikzcd}$$

hence $\mathcal{R}$ is not confluent

Huet's TRS — Distinct Normal Forms

TRS $\mathcal{R}$

$$\mathsf{f}(x,x) \to \mathsf{a} \qquad \mathsf{f}(x,\mathsf{s}(x)) \to \mathsf{b} \qquad \mathsf{c} \to \mathsf{s}(\mathsf{c})$$

Huet's TRS — Distinct Normal Forms

TRS $\mathcal{R}$

$$\mathsf{f}(x, x) \to \mathsf{a} \qquad \mathsf{f}(x, \mathsf{s}(x)) \to \mathsf{b} \qquad \mathsf{c} \to \mathsf{s}(\mathsf{c})$$

has admits non-joinable conversion:

$$\begin{tikzcd}
 & \mathsf{f}(\mathsf{c}, \mathsf{c}) \arrow[dl, bend right] \arrow[dr, bend left] & \\
\underset{\mathsf{NF}}{\underline{\mathsf{a}}} & & \mathsf{f}(\mathsf{c}, \mathsf{s}(\mathsf{c})) \arrow[d] \\
 & & \underset{\mathsf{NF}}{\underline{\mathsf{b}}}
\end{tikzcd}$$

ARI-COPS #493 — No Normal Forms...

TRS

$$
\begin{aligned}
\mathsf{br}(0, y, z) &\to y & \mathsf{p}(0) &\to 0 & x + y &\to \mathsf{br}(x, y, \mathsf{p}(x) + s(y)) \\
\mathsf{br}(\mathsf{s}(x), y, z) &\to z & \mathsf{p}(\mathsf{s}(x)) &\to x & x + y &\to \mathsf{br}(y, x, \mathsf{s}(x) + p(y))
\end{aligned}
$$

ARI-COPS #493 — No Normal Forms...

TRS

$$
\begin{array}{rclrclrcl}
\mathsf{br}(0, y, z) & \to & y & \mathsf{p}(0) & \to & 0 & x + y & \to & \mathsf{br}(x, y, \mathsf{p}(x) + s(y)) \\
\mathsf{br}(\mathsf{s}(x), y, z) & \to & z & \mathsf{p}(\mathsf{s}(x)) & \to & x & x + y & \to & \mathsf{br}(y, x, \mathsf{s}(x) + p(y))
\end{array}
$$

admits non-joinable critical pair:

$$
\begin{tikzcd}
 & x + y \arrow[dl] \arrow[dr] & \\
\underset{\text{stable}}{\underline{\mathsf{br}(x, y,}}\, \mathsf{p}(x) + \mathsf{s}(y)) & & \underset{\text{stable}}{\underline{\mathsf{br}(y, x,}}\, \mathsf{s}(x) + \mathsf{p}(y))
\end{tikzcd}
$$

ARI-COPS #493 — No Normal Forms...

TRS

$$
\begin{aligned}
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\end{aligned}
$$

admits non-joinable critical pair:

$$
\begin{tikzcd}[column sep=tiny]
& x + y \arrow[dl] \arrow[dr] & \\
\underbrace{\mathsf{br}(x, y,}_{\text{stable}} \mathsf{p}(x) + \mathsf{s}(y)) & & \underbrace{\mathsf{br}(y, x,}_{\text{stable}} \mathsf{s}(x) + \mathsf{p}(y))
\end{tikzcd}
$$

how to identify stable part?

ARI-COPS #493 — No Normal Forms...

TRS

$$
\begin{aligned}
\mathsf{br}(0, y, z) &\to y & \mathsf{p}(0) &\to 0 & x + y &\to \mathsf{br}(x, y, \mathsf{p}(x) + s(y)) \\
\mathsf{br}(\mathsf{s}(x), y, z) &\to z & \mathsf{p}(\mathsf{s}(x)) &\to x & x + y &\to \mathsf{br}(y, x, \mathsf{s}(x) + p(y))
\end{aligned}
$$

admits non-joinable critical pair:

$$
\underbrace{\mathsf{br}(x, y,}_{\text{stable}} \mathsf{p}(x) + \mathsf{s}(y)) \leftarrow x + y \rightarrow \underbrace{\mathsf{br}(y, x,}_{\text{stable}} \mathsf{s}(x) + \mathsf{p}(y))
$$

how to identify stable part? ☞ cap function

Non-Reachability by tcap

Definition

$$\mathsf{tcap}_{\mathcal{R}}(t) = \begin{cases} u & \text{if } t = f(t_1, \ldots, t_n),\ u = f(\mathsf{tcap}_{\mathcal{R}}(t_1), \ldots, \mathsf{tcap}_{\mathcal{R}}(t_n)), \text{ and (*)} \\ y & \text{otherwise} \end{cases}$$

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- (*): u is not unifiable with ℓ for any $\ell \to r \in \mathcal{R}$, and

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Non-Reachability by tcap

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- (*): u is not unifiable with ℓ for any $\ell \to r \in \mathcal{R}$, and
- y is fresh variable

Example

$$\mathsf{tcap}_{\mathcal{R}}(\mathsf{s}(\underbrace{\underbrace{x}_{y_1} + \underbrace{x}_{y_2}}_{y_3}) + \mathsf{a}) = \mathsf{s}(y_3) + \mathsf{a} \qquad \text{for } \mathcal{R} = \{x + \mathsf{s}(y) \to \mathsf{s}(x+y)\}$$

Non-Reachability by tcap

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$$\mathsf{tcap}_{\mathcal{R}}(t) = \begin{cases} u & \text{if } t = f(t_1, \ldots, t_n),\ u = f(\mathsf{tcap}_{\mathcal{R}}(t_1), \ldots, \mathsf{tcap}_{\mathcal{R}}(t_n)), \text{ and (*)} \\ y & \text{otherwise} \end{cases}$$

- (*): u is not unifiable with ℓ for any $\ell \to r \in \mathcal{R}$, and
- y is fresh variable

Example

$$\mathsf{tcap}_{\mathcal{R}}(\mathsf{s}(\underbrace{\underset{y_1}{\underline{x}} + \underset{y_2}{\underline{x}}}_{y_3}) + \mathsf{a}) = \mathsf{s}(y_3) + \mathsf{a} \qquad \text{for } \mathcal{R} = \{x + \mathsf{s}(y) \to \mathsf{s}(x + y)\}$$

Lemma

if $t \to^* u$ *then* $u = \mathsf{tcap}_{\mathcal{R}}(t)\sigma$ *for some* σ

Non-Joinability by tcap

Lemma

$t \not\downarrow u$ if $\mathsf{tcap}_{\mathcal{R}}(t)$ and $\mathsf{tcap}_{\mathcal{R}}(u)$ are not unifiable

Non-Joinability by tcap

Lemma

$t \not\downarrow u$ *if* $\mathsf{tcap}_{\mathcal{R}}(t)$ *and* $\mathsf{tcap}_{\mathcal{R}}(u)$ *are not unifiable*

Proof.

since $t' = \mathsf{tcap}_{\mathcal{R}}(t)$ and $u' = \mathsf{tcap}_{\mathcal{R}}(u)$ have no common variables,

Non-Joinability by tcap

Lemma

$t \not\downarrow u$ *if* $\mathsf{tcap}_{\mathcal{R}}(t)$ *and* $\mathsf{tcap}_{\mathcal{R}}(u)$ *are not unifiable*

Proof.

since $t' = \mathsf{tcap}_{\mathcal{R}}(t)$ and $u' = \mathsf{tcap}_{\mathcal{R}}(u)$ have no common variables,

$$t \to^* v \;{}^*{\leftarrow}\; u \implies t'\sigma = v = u'\sigma \text{ for some } \sigma$$

Non-Joinability by tcap

Lemma

$t \not\downarrow u$ *if* $\mathsf{tcap}_{\mathcal{R}}(t)$ *and* $\mathsf{tcap}_{\mathcal{R}}(u)$ *are not unifiable*

Proof.

since $t' = \mathsf{tcap}_{\mathcal{R}}(t)$ and $u' = \mathsf{tcap}_{\mathcal{R}}(u)$ have no common variables,

$$t \to^* v \ {}^*\!\!\leftarrow u \implies t'\sigma = v = u'\sigma \text{ for some } \sigma$$

hence contraposition holds $\square$

ARI-COPS #493 — tcap ?

TRS

$$\begin{aligned} \mathsf{br}(0, y, z) &\to y & \mathsf{p}(0) &\to 0 & x + y &\to \mathsf{br}(x, y, \mathsf{p}(x) + s(y)) \\ \mathsf{br}(\mathsf{s}(x), y, z) &\to z & \mathsf{p}(\mathsf{s}(x)) &\to x & x + y &\to \mathsf{br}(y, x, \mathsf{s}(x) + p(y)) \end{aligned}$$

ARI-COPS #493 — tcap ?

TRS

$$
\begin{aligned}
\mathsf{br}(0, y, z) &\to y & \mathsf{p}(0) &\to 0 & x + y &\to \mathsf{br}(x, y, \mathsf{p}(x) + s(y)) \\
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\end{aligned}
$$

by applying tcap to peak

$$
\begin{tikzcd}
& x + y \arrow[dl] \arrow[dr] & \\
t = \underbrace{\mathsf{br}(\underbrace{x}_{x_1}, \underbrace{y}_{x_2}, \underbrace{\underbrace{\mathsf{p}(\underbrace{x}_{x_3})}_{x_4} + \mathsf{s}(\underbrace{y}_{x_5})}_{x_6})}_{x_7} & & \underbrace{\mathsf{br}(\underbrace{y}_{y_1}, \underbrace{x}_{y_2}, \underbrace{\mathsf{s}(\underbrace{x}_{y_3}) + \underbrace{\mathsf{p}(\underbrace{y}_{y_4})}_{y_5}}_{y_6})}_{y_7} = u
\end{tikzcd}
$$

ARI-COPS #493 — tcap ?

TRS

$$\begin{array}{ccc}
\mathsf{br}(0, y, z) \to y & \mathsf{p}(0) \to 0 & x + y \to \mathsf{br}(x, y, \mathsf{p}(x) + s(y)) \\
\mathsf{br}(\mathsf{s}(x), y, z) \to z & \mathsf{p}(\mathsf{s}(x)) \to x & x + y \to \mathsf{br}(y, x, \mathsf{s}(x) + p(y))
\end{array}$$

by applying tcap to peak

$$x + y$$

$$t = \underbrace{\mathsf{br}(\underbrace{x}_{x_1}, \underbrace{y}_{x_2}, \underbrace{\underbrace{\mathsf{p}(\underbrace{x}_{x_3})}_{x_4} + \mathsf{s}(\underbrace{y}_{x_5})}_{x_6})}_{x_7} \qquad \underbrace{\mathsf{br}(\underbrace{y}_{y_1}, \underbrace{x}_{y_2}, \underbrace{\mathsf{s}(\underbrace{x}_{y_3}) + \underbrace{\mathsf{p}(\underbrace{y}_{y_4})}_{y_5}}_{y_6}))}_{y_7} = u$$

$\mathsf{tcap}_{\mathcal{R}}(t) = x_7$ and $\mathsf{tcap}_{\mathcal{R}}(u) = y_7$ are unifiable ...

ARI-COPS #493 — tcap and Grounding

TRS

$$\begin{aligned}
\mathsf{br}(0, y, z) &\to y & \mathsf{p}(0) &\to 0 & x + y &\to \mathsf{br}(x, y, \mathsf{p}(x) + s(y)) \\
\mathsf{br}(\mathsf{s}(x), y, z) &\to z & \mathsf{p}(\mathsf{s}(x)) &\to x & x + y &\to \mathsf{br}(y, x, \mathsf{s}(x) + p(y))
\end{aligned}$$

by applying tcap to peak

$$\begin{tikzcd}[column sep=small]
& \mathsf{c}_x + \mathsf{c}_y \arrow[dl, bend right] \arrow[dr, bend left] & \\
t = \mathsf{br}(\mathsf{c}_x, \mathsf{c}_y, \underbrace{\mathsf{p}(\mathsf{c}_x) + \mathsf{s}(\mathsf{c}_y))}_{x_1} & & \mathsf{br}(\mathsf{c}_y, \mathsf{c}_x, \underbrace{\mathsf{s}(\mathsf{c}_x) + \mathsf{p}(\mathsf{c}_y)}_{y_1}) = u
\end{tikzcd}$$

ARI-COPS #493 — tcap and Grounding

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$$\begin{aligned}
\mathsf{br}(0, y, z) &\to y & \mathsf{p}(0) &\to 0 & x + y &\to \mathsf{br}(x, y, \mathsf{p}(x) + s(y)) \\
\mathsf{br}(\mathsf{s}(x), y, z) &\to z & \mathsf{p}(\mathsf{s}(x)) &\to x & x + y &\to \mathsf{br}(y, x, \mathsf{s}(x) + p(y))
\end{aligned}$$

by applying tcap to peak

$$\begin{tikzcd}
& \mathsf{c}_x + \mathsf{c}_y \arrow[dl, bend right] \arrow[dr, bend left] & \\
t = \mathsf{br}(\mathsf{c}_x, \mathsf{c}_y, \underbrace{\mathsf{p}(\mathsf{c}_x) + \mathsf{s}(\mathsf{c}_y))}_{x_1} & & \mathsf{br}(\mathsf{c}_y, \mathsf{c}_x, \underbrace{\mathsf{s}(\mathsf{c}_x) + \mathsf{p}(\mathsf{c}_y)}_{y_1}) = u
\end{tikzcd}$$

$t \not\downarrow u$ follows from non-unifiability of $\begin{cases} \mathsf{tcap}_{\mathcal{R}}(t) = \mathsf{br}(\mathsf{c}_x, \mathsf{c}_y, x_1) \\ \mathsf{tcap}_{\mathcal{R}}(u) = \mathsf{br}(\mathsf{c}_y, \mathsf{c}_x, y_1) \end{cases}$

Summary of tcap

Definition

$\hat{t} = t\{x \mapsto \mathsf{c}_x \mid x \in \mathcal{V}\mathsf{ar}(t)\}$ where c_x is fresh constant

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$\hat{t} = t\{x \mapsto \mathsf{c}_x \mid x \in \mathcal{V}\mathsf{ar}(t)\}$ where c_x is fresh constant

Theorem (Zankl et al. 2011)

$t \not\downarrow u$ if $\mathsf{tcap}_{\mathcal{R}}(\hat{t})$ and $\mathsf{tcap}_{\mathcal{R}}(\hat{u})$ are not unifiable

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$\hat{t} = t\{x \mapsto \mathsf{c}_x \mid x \in \mathcal{V}\mathsf{ar}(t)\}$ where c_x is fresh constant

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Fact

$\mathsf{tcap}_{\mathcal{R}}(\hat{t})$ and $\mathsf{tcap}_{\mathcal{R}}(\hat{u})$ are not unifiable if t and u are distinct normal form

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Fact

$\mathsf{tcap}_{\mathcal{R}}(\hat{t})$ and $\mathsf{tcap}_{\mathcal{R}}(\hat{u})$ are not unifiable if t and u are distinct normal form

Proof.

$\mathsf{tcap}_{\mathcal{R}}(\hat{t}) = \hat{t}$ if t is ground normal form $\square$

Testing Non-Joinability by Tree Automata

Klop's Example — No Stable Part...

TRS

$$\mathsf{f}(x,x) \to \mathsf{a} \qquad \mathsf{g}(x) \to \mathsf{f}(x,\mathsf{g}(x)) \qquad \mathsf{c} \to \mathsf{g}(\mathsf{c})$$

Klop's Example — No Stable Part...

TRS

$$\mathsf{f}(x, x) \to \mathsf{a} \qquad \mathsf{g}(x) \to \mathsf{f}(x, \mathsf{g}(x)) \qquad \mathsf{c} \to \mathsf{g}(\mathsf{c})$$

admits conversion:

$$\mathsf{a} \xleftrightarrow{*} \underbrace{\mathsf{g}(\mathsf{a})}_{y_1}$$

Klop's Example — No Stable Part...

TRS

$$\mathsf{f}(x, x) \to \mathsf{a} \qquad \mathsf{g}(x) \to \mathsf{f}(x, \mathsf{g}(x)) \qquad \mathsf{c} \to \mathsf{g}(\mathsf{c})$$

admits conversion:

$$\mathsf{a} \xleftrightarrow{*} \underbrace{\mathsf{g}(\mathsf{a})}_{y_1}$$

$\mathsf{tcap}_{\mathcal{R}}(\mathsf{a}) = \mathsf{a}$ and $\mathsf{tcap}_{\mathcal{R}}(\mathsf{g}(\mathsf{a})) = y_1$ are unifiable ...

Klop's Example — Disjoint Closed Sets

TRS $\mathcal{R}$

$$\mathsf{f}(x,x) \to \mathsf{a} \qquad \mathsf{g}(x) \to \mathsf{f}(x,\mathsf{g}(x)) \qquad \mathsf{c} \to \mathsf{g}(\mathsf{c})$$

admits conversion:

$$\mathsf{a} \xleftrightarrow{*} \mathsf{g}(\mathsf{a})$$

Klop's Example — Disjoint Closed Sets

TRS $\mathcal{R}$

$$\mathsf{f}(x,x) \to \mathsf{a} \qquad \mathsf{g}(x) \to \mathsf{f}(x,\mathsf{g}(x)) \qquad \mathsf{c} \to \mathsf{g}(\mathsf{c})$$

admits conversion:

$$\mathsf{a} \xleftrightarrow{*} \mathsf{g}(\mathsf{a})$$

define

$$T = \{\mathsf{a}\} \qquad U = \{\mathsf{g}(\mathsf{a}), \mathsf{f}(\mathsf{a},\mathsf{g}(\mathsf{a})), \mathsf{f}(\mathsf{a},\mathsf{f}(\mathsf{a},\mathsf{g}(\mathsf{a}))), \dots\}$$

because T and U are disjoint and closed under rewriting, $\mathsf{a} \not\downarrow \mathsf{g}(\mathsf{a})$

Klop's Example — Disjoint Closed Sets

TRS $\mathcal{R}$

$$\mathsf{f}(x,x) \to \mathsf{a} \qquad \mathsf{g}(x) \to \mathsf{f}(x,\mathsf{g}(x)) \qquad \mathsf{c} \to \mathsf{g}(\mathsf{c})$$

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because T and U are disjoint and closed under rewriting, $\mathsf{a} \not\downarrow \mathsf{g}(\mathsf{a})$

☞ automatable if T and U are represented by tree automata

Tree Automata Construction

TRS $\mathcal{R}$:

$$\mathsf{f}(x, x) \to \mathsf{a} \qquad \mathsf{g}(x) \to \mathsf{f}(x, \mathsf{g}(x)) \qquad \mathsf{c} \to \mathsf{g}(\mathsf{c})$$

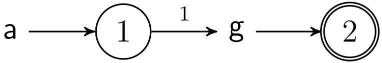
tree automaton M:

Tree Automata Construction

TRS $\mathcal{R}$:

$$\mathsf{f}(x, x) \to \mathsf{a} \qquad \mathsf{g}(x) \to \mathsf{f}(x, \mathsf{g}(x)) \qquad \mathsf{c} \to \mathsf{g}(\mathsf{c})$$

tree automaton M:



$\boxed{1}$ $\mathsf{g}(\mathsf{a}) \in \mathsf{L}(M)$

Tree Automata Construction

TRS $\mathcal{R}$:

$$\mathsf{f}(x, x) \to \mathsf{a} \qquad\qquad \mathsf{g}(x) \to \mathsf{f}(x, \mathsf{g}(x)) \qquad\qquad \mathsf{c} \to \mathsf{g}(\mathsf{c})$$

tree automaton M:

$$\begin{tikzcd} \mathsf{a} \arrow[r] & 1 \arrow[r, "1"] & \mathsf{g} \arrow[r] & \boxed{2} \end{tikzcd}$$

$\boxed{1}$ $\mathsf{g}(\mathsf{a}) \in \mathsf{L}(M)$

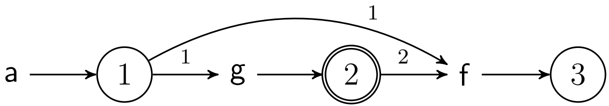
$\boxed{2}$ $\mathsf{g}(1) \to_M^* 2$ demands $\underbrace{\mathsf{f}(1, \underbrace{\mathsf{g}(1)}_{2})}_{3} \to_M^* 2$

Tree Automata Construction

TRS $\mathcal{R}$:

$$\mathsf{f}(x, x) \to \mathsf{a} \qquad \mathsf{g}(x) \to \mathsf{f}(x, \mathsf{g}(x)) \qquad \mathsf{c} \to \mathsf{g}(\mathsf{c})$$

tree automaton M:



$\boxed{1}$ $\mathsf{g}(\mathsf{a}) \in \mathsf{L}(M)$

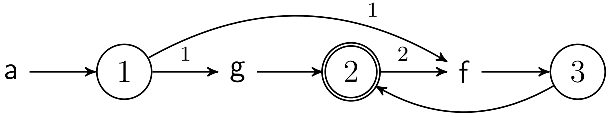
$\boxed{2}$ $\mathsf{g}(1) \to_M^* 2$ demands $\underbrace{\mathsf{f}(1, \underbrace{\mathsf{g}(1)}_{2})}_{3} \to_M^* 2$

Tree Automata Construction

TRS $\mathcal{R}$:

$$\mathsf{f}(x, x) \to \mathsf{a} \qquad \mathsf{g}(x) \to \mathsf{f}(x, \mathsf{g}(x)) \qquad \mathsf{c} \to \mathsf{g}(\mathsf{c})$$

tree automaton M:



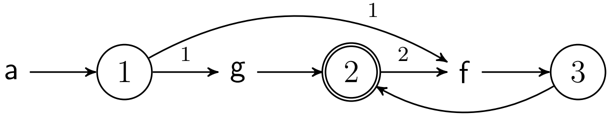
1. $\mathsf{g}(\mathsf{a}) \in \mathsf{L}(M)$
2. $\mathsf{g}(1) \to_M^* 2$ demands $\underbrace{\mathsf{f}(1, \underbrace{\mathsf{g}(1)}_{2})}_{3} \to_M^* 2$
3. completed;

Tree Automata Construction

TRS $\mathcal{R}$:

$$\mathsf{f}(x, x) \to \mathsf{a} \qquad \mathsf{g}(x) \to \mathsf{f}(x, \mathsf{g}(x)) \qquad \mathsf{c} \to \mathsf{g}(\mathsf{c})$$

tree automaton M:



1. $\mathsf{g}(\mathsf{a}) \in \mathsf{L}(M)$
2. $\mathsf{g}(1) \to_M^* 2$ demands $\underbrace{\mathsf{f}(1, \underbrace{\mathsf{g}(1)}_{2})}_{3} \to_M^* 2$
3. completed; $L(M) = \{\mathsf{g}(\mathsf{a}), \mathsf{f}(\mathsf{a}, \mathsf{g}(\mathsf{a})), \mathsf{f}(\mathsf{a}, \mathsf{f}(\mathsf{a}, \mathsf{g}(\mathsf{a}))), \ldots\}$

Summary of Tree Automata Techniques

Theorem

$t \not\downarrow u$ if there exist tree automata M and N such that

- $t \in L(M)$, $u \in L(N)$,
- T and U are closed under rewriting, and — decidable
- $T \cap U = \varnothing$ — decidable

Summary of Tree Automata Techniques

Theorem

$t \not\downarrow u$ if there exist tree automata M and N such that

- $t \in L(M)$, $u \in L(N)$,
- T and U are closed under rewriting, and — decidable
- $T \cap U = \varnothing$ — decidable

Fact (simulation of tcap by tree automata)

for every ground term t there exists tree automaton M such that

$$\mathsf{L}(M) = \{u \mid u \text{ is ground instance of } \mathsf{tcap}_{\mathcal{R}}(t)\}$$

and $\mathsf{L}(M)$ is closed under $\mathcal{R}$-rewriting

Contents: Proving (Non-)Confluence Automatically

full run of TRS category in CoCo 2024

	tool/method	YES	NO
	CSI	272	205
	ACP	257	168
	Hakusan	139	74
	CONFident	108	142
	FORT-h	34	89
$\boxed{1}$	rule labeling	133	—
$\boxed{2}$	persistency decomposition (RL+KB)	$\geqslant 150$	—
$\boxed{3}$	generating convertible pairs	—	—
$\boxed{4}$	**non-joinability test**	—	207

References for Tree Automata Techniques

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  Decidable Approximations of Sets of Descendants and Sets of Normal Forms.
  9th RTA, LNCS 1379, pp. 151–165, 1998.

- B. Felgenhauer and R. Thiemann.
  Reachability, Confluence, and Termination Analysis with State-Compatible Automata. I&C 253:467–483, 2017.