SAT/SMT Solving and Applications in Rewriting

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Outline

Exercise

- LPO is a relation on terms $\succ_{LPO}$ ($\succ$ for short), parametrized by precedence p

$$\frac{s_i = t \vee s_i \succ t}{s = f(s_1, \ldots, s_n) \succ t} \quad \text{(sub)}$$

$$\frac{p(f) > p(g) \quad \forall i \in \{1, \ldots, m\}.\ s \succ t_i}{s = f(\ldots) \succ g(t_1, \ldots, t_m) = t} \quad \text{(prec)}$$

$$\frac{\forall j \in \{1, \ldots, i-1\}.\ s_j = t_j \quad s_i \succ t_i \quad \forall j \in \{i+1, \ldots, n\}.\ s \succ t_i}{s = f(s_1, \ldots, s_n) \succ f(t_1, \ldots, t_n) = t} \quad \text{(lex)}$$

- given some argument filter π define
  - $\pi(x) = x$
  - $\pi(f(t_1, \ldots, t_n)) = \begin{cases} \pi(t_i), & \text{if } \pi(f) = i \\ f([\pi(t_i) \mid i \leftarrow [1..n], i \in \pi(f)]), & \text{if } \pi(f) \text{ is a set} \end{cases}$
- exercise: given s and t, encode "$\exists \pi.\ \pi(s) \succ_{LPO(p)} \pi(t)$" as SAT problem

Towards an Encoding of $\ulcorner \pi(s) \succ \pi(t) \urcorner$

- encoding is based on variables $i \in \pi(f)$, $set(f)$, $\ulcorner \pi(s_i) \succ \pi(t_j) \urcorner$, $\ulcorner \pi(s_i) = \pi(t_j) \urcorner$ (and precedence encoding as before)
- $\ulcorner \pi(s_i) \succeq \pi(t_j) \urcorner := \ulcorner \pi(s_i) \succ \pi(t_j) \urcorner \vee \ulcorner \pi(s_i) = \pi(t_j) \urcorner$
- add the following constraint about $\ulcorner \pi(s_i) = \pi(t_j) \urcorner$
- $\ulcorner \pi(x) = \pi(y) \urcorner \to \bot$ if $x \neq y$
- $\ulcorner \pi(f(s_1, \ldots, s_n)) = \pi(t_j) \urcorner \to \neg set(f) \to i \in \pi(f) \to \ulcorner \pi(s_i) = \pi(t_j) \urcorner$
- $\ulcorner \pi(s_i) = \pi(g(t_1, \ldots, g_m)) \urcorner \to \neg set(g) \to j \in \pi(g) \to \ulcorner \pi(s_i) = \pi(t_j) \urcorner$
- $\neg set(f) \to \ulcorner exactlyOne(1 \in \pi(f), \ldots, n \in \pi(f)) \urcorner$ for n-ary f
- $\ulcorner \pi(f(\ldots)) = \pi(y) \urcorner \to \neg set(f)$
- $\ulcorner \pi(x) = \pi(g(\ldots)) \urcorner \to \neg set(g)$
- $\ulcorner \pi(f(\ldots)) = \pi(g(\ldots)) \urcorner \to \neg set(f) \vee \neg set(g)$ if $f \neq g$
- $\ulcorner \pi(f(s_1, \ldots, s_n)) = \pi(f(t_1, \ldots, t_n)) \urcorner \to set(f) \to i \in \pi(f) \to \ulcorner \pi(s_i) = \pi(t_i) \urcorner$

An Encoding of $\ulcorner \pi(s) \succ \pi(t) \urcorner$

- add all of the following constraints
- $\ulcorner \pi(s) \succ \pi(t) \urcorner$
- $\ulcorner \pi(x) \succ \pi(t_j) \urcorner \to \bot$
- $\ulcorner \pi(f(s_1, \ldots, s_n)) \succ \pi(t_j) \urcorner \to \neg \mathit{set}(f) \to i \in \pi(f) \to \ulcorner \pi(s_i) \succ \pi(t_j) \urcorner$
- $\ulcorner \pi(s_i) \succ \pi(g(t_1, \ldots, t_m)) \urcorner \to \neg \mathit{set}(g) \to j \in \pi(g) \to \ulcorner \pi(s_i) \succ \pi(t_j) \urcorner$
- $\ulcorner \pi(f(s_1, \ldots, s_n)) \succ \pi(y) \urcorner \to \mathit{set}(f) \to \bigvee_i (i \in \pi(f) \wedge \ulcorner \pi(s_i) \succeq \pi(y) \urcorner)$
- $\ulcorner \pi(f(s_1, \ldots, s_n)) \succ \pi(g(t_1, \ldots, t_m)) \urcorner \to \mathit{set}(f) \to \mathit{set}(g) \to$
  $(\ulcorner p(f) > p(g) \urcorner \wedge \bigwedge_j (j \in \pi(g) \to \ulcorner \pi(f(s_1, \ldots, s_n)) \succ \pi(t_j) \urcorner))$
  $\vee \bigvee_i (i \in \pi(f) \wedge \ulcorner \pi(s_i) \succeq \pi(g(t_1, \ldots, t_m)) \urcorner)$ if $f \neq g$
- $\ulcorner \pi(f(s_1, \ldots, s_n)) \succ \pi(f(t_1, \ldots, t_n)) \urcorner \to \mathit{set}(f) \to$
  $\bigvee_i (i \in \pi(f) \wedge \ulcorner \pi(s_i) \succeq \pi(f(t_1, \ldots, t_n)) \urcorner)$
  $\vee \bigvee_i (i \in \pi(f) \wedge \ulcorner \pi(s_i) \succ \pi(t_i) \urcorner \wedge \bigwedge_{j<i} (j \in \pi(f) \to \ulcorner \pi(s_j) = \pi(t_j) \urcorner) \wedge$
  $\bigwedge_{j>i} (j \in \pi(f) \to \ulcorner \pi(f(s_1, \ldots, s_n)) \succ \pi(t_j) \urcorner))$

Outline

Definition (Knuth–Bendix Order (KBO))

- let $p : \mathcal{F} \to \mathbb{N}$ be some precedence
- let $w_0 \in \mathbb{N} \setminus \{0\}$ and let $w : \mathcal{F} \to \mathbb{N}$ be some weight function
- define $w(t) = \begin{cases} w_0, & \text{if } t \text{ is a variable} \\ w(f) + w(t_1) + \ldots + w(t_n), & \text{if } t = f(t_1, \ldots, t_n) \end{cases}$
- KBO is a relation on terms $\succ_{KBO}$ ($\succ$ for short), defined by these inference rules

$$\frac{\mathcal{V}(s) \supseteq \mathcal{V}(t) \quad w(s) > w(t)}{s \succ t} \quad \text{(weight)}$$

$$\frac{\mathcal{V}(s) \supseteq \mathcal{V}(t) \quad w(s) \geq w(t)}{s = f(\ldots) \succ x = t} \quad \text{(variable)}$$

$$\frac{\mathcal{V}(s) \supseteq \mathcal{V}(t) \quad w(s) \geq w(t) \quad p(f) > p(g)}{s = f(\ldots) \succ g(\ldots) = t} \quad \text{(precedence)}$$

$$\frac{\mathcal{V}(s) \supseteq \mathcal{V}(t) \quad w(s) \geq w(t) \quad \forall j \in \{1, \ldots, i-1\}.\, s_j = t_j \quad s_i \succ t_i}{s = f(s_1, \ldots, s_n) \succ f(t_1, \ldots, t_n) = t} \quad \text{(lexicographic)}$$

Definition (Admissibility)

KBO parameters are admissible if

- for all constants f: $w(f) \geq w_0$, and
- for all unary symbols f: $w(f) = 0$ implies that $p(f) > p(g)$ for all $g \neq f$

Theorem

every KBO with admissible parameters is a reduction order

Task: given TRS, search for KBO parameters

- search for suitable rules is easier than LPO: no overlap
- problem: calculation and comparisons of weights requires arithmetic
- solution:
  - switch from SAT to SMT (SAT modulo theories)
  - required theory for KBO: linear arithmetic

SAT Modulo Theories (SMT)

- theory defines sorted first-order terms with standard semantics
- theory atom is term of Boolean sort
- SMT formulas are formulas as in propositional case, extended by theory atoms
- SMT solving: given SMT formula, determine whether there is a satisfying assignment
- such an assignment may contain theory variables and propositional variables

Example (Theories)

- LRA: linear real arithmetic; NRA: non-linear real arithmetic
  - domain: $\mathbb{R}$
  - arbitrary addition; in linear case multiplication only by constants
  - example formula: $a \wedge x + 2y > 5 \to \neg(2x - 7 = 5z) \vee z \geq \frac{14}{5} \vee \neg b$
  - a solution: $\alpha(a) = \alpha(b) = \mathsf{T}$, $\alpha(x) = 3$, $\alpha(y) = 2$, $\alpha(z) = -\frac{13}{5}$
- LIA: linear integer arithmetic; NIA: non-linear integer arithmetic
  - same as LRA and NRA, except that theory variables and constants are restricted to $\mathbb{Z}$

Definition (KBO encoding into LIA)

- encoding uses integer variables $p(f)$, $w(f)$, w_0 and $w(t)$
- constraints are now straight-forward for TRS $\mathcal{R}$ over signature $\mathcal{F}$
- $w(f) \geq 0$ for all $f \in \mathcal{F}$
- $w(f) \geq w_0$ for all constants f
- $w(f) = 0 \to \bigwedge_{g \neq f} p(f) > p(g)$ for all unary f
- $w(x) = w_0$ for all variables x in $\mathcal{R}$
- $w(f(t_1, \ldots, t_n)) = w(f) + w(t_1) + \cdots + w(t_n)$ for all subterms in $\mathcal{R}$
- $\ulcorner s > t \urcorner := \begin{cases} \bot, & \text{if } \mathcal{V}(s) \not\supseteq \mathcal{V}(t) \\ w(s) > w(t) \vee w(s) \geq w(t) \wedge \langle s > t \rangle, & \text{if } \mathcal{V}(s) \supseteq \mathcal{V}(t) \end{cases}$
- $\langle f(\ldots) > x \rangle := \top$
- $\langle x > t \rangle := \bot$
- $\langle f(\ldots) > g(\ldots) \rangle := p(f) > p(g)$, if $f \neq g$
- $\langle f(s_1, \ldots, s_n) > f(t_1, \ldots, t_n) \rangle := \begin{cases} \bot, & \text{if } (s_1, \ldots, s_n) = (t_1, \ldots, t_n) \\ \ulcorner s_i > t_i \urcorner, & \text{if } (s_1, \ldots, s_{i-1}) = (t_1, \ldots, t_{i-1}) \text{ and } s_i \neq t_i \end{cases}$

Exercise

Design a LIA encoding that searches for an argument filter π and KBO parameters; in particular you need an encoding for constraints of the form

$$\ulcorner \pi(s) \succ_{KBO} \pi(t) \urcorner$$

hints

1.
2.

Outline

Applying an SMT Solver

Two Alternatives

1. use any SMT solver and standard format
   - SMT-Lib is widely used format
     `https://smt-lib.org/language.shtml`
   - for successful SMT solvers, look at competition
     `https://smt-comp.github.io/2024/`
   - demos have been tested with Z3 (Microsoft Research)
     `https://github.com/Z3Prover/z3`
2. use some language-binding for your programming language and SMT solver
   - simple-smt: `https://hackage.haskell.org/package/simple-smt`
   - sbv: `https://hackage.haskell.org/package/sbv`
   - Z3-bindings: `https://github.com/Z3Prover/z3#z3-bindings`

SMT-LIB 2 Format – KBO for $\{plus(s(x),y) \to s(plus(x,y)); \quad plus(0,y) \to y\}$ (1/4)

```
(set-logic QF_LIA)                ; QF_LIA, QF_LRA, QF_NIA, QF_NRA; QF = quantifier free
(declare-fun w0 () Int )          ; declare theory variables (type: Int or Real)
                                  ; declare propositional variables (type: Bool)

(declare-fun wf_plus () Int )     ; weights for symbols
(declare-fun wf_s () Int )

(declare-fun pf_plus () Int )     ; precedences
(declare-fun pf_s () Int )

(declare-fun w_s__x () Int )      ; weights for terms, all variables = x
(declare-fun w_plus__s__x__x () Int )
(declare-fun w_plus__x__x () Int )
(declare-fun w_s__plus__x__x () Int )
```


SMT-LIB 2 Format – KBO for $\{plus(s(x),y) \to s(plus(x,y)); \quad plus(0,y) \to y\}$ (2/4)

```
; assertion are added using S-expressions
; predefined on Booleans:  and, or, not, true, false, => (implication), = (equivalence)
; predefined on numbers: numerals, +, *, -   (beware: (- 5), not: -5)
; predefined comparisons: =, <, >, >=, <=

(assert (and (<= 0 pf_plus ) (<= pf_plus 1 ) ) )  ; restrict precedence range
(assert (and (<= 0 pf_s ) (<= pf_s 1 ) ) )

(assert (> w0 0 ) )                               ; admissibility
(assert (>= wf_s 0 ) )
(assert (>= wf_plus 0 ) )
(assert (=> (= wf_s 0 ) (> pf_s pf_plus ) ) )

(assert (= w_s__x (+ wf_s w0 ) ) )                ; weight computation
(assert (= w_plus__s__x__x (+ wf_plus w_s__x w0 ) ) )
(assert (= w_plus__x__x (+ wf_plus w0 w0 ) ) )
(assert (= w_s__plus__x__x (+ wf_s w_plus__x__x ) ) )
```


SMT-LIB 2 Format – KBO for $\{plus(s(x),y) \to s(plus(x,y)); \quad plus(0,y) \to y\}$ (3/4)

```
; rule constraints
(assert (or (> w_plus__s__x__x w_s__plus__x__x )
            (and (>= w_plus__s__x__x w_s__plus__x__x ) (> pf_plus pf_s ) ) ) )
(assert true )

(check-sat )                      ; invoke solver

(get-value (w0 ) )                ; extract solution for variable w0
(get-value (wf_plus ) )           ; weight of plus
(get-value (pf_plus ) )           ; precedence of plus
(get-value (wf_s ) )              ; weight of s
(get-value (pf_s ) )              ; precedence of s
; precedence and weight of symbol 0 was not required -> set to defaults: 0 and w0
```

Remarks

- declarations and assertions can be mixed
- SMT-Lib also supports incremental invocations, cf. assertion stacks, push, pop

SMT-LIB 2 Format – KBO for $\{plus(s(x), y) \to s(plus(x, y)); \quad plus(0, y) \to y\}$ (4/4)

- store encoding in file, e.g., `kbo_plus_encoding.smt2`
- invoke SMT solver, e.g., z3

```
$ z3 kbo_plus_encoding.smt2

sat
((w0 1))
((wf_plus 0))
((pf_plus 2))
((wf_s 1))
((pf_s 1))
```


SMT-Solving via Language-Bindings

- have a look at `demos/miniTT/src/KBO.hs`
- there, `simple-smt` is used
- a bit more complex to use than SAT-MiniSat
  - variables need to be declared before usage
  - variable names must be valid
    (encoding must take care of forbidden characters and keywords in SMT, cf. `stringToSMT`, `wSym`, `pSym`, ...)

Design Choices

- which theory to select?

fragment	LRA	LIA	NRA	NIA
quantifier-free	NPC	NPC	$O(2^{2^n})$	undecidable
qf, only conjunctions	P	NPC	$O(2^{2^n})$	undecidable
allowing quantors	decidable	decidable	$O(2^{2^n})$	undecidable

  table indicates that LRA is theoretically the easiest (LRA is convex, LIA is not)
- choice should be backed by experiments
  - KBO can also be encoded in LRA instead of LIA (`miniTT/src/KBO2.hs`)
  - intuition: LRA should be faster than LIA
  - experiments: KBO-LRA is 2x slower than KBO-LIA on termination problem database
- another tradeoff: encoding time vs solving time

Outline

SMT Problem

decide satisfiability of formulas in

propositional logic + domain-specific background theories

Two Approaches

1. eager approach:

   translate formula into equisatisfiable propositional formula

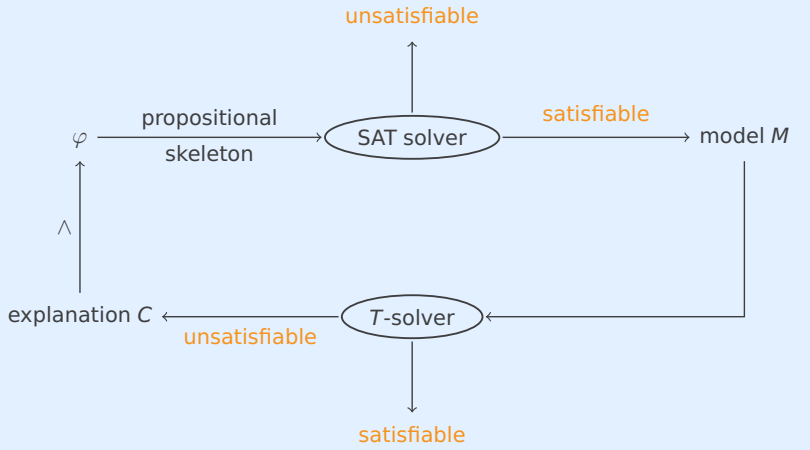
2. lazy approach:

   combine SAT solver with specialized solvers for background theories

Terminology

theory solver for T (T-solver) is procedure for deciding T-satisfiability of conjunction of quantifier-free literals

SMT Solving: Lazy Approach



Example

formula $\underset{a}{x = 1} \wedge (\neg\underset{b}{(y = 1)} \vee \neg\underset{c}{(x + 2y = 3)}) \wedge \underset{d}{x + y = 2}$ is unsatisfiable

- input to SAT solver (propositional skeleton)

$$a \wedge (\neg b \vee \neg c) \wedge d \wedge (\neg a \vee b \vee \neg d) \wedge (\neg a \vee \neg b \vee c)$$

blocking clause

- SAT solver reports unsatisfiable

$$a \wedge b \wedge \neg c \wedge d$$

- input to LIA solver

$$x = 1 \wedge y = 1 \wedge x + 2y \neq 3 \wedge x + y = 2$$

- LIA solver reports unsatisfiable

most state-of-the-art SMT solvers use DPLL(T) (including CDCL extensions)

general framework for lazy SMT solving with theory propagation

Definitions

first-order theory T, formulas F and G, list of literals M

- F is T-satisfiable if $F \wedge T$ is satisfiable
- $F \models_T G$ if $F \wedge \neg G$ is not T-satisfiable
- $F \equiv_T G$ if $F \models_T G$ and $G \models_T F$
- $M = l_1, \ldots, l_k$ is T-consistent if $l_1 \wedge \cdots \wedge l_k$ is T-satisfiable

Definition

DPLL(T) consists of DPLL rules unit propagate, decide, fail, restart and

- T-backjump $\quad M \overset{d}{l} N \parallel F, C \implies M\, l' \parallel F, C$

  if $M \overset{d}{l} N \models \neg C$ and $\exists$ clause $C' \vee l'$ such that
  - $F, C \models_T C' \vee l'$ and $M \models \neg C'$
  - l' is undefined in M and l' or l'^c occurs in F or in $M \overset{d}{l} N$
- T-learn $\quad M \parallel F \implies M \parallel F, C$

  if $F \models_T C$ and all atoms of C occur in M or F
- T-forget $\quad M \parallel F, C \implies M \parallel F$

  if $F \models_T C$
- T-propagate $\quad M \parallel F \implies M\, l \parallel F$

  if $M \models_T l$, l is undefined in M, and l or l^c occurs in F

Example

(EUF) formula $\quad \underset{1}{g(a) = c} \wedge (\neg(\underset{2}{f(g(a))} = f(c)) \vee \underset{3}{g(a) = d}) \wedge \neg(\underset{4}{c = d})$

	$\parallel$ 1, $\neg 2 \vee 3$, $\neg 4$		
$\implies$	1 $\parallel$ 1, $\neg 2 \vee 3$, $\neg 4$		unit propagate
$\implies$	1 $\neg 4$ $\parallel$ 1, $\neg 2 \vee 3$, $\neg 4$		unit propagate
$\implies$	1 $\neg 4$ $\overset{d}{\neg 2}$ $\parallel$ 1, $\neg 2 \vee 3$, $\neg 4$		decide
$\implies$	1 $\neg 4$ $\overset{d}{\neg 2}$ $\parallel$ 1, $\neg 2 \vee 3$, $\neg 4$, $\neg 1 \vee 2$		T-learn
$\implies$	1 $\neg 4$ 2 $\parallel$ 1, $\neg 2 \vee 3$, $\neg 4$, $\neg 1 \vee 2$		T-backjump
$\implies$	1 $\neg 4$ 2 3 $\parallel$ 1, $\neg 2 \vee 3$, $\neg 4$, $\neg 1 \vee 2$		unit propagate
$\implies$	1 $\neg 4$ 2 3 $\parallel$ 1, $\neg 2 \vee 3$, $\neg 4$, $\neg 1 \vee 2$, $\neg 1 \vee \neg 2 \vee \neg 3 \vee 4$		T-learn
$\implies$	fail-state		fail

Remark

lazy SMT approach is modeled in DPLL(T) as follows:

if state $M \parallel F$ is reached such that unit propagate, decide, fail, T-backjump are not applicable

check T-consistency of M with T-solver

1. if M is T-consistent then F is T-satisfiable
2. if M is not T-consistent then $\quad F \models_T \neg(l_1 \wedge \cdots \wedge l_k)$ for some literals $l_1, \ldots, l_k$ in M

   add blocking clause $\neg l_1 \vee \cdots \vee \neg l_k$ by T-learn and apply restart

Improvements

1. apply fail or T-backjump after T-learn (instead of restart)
2. check T-consistency of M or apply T-propagate before decide
3. find small unsatisfiable cores to minimize k in blocking clauses

Outline

Further Reading

- Harald Zankl, Nao Hirokawa, and Aart Middeldorp
  KBO Orientability
  Journal of Automated Reasoning 43(2), pp. 173 – 201, 2009
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  SAT Solving for Termination Proofs with Recursive Path Orders and Dependency Pairs
  Journal of Automated Reasoning, 49(1), pp. 53 – 93, 2012
- Robert Nieuwenhuis, Albert Oliveras, and Cesare Tinelli
  Solving SAT and SAT Modulo Theories: From an Abstract Davis–Putnam–Logemann–Loveland Procedure to DPLL(T)
  Journal of the ACM 53(6), pp. 937 – 977, 2006