Coherence in Non-Terminating Linear Polygraphs

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1 Introduction

In equational theories, coherence rules guarantee computational properties of the axioms of the theory. In computer science, logic and algebra, solving a coherence problem amounts to finding the coherence rules of a given theory. Several works were developed to study coherence problems by rewriting methods [7]. These methods use Squier's Theorem [12], which allows the construction of a coherent presentation of a monoid from a convergent rewriting system presenting this monoid. More precisely, a family of confluence diagrams associated to all critical pairs for a convergent presentation of a monoid generates the homotopical syzygies for this monoid. These syzygies express coherence relations between parallel reduction paths. The termination condition for the construction of coherent presentations has been weakened in [3, 4]. Squier's Theorem finds many applications in representation theory, like the study of Artin monoids [5] or plactic monoids [8].

The tools of linear rewriting can also be used to solve coherence problems for algebras or linear categories. Gröbner bases are an important example of such a tool, see [11]. Termination for Gröbner bases comes from monomial orders. These orders yield terminating sets of rules for algebras which can be completed into confluent ones by using Buchberger's procedure, making convergent presentations. Those convergent presentations can then be used to construct coherent presentations of algebras by choosing a family of confluence diagrams of the critical pairs. In [6], the condition of having a monomial order is weakened into termination without compatibility with a monomial order. This result corresponds to a linear version of Squier's Theorem. However convergence is not always easy to obtain, like in the case of Hecke algebras [9] or others important families of algebras. Thus, we want to weaken the convergence condition to keep only confluence and quasi-termination, a condition weaker than termination but sufficient to treat the case of Hecke algebras.

For example linear Squier's Theorem can be used to construct a coherent presentation for the non commutative algebra presented by the linear polygraph with one 0-cell, three 1-cells x, y and z, and one 2-cell $xyz \Rightarrow x^3 + y^3 + z^3$. This presentation yields a terminating and confluent rewriting system. Thus, by linear Squier's Theorem, we can construct a coherent presentation for this algebra. Now, consider the algebra $\mathbf{A}$ presented by the linear polygraph Σ with one 0-cell, two 1-cells x and y and one 2-cell $xy \Rightarrow x^2 + y^2$. The linear polygraph Σ is not terminating because of the rewriting sequence $x^2y \Rightarrow x^3 + xy^2 \Rightarrow x^3 + x^2y + y^3$. This prevents us from finding a coherent presentation of $\mathbf{A}$ with the given linear polygraph by using linear Squier's Theorem.

Some algebras, like Hecke algebras, do not admit any finite convergent presentation on a fixed set of generators, see [10]. To construct coherent presentations of those algebras, one needs to weaken the termination hypothesis of the linear Squier's Theorem into quasi-termination and choose orientations of rules which make confluence trivial at the cost of losing termination. This

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is in essence our main result 4.5. However, the presence of a linear structure makes the problem harder than in the set-wise setting.

In section 2, we recall the notions of linear polygraphs presented in [2]. We introduce multiple notions of decreasingness for linear polygraphs. We then present in section 3 a criterion of local confluence from quasi-termination and confluence of the critical branchings 3.1. In section 4, we recall from [1] the notion of coherent presentation for linear categories and introduce the different notions of loops for linear polygraphs. We then prove our main result 4.5. We illustrate this result by two examples. The first example is a case of Hecke algebra in which we orient a rule to obtain quasi-termination. The second example treats the case of $\mathbf{A} = \langle x, y \mid xy = x^2 + y^2 \rangle$.

2 Linear labelled two-dimensional polygraphs

2.1. Linear $(2,2)$-polygraphs and rewriting. Let R be a commutative ring. An R-*linear* $(1,1)$-*category*, or linear $(1,1)$-category if not ambiguous, is a 1-category enriched in R-modules. A *(R-)linear* $(2,2)$-*polygraph* is a triple $\Sigma = (\Sigma_0, \Sigma_1, \Sigma_2)$, where (Σ_0, Σ_1) is a 1-polygraph and Σ_2 is a globular extension of the free R-linear category Σ_1^ℓ over Σ_1. The (R-)linear $(2,2)$-polygraph Σ is called *left-monomial* if all elements of Σ_2 have a source in Σ_1^*, for such a polygraph *monomial* is a 1-cell of Σ_1^*. We will respectively use the terms *linear category* and *linear polygraph* in place of linear $(1,1)$-category and linear $(2,2)$-polygraph. From now on, we assume that all linear polygraphs are left-monomial.

Let Σ be a linear polygraph. A *rewriting step* of Σ is a 2-cell of the form $w + \lambda u\varphi v$ where u and v are monomials, φ is a 2-cell of Σ_2, w is a linear combination of monomials which does not contain $us_1(\varphi)v$ and λ is a non zero scalar. We say that Σ is *quasi-terminating* if any infinite rewriting path of Σ contains infinitely many occurrences of a same 1-cell.

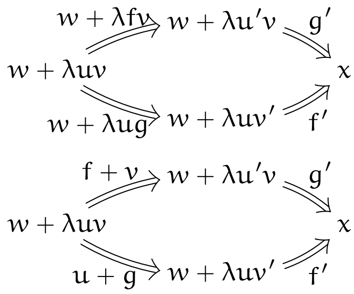
A linear polygraph Σ is *(scalar) exponentiation free* if no monomial m of Σ^ℓ can be rewritten into $\lambda m + f$ for some scalar λ other than 0 or 1 and some non zero 1-cell f which does not contain m in its monomial decomposition. Note that if Σ is quasi-terminating, exponentiation freedom is equivalent to the the fact that for every monomial m rewriting into a 1-cell f containing m in its monomial decomposition, we have $f = m$.

2.2. Labelled linear polygraphs. A *labelled linear polygraph* is the data of a linear polygraph Σ, a set W and a map ψ from $\Sigma_{\mathtt{stp}}$ to W. We say that ψ is *whisker-compatible* if for any rewriting steps f and g such that $\psi(f) \prec \psi(g)$, we have $\psi(u_1 f u_2 + v) \prec \psi(u_1 g u_2 + v)$ for any composable 1-cells u_1 and u_2 in Σ_1^ℓ and any 1-cell v of Σ_1^ℓ such that $u_1 f u_2 + v$ and $u_1 g u_2 + v$ are rewriting steps.

A *well-founded labelled linear polygraph* is a data $(\Sigma, W, \prec, \psi)$ made of a linear 2-polygraph Σ, a set W, a well-founded order $\prec$ on W and a map $\psi : \Sigma_{\mathtt{stp}} \longrightarrow W$. The map ψ is called a *well-founded labelling* of Σ and associates to a rewriting step f a *label* $\psi(f)$. Given a rewriting sequence f, we denote by $L^W(f)$ the set of labels of rewriting steps in f. Note that two distinct rewriting sequences f and g can correspond to a same 2-cell in Σ_2^ℓ despite $L^W(f)$ and $L^W(g)$ being distinct. We say a linear polygraph is *strictly decreasing* if there exists a well-founded labelling $(W, \prec, \psi)$ of Σ such that all local branchings (f, g) of Σ can be completed into a confluence diagram $(f \cdot f', g \cdot g')$ such that all elements of $L^W(f')$ are lower than $\psi(f)$ and all elements of $L^W(g')$ are lower than $\psi(g)$. Note that strict decreasingness is a particular case of the decreasingness property defined in [13].

2.3. Local branchings of a linear polygraph. An aspherical branching of Σ is a local branching (f, g) such that $f = g$. A *Peiffer branching* of Σ is a local branching $(w+\lambda f v, w+\lambda u g)$ with 1-source uv where u, v are monomials,f, g are in Σ_{stp}, λ is a non zero scalar and w is a linear combination of monomials which does not contain uv. An *additive branching* of Σ is a local branching $(f + v, u + g)$ with 1-source $u + v$ where u and v do not have any common monomial in their decomposition and f, g are in Σ_{stp}. An *overlapping branching* of Σ is a local branching that is not aspherical, Peiffer or additive. An overlapping branching is called a *critical branching* if its source is a minimal monomial.

Let Σ be a linear polygraph and let Γ be a globular extension of the linear $(2, 1)$-category Σ^{ℓ}. The linear polygraph Σ is *(strictly) Peiffer decreasing with respect to* Γ if there exists a well-founded labelling $(W, \prec, \psi)$ such that Σ is decreasing with respect to the labelling $(W, \prec, \psi)$, for any Peiffer branching $(w + \lambda f v, w + \lambda u g)$, there exists a (strictly) decreasing confluence diagram such that$(w + \lambda f v) \star_1 f' \equiv_\Gamma (w + \lambda u g) \star_1 g'$ and for any additive branching $(f+v, u+g)$, there exists a (strictly) decreasing confluence diagram such that $(f + v) \star_1 f' \equiv_\Gamma (u + g) \star_1 g'$.



3 Decreasingness and confluence

A decreasing linear polygraph is confluent. This result has the same proof than the abstract rewriting result of [13]. Newman's lemma is also the same as the one for 2-polygraphs. However, the critical pair lemma is different than the one for 2-polygraphs as proved in [6].

3.1. Lemma. *Let Σ be a quasi-terminating exponentiation free linear polygraph. The two following properties hold:*

– *all additive branchings of Σ are confluent,*

– *for any 1-cell u of Σ_1^{ℓ}, if all critical branchings of Σ are confluent and Σ is locally confluent at every 1-cell v such that $v \prec u$, then, Σ is locally confluent at u.*

This Lemma can be proved using the same proof sketch that [6] by checking all families of local branchings. Aspherical branchings being always confluent, we prove that all additive and Peiffer branchings of Σ are confluent. We then prove that all overlapping branchings are confluent by using the confluence of critical branchings.

In this way, we prove the following Theorem by well-founded induction on the order $\prec$.

3.2. Theorem. *Let Σ be a quasi-terminating exponentiation free linear polygraph. Then Σ is locally confluent if and only if all its critical branchings are confluent.*

3.3. A counterexample without exponentiation freedom. Exponentiation freedom is a necessary condition to Lemma 3.1. Let us consider for example the algebra presented by the linear polygraph with generating 1-cells x, y, z, t and rules $xz \Rightarrow xy$, $yt \Rightarrow \lambda zt$. This linear polygraph is not exponentiation free if $\lambda \notin \{0, 1\}$. For example, if $\lambda = -1$, we have the infinite rewriting sequence $xyt \Rightarrow -xzt \Rightarrow -xyt \Rightarrow xzt \Rightarrow \ldots$ or, if $\lambda = 2$ in a field of characteristic 0, we do not even have quasi-termination because of the infinite rewriting sequence $xyt \Rightarrow 2xzt \Rightarrow 2xyt \Rightarrow 4xzt \Rightarrow \ldots$. In this linear polygraph, the additive branching of source $xyt + xzt$ is not confluent, see [6].

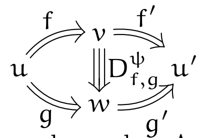
4 Coherence by decreasingness in linear categories

4.1. Coherent presentation of a linear category. An R-*linear* $(2,2)$-*category*, or linear $(2,2)$-category if not ambiguous, is a 1-category enriched in R-linear categories. A *(*R-*)linear polygraph* is a triple $\Sigma = (\Sigma_0, \Sigma_1, \Sigma_2, \Sigma_3)$, where $(\Sigma_0, \Sigma_1, \Sigma_2)$ is a linear polygraph and Σ_3 is a globular extension of the free R-linear $(2,2)$-category Σ_2^ℓ over Σ_2. A *coherent presentation* of a linear category $\mathcal{C}$ is a linear $(3,2)$-polygraph Σ such that the linear polygraph $(\Sigma_0, \Sigma_1, \Sigma_2)$ presents $\mathcal{C}$ and the quotient of the free $(2,2)$-linear category over $(\Sigma_0, \Sigma_1, \Sigma_2)$ by Σ_3 is an aspherical $(2,1)$-module.

4.2. Loops. A 2-*loop* in the linear $(2,2)$-category Σ_2^ℓ is a 2-cell f of Σ_2^ℓ such that $s_1(f) = t_1(f)$. Two 2-loops f and g in Σ_2^ℓ are *equivalent* if there exist a decomposition $f = f_1 \star_1 \ldots \star_1 f_p$, where f_i is a rewriting step of Σ for any $1 \leqslant i \leqslant p$, and a circular permutation σ such that $g = f_{\sigma(1)} \star_1 \ldots \star_1 f_{\sigma(p)}$. This defines an equivalence relation on 2-loops of Σ_2^ℓ. We denote by $\mathcal{L}(f)$ the equivalence class of a 2-loop f in Σ_2^ℓ for this relation. A 2-loop f in Σ_2^ℓ is *minimal with respect to* 1-*composition*, if any decomposition $f = g \star_1 h \star_1 k$ in Σ_2^ℓ with h a 2-loop implies that h is either an identity or equal to f. A 2-loop f in Σ_2^ℓ is *minimal by context*, if there is no decomposition $f = ugv + h$, where u and v are nonidentity 1-cells in Σ_1^*, g is a loop in Σ_2^ℓ and h is a nonzero loop. A 2-loop f in Σ_2^ℓ is *elementary* if it is minimal both with respect to 1-composition and by context.

4.3. Generating decreasing confluences. We denote by $\mathcal{E}(\Sigma)$ the set of equivalence classes of elementary 2-loops of Σ_2^ℓ. A *loop extension* of Σ is a globular extension of the $(2,1)$-category $\Sigma_2^\top$ made of a family of 3-cells $A_\alpha : \alpha \Rrightarrow 1_{s_1(\alpha)}$ indexed by exactly one α for each equivalence class in $\mathcal{E}(\Sigma)$.

Let (Σ, ψ) be a decreasing linear polygraph. A *family of generating decreasing confluences of* Σ *with respect to* ψ is a globular extension of the Σ_2^ℓ that contains, for every critical branching $(f, g) : u \Rightarrow (v, w)$ of Σ, exactly one 3-cell $D_{f,g}^\psi$ of the following form and where the confluence diagram $(f \cdot f', g \cdot g')$ is decreasing with respect to ψ.



4.4. Squier's decreasing completion. Let (Σ, ψ) be a decreasing linear polygraph. A *Squier's decreasing completion of* Σ *with respect to* ψ is a linear $(3,2)$-polygraph that extends the 2-polygraph Σ by a globular extension $\mathcal{O}(\Sigma, \psi) \cup \mathcal{L}(\Sigma)$ where $\mathcal{O}(\Sigma, \psi)$ is a chosen family of generating decreasing confluences with respect to ψ and $\mathcal{L}(\Sigma)$ is a loop extension of Σ defined in [4, 2.2.5.]. If (Σ, ψ) is a strictly decreasing 2-polygraph, a *strictly decreasing Squier's completion* is a Squier's decreasing completion, whose the generating decreasing confluences are required strict.

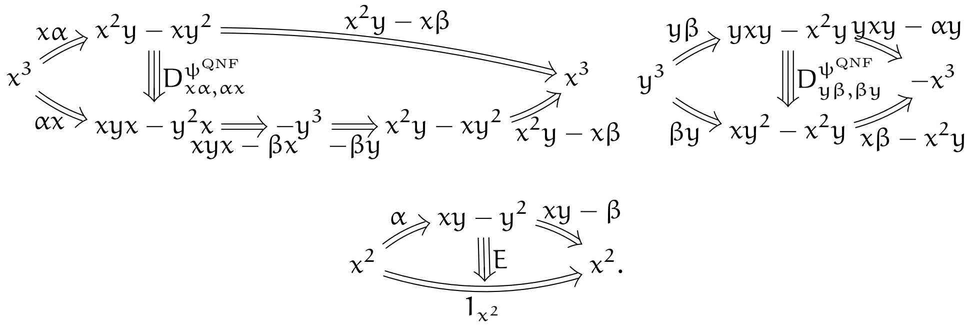
4.5. Theorem. *Let* Σ *be a quasi-terminating exponentiation free linear polygraph. Let* $\mathcal{D}(\Sigma, \psi)$ *be a strictly decreasing Squier's completion of* Σ *with respect to a labelling to the quasi-normal form (or QNF labelling)* $(W, \psi, \prec)$*. If* ψ *is whisker compatible, and* ψ *is strictly Peiffer decreasing with respect to* $\mathcal{D}(\Sigma, \psi)$*, then the strictly decreasing Squier's completion* $\mathcal{D}(\Sigma, \psi)$ *is a coherent presentation of the linear category presented by* Σ*.*

4.6. Examples. **1)** The Hecke algebra of the monoid $\mathbf{B}_3^+$ is the $\mathbb{Z}[q, q^{-1}]$-algebra generated by two elements s and t subject to the relations $sts = tst$, $s^2 = (1-q)s$, $t^2 = (1-q)t$. We consider the presentation of this algebra by the following $\mathbb{Z}[q, q^{-1}]$-linear polygraph:

$$\Lambda(\mathbf{B}_3^+) = \left\langle s, t \mid sts \Rightarrow tst,\ tst \Rightarrow sts,\ s^2 \Rightarrow (1-q)s,\ t^2 \Rightarrow (1-q)t \right\rangle.$$

The linear polygraph $\Lambda(\mathbf{B}_3^+)$ is quasi-terminating. Let ψ^{QNF} be the QNF labelling of $\Lambda(\mathbf{B}_3^+)$ defined by choosing the quasi-normal forms as the linear combination of monomials of the form $q^n(sts)^N\nu$ where $(sts)^N\nu$ is a word not containing the letter q and defined as in [4, Example 2.4.10.]. This labelling is whisker compatible and strictly Peiffer decreasing with respect to the decreasing Squier's completion $\mathcal{D}(\Sigma, \psi^{\mathrm{QNF}})$ associated to ψ^{QNF}. This globular extension contains one 3-cell for each critical branching with the sources $stst$, $tsts$, $ststs$, $tstst$, $ssts$, $stss$, $ttst$, $tstt$, sss, ttt, and one 3-cell associated to the only elementary 2-loop up to equivalence $sts \Rightarrow tst \Rightarrow sts$.

2) Let $\mathbb{K}$ be a field and $\mathbf{A}$ be the $\mathbb{K}$-algebra generated by two elements x and y subject to the relation $xy = x^2 + y^2$. The $\mathbb{K}$-linear polygraph $\Lambda = \left\langle x, y \mid \alpha : x^2 \Rightarrow xy - y^2,\ \beta : y^2 \Rightarrow xy - x^2 \right\rangle$. is a confluent and quasi-terminating presentation of $\mathbf{A}$. The linear polygraph Λ has only one elementary 2-loop up to equivalence and two critical branchings. By choosing the QNF labelling on Λ induced by the deglex order $y > x$, we get the following Squier's decreasing completion for Λ



By Theorem 4.5, this completion yields a coherent presentation of the algebra $\mathbf{A}$.

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