A Semantic Criterion for Proving Infeasibility in Conditional Rewriting

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1 Introduction

In the literature about confluence and termination of (conditional) rewriting, a number of important issues can be investigated as feasibility tests specified as sequences $s_1 \rightarrow^* t_1, \ldots, s_n \rightarrow^* t_n$ of reachability conditions $s_i \rightarrow^* t_i$ where (in contrast to the usual reachability problems) the instantiation of variables in terms $s_i$ and $t_i$ is allowed for all $1 \leq i \leq n$. Given a Conditional Term Rewriting System (CTRS) $R$, a sequence $s_1 \rightarrow^* t_1, \ldots, s_n \rightarrow^* t_n$ is $R$-feasible if there is a substitution $\sigma$ such that for all $i$, $1 \leq i \leq n$, $\sigma(s_i) \rightarrow^*_R \sigma(t_i)$. In the realm of oriented CTRSs, with rules of the form $\ell \rightarrow r \leftarrow c$ (with $c$ usually written $s_1 \rightarrow t_1, \ldots, s_n \rightarrow t_n$ when the usual rewriting semantics for the evaluation of the conditions $s_i \rightarrow t_i$ is assumed), the negation of this property, i.e., the $R$-infeasibility of a sequence $s_1 \rightarrow^* t_1, \ldots, s_n \rightarrow^* t_n$ as above, can be used to (i) disable the use of conditional rules $\ell \rightarrow r \leftarrow c$ from $R$ in reductions, (ii) discard conditional critical pairs $s \downarrow t \leftarrow c$ in the analysis of confluence of CTRSs [11, 12, 13, 14], (iii) discard conditional dependency pairs $u \rightarrow v \leftarrow c$ in the analysis of operational termination of CTRSs [9], or (iv) prove the non-joinability of terms $s$ and $t$, possibly coming from conditional or unconditional critical pairs, in (C)TRSs: the $R$-infeasibility of $s \rightarrow^* x, t \rightarrow^* x$ (with $x$ not occurring in $s$ or $t$) implies the non-joinability of $s$ and $t$.

In this paper we apply the semantic approach to the analysis of properties of CTRSs in [5] to prove infeasibility: an $R$-infeasibility problem is translated into the problem of finding a model $A$ of the set of sentences $\overline{R}$ representing the operational semantics of the CTRS $R$ plus a sentence $\neg(\exists \vec{x}) (s_1 \rightarrow^* t_1 \land \cdots \land s_n \rightarrow^* t_n)$ where $\rightarrow^*$ is viewed now as a predicate symbol in the underlying first-order language and can be freely interpreted in a first-order structure $A$ [4]. If $A \models \overline{R} \cup \{\neg(\exists \vec{x}) (s_1 \rightarrow^* t_1 \land \cdots \land s_n \rightarrow^* t_n)\}$ holds in the usual logical sense, i.e., the structure $A$ satisfies all sentences in the right-hand side of `$\models$’, then $s_1 \rightarrow^* t_1, \ldots, s_n \rightarrow^* t_n$ is $R$-infeasible. In the examples discussed in this paper, such first-order structures have been synthesized using AGES, a tool for the automatic generation of models of first-order theories [3] which implements the methods described in [6]. Besides, we also introduce a number of auxiliary results which can be combined with this and other approaches in proofs of infeasibility, and hence in any application to prove confluence and operational termination of CTRSs.

2 Infeasibility problems

Borrowing [11, Definition 7.1.8(3)] for feasibility of Conditional Critical Pairs (CCPs, see [11, Definition 7.1.8(1)]), we introduce the following.

Definition 1. Let $R$ be a CTRS. A sequence $s_1 \rightarrow^* t_1, \ldots, s_n \rightarrow^* t_n$, where $s_i$ and $t_i$ are terms for all $1 \leq i \leq n$ is called a feasibility sequence. It is called $R$-feasible if there is a substitution $\sigma$ such that for all $1 \leq i \leq n$, $\sigma(s_i) \rightarrow^*_R \sigma(t_i)$. Otherwise, it is called $R$-infeasible.

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For instance, (Repl) actually establishes that, for every rule $A$,

$$f(x_1, \ldots, x_i, \ldots, x_k) \to f(x_1, \ldots, y_i, \ldots, x_k)$$

for all $f \in \mathcal{F}$ and $1 \leq i \leq k = \text{arity}(f)$.

Following [7], in Definition 1 we write $s \rightarrow^{*}_R t$ or $s \rightarrow_R t$ for terms $s$ and $t$ iff there is a proof tree for $s \rightarrow t$ (resp. $s \rightarrow t$) using $\mathcal{R}$ in the inference system of Figure 1.

**Remark 2.** All rules in the inference system in Figure 1 are schematic in that each inference rule $B_1, \ldots, B_n$ can be used under any instance $\sigma(B_1) \cdots \sigma(B_n)$ of the rule by a substitution $\sigma$.

For instance, (Repl) actually establishes that, for every rule $\ell \rightarrow r \leftarrow s_1 \rightarrow t_1, \ldots, s_n \rightarrow t_n$ in the CTRS $\mathcal{R}$, every instance $\sigma(\ell)$ by a substitution $\sigma$ rewrites into $\sigma(r)$ provided that, for each $s_i \rightarrow t_i$, with $1 \leq i \leq n$, the reachability condition $\sigma(s_i) \rightarrow^* \sigma(t_i)$ can be proved.

Now, we can say (as usual) that a critical pair $s \downarrow t \leftarrow c$ or a rule $\ell \rightarrow r \leftarrow c$, where $c$ is $s_1 \rightarrow t_1, \ldots, s_n \rightarrow t_n$ is $\mathcal{R}$-infeasible if $s_1 \rightarrow^* t_1, \ldots, s_n \rightarrow^* t_n$ is $\mathcal{R}$-infeasible. We can also see the usual joinability problem $s \downarrow t$, i.e., the existence of a term $u$ such that $s \rightarrow^*_R u$ and $t \rightarrow^*_R u$, as an specific feasibility sequence.

**Proposition 3.** Let $\mathcal{R}$ be a CTRS, $s, t$ be terms, and $x$ be a fresh variable not occurring in $s$ or $t$. If $s$ and $t$ are joinable, then $s \rightarrow^* x, t \rightarrow^* x$ is $\mathcal{R}$-feasible. If $s$ and $t$ are ground, then the $\mathcal{R}$-feasibility of $s \rightarrow^* x, t \rightarrow^* x$ implies joinability of $s$ and $t$.

An immediate consequence of the previous observation is that the $\mathcal{R}$-infeasibility of $s \rightarrow^* x, t \rightarrow^* x$ implies the non-joinability of $s$ and $t$. Aoto has investigated methods for proving non-joinability of ground terms in TRSs [1]. Actually, Aoto makes the following interesting observation: for Term Rewriting Systems $\mathcal{R}$, the so-called usable rules for reachability $U(\mathcal{R}, s)$ associated to a term $s$ (roughly speaking an overapproximation of the set of rules that can be applied to $s$ and then to any other term introduced by the application of a rule, see [1, Definition 3]) are the only ones we need to consider in any rewriting sequence starting from $s$, i.e., $s \rightarrow^*_R t$ iff $s \rightarrow^*_U(\mathcal{R}, s) t$ (see [1, Lemma 4]). Actually, this holds as well for the usable rules $U(\mathcal{R}, t)$ for CTRSs $\mathcal{R}$ and terms $s$ introduced in [10, Definition 11]. Thus, we have the following easy corollary of this fact, where, given terms $s_1, \ldots, s_n$, we let $U(\mathcal{R}, s_1, \ldots, s_n) = \bigcup_{i=1}^n U(\mathcal{R}, s_i)$.

**Proposition 4.** Let $\mathcal{R}$ be a CTRS and $s_1 \rightarrow^* t_1, \ldots, s_n \rightarrow^* t_n$ be a feasibility sequence. If the sequence is $U(\mathcal{R}, s_1, \ldots, s_n)$-feasible, then it is $\mathcal{R}$-feasible. If terms $s_i$ are ground for all $1 \leq i \leq n$ and the sequence is $\mathcal{R}$-feasible, then it is $U(\mathcal{R}, s_1, \ldots, s_n)$-feasible.

Thus, we can prove $\mathcal{R}$-infeasibility of critical pairs and rules by proving the $U(\mathcal{R}, s_1, \ldots, s_n)$-infeasibility of the corresponding feasibility sequence $s_1 \rightarrow^* t_1, \ldots, s_n \rightarrow^* t_n$, provided that terms $s_i$ are ground for all $1 \leq i \leq n$.
\[
(\forall x) \ x \to^* x \tag{4}
\]
\[
(\forall x, y, z) \ x \to y \land y \to^* z \Rightarrow x \to^* z \tag{5}
\]
\[
a \to b \tag{8}
\]
\[
f(a) \to b \tag{9}
\]
\[
(\forall x, y) \ x \to y \Rightarrow f(x) \to f(y) \tag{6}
\]
\[
(\forall x) \ f(x) \to^* x \Rightarrow g(x) \to g(a) \tag{10}
\]

Figure 2: First-order theory for \( \mathcal{R} \) in Example 5

3 A semantic criterion for infeasibility

In the logic of CTRSs, with binary predicates \( \to \) and \( \to^* \), the (first-order) theory \( \overline{\mathcal{R}} \) for a CTRS \( \mathcal{R} = (\mathcal{F}, \mathcal{R}) \) is obtained from the inference rules in Figure 1 by specializing (Cong)\(_{f,i} \) for each \( f \in \mathcal{F} \) and \( 1 \leq i \leq ar(f) \) and (Repl)\(_{r} \) for all \( r : \ell \to r \Leftarrow c \in \mathcal{R} \). Inference rules \( \frac{a_i}{a} \Rightarrow b \) become universally quantified implications \( B_1 \land \cdots \land B_n \Rightarrow A \) [8, Section 2].

Example 5. Consider the following CTRS \( \mathcal{R} \) [2, page 46]:

\[
a \to b \tag{1}
\]
\[
g(x) \rightarrow g(a) \Leftarrow f(x) \rightarrow x \tag{3}
\]
\[
f(a) \rightarrow b \tag{2}
\]

Figure 2 shows its associated theory \( \overline{\mathcal{R}} \).

By a structure \( \mathcal{A} \) for a first-order logic language we mean an interpretation of the function and predicate symbols of the language \( (f, g, \ldots \) and \( P, Q, \ldots, \) respectively) as mappings \( f^\mathcal{A}, g^\mathcal{A}, \ldots \) and relations \( P^\mathcal{A}, Q^\mathcal{A}, \ldots \) on a given set (carrier) also denoted \( \mathcal{A} \). Then, the usual interpretation of first-order formulas with respect to the structure is considered. A model for a set \( \mathcal{S} \) of first-order sentences (i.e., formulas whose variables are all quantified) is just a structure that makes them all true, written \( \mathcal{A} \models \mathcal{S} \) see [4].

Proposition 6 (Semantic criterion for infeasibility). Let \( \mathcal{R} \) be a CTRS, \( s_1 \to^* t_1, \ldots, s_n \to^* t_n \) a feasibility sequence, and \( \mathcal{A} \) be a structure with nonempty domain. If \( \mathcal{A} \models \overline{\mathcal{R}} \cup \{ \neg (\exists \overline{x}) (s_1 \to^* t_1 \land \cdots \land s_n \to^* t_n) \} \), then the sequence is \( \mathcal{R} \)-infeasible.

Proof. By contradiction. Assume that the sequence is \( \mathcal{R} \)-feasible. Then, there is a substitution \( \sigma \) such that, for all \( 1 \leq i \leq n \), \( \sigma(s_i) \to^* \overline{\mathcal{R}} \sigma(t_i) \). Since \( \mathcal{A} \) is a model of \( \overline{\mathcal{R}} \), by correctness we have that, for all \( 1 \leq i \leq n \), \( \mathcal{A} \models (\forall \overline{y}_i) (\sigma(s_i) \to^* \sigma(t_i)) \), where \( \overline{y}_i \) are the variables in \( \text{Var}(\sigma(s_i)) \cup \text{Var}(\sigma(t_i)) \). Therefore,

\[
\mathcal{A} \models (\forall \overline{y}) (\sigma(s_1) \to^* \sigma(t_1) \land \cdots \land \sigma(s_n) \to^* \sigma(t_n)) \tag{11}
\]

with \( \overline{y} \) the variables in \( \bigcup_{i=1}^{n} \text{Var}(\sigma(s_i)) \cup \text{Var}(\sigma(t_i)) \). Hence, for all valuations \( \nu : \overline{y} \to \mathcal{A} \), the interpretation of the universally quantified formula in (11) in the structure \( \mathcal{A} \), i.e., \( \nu(\sigma(s_1) \to^* \sigma(t_1) \land \cdots \land \sigma(s_n) \to^* \sigma(t_n))_\mathcal{A}^\nu \), is true. Let \( \overline{x} \) be the variables in \( \bigcup_{i=1}^{n} \text{Var}(s_i) \cup \text{Var}(t_i) \). Since \( \mathcal{A} \) has a nonempty domain, given an arbitrary valuation \( \nu' : \overline{y} \to \mathcal{A} \), there is a valuation \( \nu'(x) = [\sigma(x)]_\mathcal{A}^\nu \) for all variable \( x \) in \( \overline{x} \), such that \( [s_1 \to^* t_1 \land \cdots \land s_n \to^* t_n]_\mathcal{A}^\nu \) is true. This contradicts our assumption \( \mathcal{A} \models \neg (\exists \overline{x}) (s_1 \to^* t_1 \land \cdots \land s_n \to^* t_n) \).

In the following, we assume that the signature \( \mathcal{F} \) of any CTRS to be used with Proposition 6 contains at least a (dummy) constant symbol so that any associated structure \( \mathcal{A} \) has a nonempty domain. Models to be used in Proposition 6 for infeasibility checking can be automatically
generated from the CTRS \( \mathcal{R} \) and sentence at stake, i.e., \( \neg(\exists x) (s_1 \rightarrow^* t_1 \land \cdots \land s_n \rightarrow^* t_n) \), by using (as we do in the following examples) a tool like AGES [3].

**Example 7** (Infeasible rule). The following structure \( \mathcal{A} \) over \( \mathbb{N} - \{0\} \):

\[
\begin{align*}
a^4 &= 1 & b^4 &= 2 & f^4(x) &= x + 1 & g^4(x) &= 1 \\
x \rightarrow^A y &\iff y \geq x & x (\rightarrow^*)^A y &\iff y \geq x
\end{align*}
\]

is a model of \( \overline{\mathcal{R}} \cup \{\neg(\exists x) f(x) \rightarrow^* x\} \) for \( \mathcal{R} \) in Example 5 (see \( \overline{\mathcal{R}} \) in Figure 2).

Thus, rule (3) is proved \( \mathcal{R} \)-infeasible. Using this observation it is not difficult to see that \( \mathcal{R} \) is operationally terminating, i.e., no term \( t \) has an infinite proof tree using the inference system in Figure 1 [7].

**Example 8** (Infeasible critical pair). The following CTRS [14, Example 23]

\[
\begin{align*}
g(x) &\rightarrow f(x,x) & (12) \\
g(x) &\rightarrow g(x) \leftarrow g(x) \rightarrow f(a,b) & (13)
\end{align*}
\]

has a conditional critical pair \( f(x,x) \downarrow g(x) \leftarrow g(x) \rightarrow f(a,b) \). The following structure \( \mathcal{A} \) over the finite domain \( \{0, 1\} \):

\[
\begin{align*}
a^4 &= 1 & b^4 &= c^4 &= 0 & f^4(x,y) &= \begin{cases} x - y + 1 & \text{if } x \geq y \\ y - x + 1 & \text{otherwise} \end{cases} \\
g^4(x) &= 1 & x \rightarrow^A y &\iff x = y & x (\rightarrow^*)^A y &\iff x \geq y
\end{align*}
\]

is a model \( \overline{\mathcal{R}} \cup \{\neg(\exists x) g(x) \rightarrow^* f(a,b)\} \). Thus, the critical pair is infeasible. In [14, Example 23] this is proved by using unification tests together with a transformation. It is discussed that the alternative tree automata techniques investigated in the paper do not work for this example.

**Example 9** (Infeasible rule). Consider the following CTRS [14, Example 17]

\[
\begin{align*}
h(x) &\rightarrow a & (14) \\
g(x) &\rightarrow a \leftarrow h(x) \rightarrow b & (16) \\
g(x) &\rightarrow x & (15) \\
c &\rightarrow c & (17)
\end{align*}
\]

The following structure \( \mathcal{A} \) over \( \mathbb{N} \):

\[
\begin{align*}
a^4 &= 0 & b^4 &= c^4 &= 1 & f^4(x) &= x + 2 & h^4(x) &= 0 \\
x \rightarrow^A y &\iff x \geq y & x (\rightarrow^*)^A y &\iff x \geq y
\end{align*}
\]

is a model \( \overline{\mathcal{R}} \cup \{\neg(\exists x) h(x) \rightarrow^* b\} \). Therefore, rule (16) is proved \( \mathcal{R} \)-infeasible. In [14, Example 17] this is proved by using tree automata techniques. It is also shown that the alternative technique investigated in the paper (the use of unification tests) does not work in this case.

**Example 10** (Non-joinable terms). Consider the following CTRS [11, Example 7.3.3]:

\[
\begin{align*}
a &\rightarrow b & (18) \\
f(x) &\rightarrow c \leftarrow x \rightarrow a & (19)
\end{align*}
\]

Although there is no critical pair, the system is not (locally) confluent because \( f(a) \rightarrow^R f(b) \) and \( f(a) \rightarrow^R c \) but \( c \) and \( f(b) \) are not joinable. The following structure \( \mathcal{A} \) over \( \mathbb{N} \cup \{-1\} \):

\[
\begin{align*}
a^4 &= 0 & b^4 &= -1 & c^4 &= 1 & f^4(x) &= x + 1 \\
x \rightarrow^A y &\iff x = y & x (\rightarrow^*)^A y &\iff x = y
\end{align*}
\]

is a model of \( \overline{\mathcal{U}(\mathcal{R}, f(b), c)} \cup \{\neg(\exists x) (f(b) \rightarrow^* x \land c \rightarrow^* x)\} \), where \( \mathcal{U}(\mathcal{R}, f(b), c) = \{\text{(19)}\} \). Therefore, \( f(b) \) and \( c \) are proved not joinable.

\footnote{In AGES the universally quantified version \( (\forall x) (s_1 \rightarrow^* t_1 \land \cdots \land s_n \rightarrow^* t_n) \) of \( (\exists x) (s_1 \rightarrow^* t_1 \land \cdots \land s_n \rightarrow^* t_n) \) is used; actually, only \( (s_1 \rightarrow^* t_1 \land \cdots \land s_n \rightarrow^* t_n) \) is introduced (universal quantification is assumed).}
4 Conclusions

We have presented a semantic approach to prove infeasibility in conditional rewriting. Interestingly, with such a semantic approach we could handle many examples coming from papers developing different specific techniques to deal with these problems. In particular, we could deal with all examples solved in [13, 14] (some of them reported in our examples above; note that these papers explore several alternative methods and, as reported by the authors, some of them fail in specific examples which require a different approach). We also dealt with all Aoto’s examples in [1] in combination with his usable rules refinement (see Proposition 4).

We do not have a dedicated, fully automated ‘infeasibility’ checker yet. Instead we just encode the problem we want to deal with (e.g., infeasibility of a critical pair, or rule; or non-joinability) as an specific infeasibility sequence and then use AGES to apply Proposition 6. The generation of a model is completely automatic, but in order to succeed on a given problem, we often require to manually change special settings like the generation of piecewise interpretations (like in Example 8), usually ‘expensive’ and not part of the default configuration. Thus, further investigation developing heuristics for an efficient use of the technique, possibly in combination with other existing techniques, would be necessary.

References