A Semantic Criterion for Proving Infeasibility in Conditional Rewriting *

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1 Introduction

In the literature about *confluence* and *termination* of (conditional) rewriting, a number of important issues can be investigated as *feasibility tests* specified as sequences $s_1 \to^* t_1, \ldots, s_n \to^* t_n$ of reachability conditions $s_i \to^* t_i$ where (in contrast to the usual reachability problems) the *instantiation* of variables in terms s_i and t_i is allowed for all $1 \leq i \leq n$. Given a *Conditional Term Rewriting System* (CTRS) $\mathcal{R}$, a sequence $s_1 \to^* t_1, \ldots, s_n \to^* t_n$ is $\mathcal{R}$-*feasible* if there is a substitution σ such that for all i, $1 \leq i \leq n$, $\sigma(s_i) \to^*_{\mathcal{R}} \sigma(t_i)$. In the realm of *oriented* CTRSs, with rules of the form $\ell \to r \Leftarrow c$ (with c usually written $s_1 \to t_1, \ldots, s_n \to t_n$ when the usual *rewriting semantics* for the evaluation of the conditions $s_i \to t_i$ is assumed), the *negation* of this property, i.e., the $\mathcal{R}$-*infeasibility* of a sequence $s_1 \to^* t_1, \ldots, s_n \to^* t_n$ as above, can be used to (i) disable the use of *conditional rules* $\ell \to r \Leftarrow c$ from $\mathcal{R}$ in reductions, (ii) discard *conditional critical pairs* $s \downarrow t \Leftarrow c$ in the analysis of *confluence* of CTRSs [11, 12, 13, 14], (iii) discard *conditional dependency pairs* $u \to v \Leftarrow c$ in the analysis of *operational termination* of CTRSs [9], or (iv) prove the *non-joinability* of terms s and t, possibly coming from conditional or unconditional critical pairs, in (C)TRSs: the $\mathcal{R}$-infeasibility of $s \to^* x, t \to^* x$ (with x not occurring in s or t) implies the non-joinability of s and t.

In this paper we apply the semantic approach to the analysis of properties of CTRSs in [5] to prove infeasibility: an $\mathcal{R}$-infeasibility problem is translated into the problem of finding a *model* $\mathcal{A}$ of the set of sentences $\overline{\mathcal{R}}$ representing the operational semantics of the CTRS $\mathcal{R}$ plus a sentence $\neg(\exists\vec{x})\ (s_1 \to^* t_1 \wedge \cdots \wedge s_n \to^* t_n)$ where $\to^*$ is viewed now as a predicate symbol in the underlying first-order language and can be freely interpreted in a first-order structure $\mathcal{A}$ [4]. If $\mathcal{A} \models \overline{\mathcal{R}} \cup \{\neg(\exists\vec{x})\ (s_1 \to^* t_1 \wedge \cdots \wedge s_n \to^* t_n)\}$ holds in the usual logical sense, i.e., the structure $\mathcal{A}$ satisfies all sentences in the right-hand side of '$\models$', then $s_1 \to^* t_1, \cdots, s_n \to^* t_n$ is $\mathcal{R}$-infeasible. In the examples discussed in this paper, such first-order structures have been synthesized using AGES, a tool for the automatic generation of models of first-order theories [3] which implements the methods described in [6]. Besides, we also introduce a number of auxiliary results which can be combined with this and other approaches in proofs of infeasibility, and hence in any application to prove confluence and operational termination of CTRSs.

2 Infeasibility problems

Borrowing [11, Definition 7.1.8(3)] for feasibility of Conditional Critical Pairs (CCPs, see [11, Definition 7.1.8(1)]), we introduce the following.

Definition 1. *Let $\mathcal{R}$ be a CTRS. A sequence $s_1 \to^* t_1, \ldots, s_n \to^* t_n$, where s_i and t_i are terms for all $1 \leq i \leq n$ is called a* feasibility sequence. *It is called $\mathcal{R}$-*feasible *if there is a substitution σ such that for all $1 \leq i \leq n$, $\sigma(s_i) \to^*_{\mathcal{R}} \sigma(t_i)$. Otherwise, it is called $\mathcal{R}$-*infeasible.

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(Refl)	$\dfrac{}{x \to^* x}$	(Cong)	$\dfrac{x_i \to y_i}{f(x_1,\ldots,x_i,\ldots,x_k) \to f(x_1,\ldots,y_i,\ldots,x_k)}$ for all $f \in \mathcal{F}$ and $1 \leq i \leq k = arity(f)$
(Tran)	$\dfrac{x \to z \qquad z \to^* y}{x \to^* y}$	(Repl)	$\dfrac{s_1 \to^* t_1 \quad \cdots \quad s_n \to^* t_n}{\ell \to r}$ for $\ell \to r \Leftarrow s_1 \to t_1,\ldots,s_n \to t_n \in \mathcal{R}$

Figure 1: Inference rules for conditional rewriting with a CTRS $\mathcal{R}$ with signature $\mathcal{F}$

Following [7], in Definition 1 we write $s \to^*_{\mathcal{R}} t$ or $s \to_{\mathcal{R}} t$ for terms s and t iff there is a proof tree for $s \to^* t$ (resp. $s \to t$) using $\mathcal{R}$ in the inference system of Figure 1.

Remark 2. *All rules in the inference system in Figure 1 are* schematic *in that each inference rule* $\frac{B_1 \ \cdots \ B_n}{A}$ *can be used under any* instance $\frac{\sigma(B_1) \ \cdots \ \sigma(B_n)}{\sigma(A)}$ *of the rule by a substitution σ. For instance, (Repl) actually establishes that, for every rule $\ell \to r \Leftarrow s_1 \to t_1,\ldots,s_n \to t_n$ in the CTRS $\mathcal{R}$, every instance $\sigma(\ell)$ by a substitution σ rewrites into $\sigma(r)$ provided that, for each $s_i \to t_i$, with $1 \leq i \leq n$, the reachability condition $\sigma(s_i) \to^* \sigma(t_i)$ can be* proved.

Now, we can say (as usual) that a critical pair $s \downarrow t \Leftarrow c$ or a rule $\ell \to r \Leftarrow c$ where c is $s_1 \to t_1,\ldots,s_n \to t_n$ is $\mathcal{R}$-infeasible if $s_1 \to^* t_1,\ldots,s_n \to^* t_n$ is $\mathcal{R}$-infeasible. We can also see the usual joinability problem $s \downarrow t$, i.e, the existence of a term u such that $s \to^*_{\mathcal{R}} u$ and $t \to^*_{\mathcal{R}} u$, as an specific feasibility sequence.

Proposition 3. *Let $\mathcal{R}$ be a CTRS, s,t be terms, and x be a fresh variable not occurring in s or t. If s and t are joinable, then $s \to^* x, t \to^* x$ is $\mathcal{R}$-feasible. If s and t are* ground*, then the $\mathcal{R}$-feasibility of $s \to^* x, t \to^* x$ implies joinability of s and t.*

An immediate consequence of the previous observation is that the $\mathcal{R}$-infeasibility of $s \to^* x, t \to^* x$ implies the non-joinability of s and t. Aoto has investigated methods for proving non-joinability of *ground* terms in TRSs [1]. Actually, Aoto makes the following interesting observation: for Term Rewriting Systems $\mathcal{R}$, the so-called *usable rules for reachability* $\mathcal{U}(\mathcal{R}, s)$ associated to a term s (roughly speaking an *overapproximation* of the set of rules that can be applied to s and then to any other term introduced by the application of a rule, see [1, Definition 3]) are the only ones we need to consider in any rewriting sequence starting from s, i.e., $s \to^*_{\mathcal{R}} t$ iff $s \to^*_{\mathcal{U}(\mathcal{R},s)} t$ (see [1, Lemma 4]). Actually, this holds as well for the usable rules $\mathcal{U}(\mathcal{R}, t)$ for CTRSs $\mathcal{R}$ and terms s introduced in [10, Definition 11]. Thus, we have the following easy corollary of this fact, where, given terms $s_1,\ldots,s_n$, we let $\mathcal{U}(\mathcal{R}, s_1,\ldots,s_n) = \bigcup_{i=1}^{n} \mathcal{U}(\mathcal{R}, s_i)$.

Proposition 4. *Let $\mathcal{R}$ be a CTRS and $s_1 \to^* t_1,\ldots,s_n \to^* t_n$ be a feasibility sequence. If the sequence is $\mathcal{U}(\mathcal{R}, s_1,\ldots,s_n)$-feasible, then it is $\mathcal{R}$-feasible. If terms s_i are* ground *for all $1 \leq i \leq n$ and the sequence is $\mathcal{R}$-*feasible, *then it is $\mathcal{U}(\mathcal{R}, s_1,\ldots,s_n)$*-feasible.

Thus, we can prove $\mathcal{R}$-infeasibility of critical pairs and rules by proving the $\mathcal{U}(\mathcal{R}, s_1,\ldots,s_n)$-*infeasibility* of the corresponding feasibility sequence $s_1 \to^* t_1,\ldots,s_n \to^* t_n$, provided that terms s_i are ground for all $1 \leq i \leq n$.

$$(\forall x)\ \ x \to^* x \quad (4)$$
$$(\forall x,y,z)\ \ x \to y \wedge y \to^* z \Rightarrow x \to^* z \quad (5)$$
$$(\forall x,y)\ \ x \to y \Rightarrow \mathsf{f}(x) \to \mathsf{f}(y) \quad (6)$$
$$(\forall x,y)\ \ x \to y \Rightarrow \mathsf{g}(x) \to \mathsf{g}(y) \quad (7)$$
$$\mathsf{a} \to \mathsf{b} \quad (8)$$
$$\mathsf{f}(\mathsf{a}) \to \mathsf{b} \quad (9)$$
$$(\forall x)\ \ \mathsf{f}(x) \to^* x \Rightarrow \mathsf{g}(x) \to \mathsf{g}(\mathsf{a}) \quad (10)$$

Figure 2: First-order theory for $\mathcal{R}$ in Example 5

3 A semantic criterion for infeasibility

In the logic of CTRSs, with binary *predicates* $\to$ and $\to^*$, the (first-order) theory $\overline{\mathcal{R}}$ for a CTRS $\mathcal{R} = (\mathcal{F}, R)$ is obtained from the inference rules in Figure 1 by *specializing* $(Cong)_{f,i}$ for each $f \in \mathcal{F}$ and i, $1 \leq i \leq ar(f)$ and $(Repl)_\rho$ for all $\rho : \ell \to r \Leftarrow c \in R$. Inference rules $\frac{B_1 \cdots B_n}{A}$ become universally quantified *implications* $B_1 \wedge \cdots \wedge B_n \Rightarrow A$ [8, Section 2].

Example 5. *Consider the following CTRS $\mathcal{R}$ [2, page 46]:*

$$\mathsf{a} \to \mathsf{b} \quad (1) \qquad \mathsf{g}(x) \to \mathsf{g}(\mathsf{a}) \Leftarrow \mathsf{f}(x) \to x \quad (3)$$
$$\mathsf{f}(\mathsf{a}) \to \mathsf{b} \quad (2)$$

Figure 2 shows its associated theory $\overline{\mathcal{R}}$.

By a structure $\mathcal{A}$ for a first-order logic language we mean an interpretation of the function and predicate symbols of the language ($f, g, \ldots$ and $P, Q, \ldots$, respectively) as mappings $f^{\mathcal{A}}, g^{\mathcal{A}}, \ldots$ and relations $P^{\mathcal{A}}, Q^{\mathcal{A}}, \ldots$ on a given set (carrier) also denoted $\mathcal{A}$. Then, the usual interpretation of first-order formulas with respect to the structure is considered. A model for a set $\mathcal{S}$ of first-order sentences (i.e., formulas whose variables are all *quantified*) is just a structure that makes them all true, written $\mathcal{A} \models \mathcal{S}$ see [4].

Proposition 6 (Semantic criterion for infeasibility)**.** *Let $\mathcal{R}$ be a CTRS, $s_1 \to^* t_1, \ldots, s_n \to^* t_n$ be a feasibility sequence, and $\mathcal{A}$ be a structure with nonempty domain. If $\mathcal{A} \models \overline{\mathcal{R}} \cup \{\neg(\exists \vec{x})\,(s_1 \to^* t_1 \wedge \cdots \wedge s_n \to^* t_n)\}$, then the sequence is $\mathcal{R}$-infeasible.*

Proof. By contradiction. Assume that the sequence is $\mathcal{R}$-feasible. Then, there is a substitution σ such that, for all $1 \leq i \leq n$, $\sigma(s_i) \to^*_{\mathcal{R}} \sigma(t_i)$. Since $\mathcal{A}$ is a model of $\overline{\mathcal{R}}$, by correctness we have that, for all $1 \leq i \leq n$, $\mathcal{A} \models (\forall \vec{y}_i)\ \sigma(s_i) \to^* \sigma(t_i)$, where $\vec{y}_i$ are the variables in $\mathcal{V}ar(\sigma(s_i)) \cup \mathcal{V}ar(\sigma(t_i))$. Therefore,

$$\mathcal{A} \models (\forall \vec{y})\ (\sigma(s_1) \to^* \sigma(t_1) \wedge \cdots \wedge \sigma(s_n) \to^* \sigma(t_n)) \tag{11}$$

with $\vec{y}$ the variables in $\bigcup_{i=1}^n \mathcal{V}ar(\sigma(s_i)) \cup \mathcal{V}ar(\sigma(t_i))$. Hence, for all valuations $\nu : \vec{y} \to \mathcal{A}$, the interpretation of the universally quantified formula in (11) in the structure $\mathcal{A}$, i.e., $[\sigma(s_1) \to^* \sigma(t_1) \wedge \cdots \wedge \sigma(s_n) \to^* \sigma(t_n)]^{\mathcal{A}}_\nu$, is *true*. Let $\vec{x}$ be the variables in $\bigcup_{i=1}^n \mathcal{V}ar(s_i) \cup \mathcal{V}ar(t_i)$. Since $\mathcal{A}$ has a nonempty domain, given an arbitrary valuation $\nu : \vec{y} \to \mathcal{A}$, there is a valuation $\nu' : \vec{x} \to \mathcal{A}$ given by $\nu'(x) = [\sigma(x)]^{\mathcal{A}}_\nu$ for all variable x in $\vec{x}$, such that $[s_1 \to^* t_1 \wedge \cdots \wedge s_n \to^* t_n]^{\mathcal{A}}_{\nu'}$ is true. This contradicts our assumption $\mathcal{A} \models \neg(\exists \vec{x})\,(s_1 \to^* t_1 \wedge \cdots \wedge s_n \to^* t_n)$. □

In the following, we assume that the signature $\mathcal{F}$ of any CTRS to be used with Proposition 6 contains at least a (dummy) *constant symbol* so that any associated structure $\mathcal{A}$ has a nonempty domain. Models to be used in Proposition 6 for infeasibility checking can be automatically generated from the CTRS $\mathcal{R}$ and sentence at stake, i.e., $\neg(\exists\vec{x})\,(s_1 \to^* t_1 \wedge \cdots \wedge s_n \to^* t_n)$, by using (as we do in the following examples) a tool like AGES [3].[1]

Example 7 (Infeasible rule)**.** *The following structure $\mathcal{A}$ over $\mathbb{N} - \{0\}$:*

$$\begin{array}{cccc} \mathsf{a}^{\mathcal{A}} = 1 & \mathsf{b}^{\mathcal{A}} = 2 & \mathsf{f}^{\mathcal{A}}(x) = x+1 & \mathsf{g}^{\mathcal{A}}(x) = 1 \\ x \to^{\mathcal{A}} y \Leftrightarrow y \geq x & x\,(\to^*)^{\mathcal{A}}\,y \Leftrightarrow y \geq x & & \end{array}$$

is a model of $\overline{\mathcal{R}} \cup \{\neg(\exists x)\,\mathsf{f}(x) \to^* x\}$ for $\mathcal{R}$ in Example 5 (see $\overline{\mathcal{R}}$ in Figure 2). Thus, rule (3) is proved $\mathcal{R}$-infeasible. Using this observation it is not difficult to see that $\mathcal{R}$ is operationally terminating, *i.e., no term t has an infinite proof tree using the inference system in Figure 1 [7].*

Example 8 (Infeasible critical pair)**.** *The following CTRS $\mathcal{R}$ [14, Example 23]*

$$\mathsf{g}(x) \to \mathsf{f}(x,x) \tag{12}$$

$$\mathsf{g}(x) \to \mathsf{g}(x) \Leftarrow \mathsf{g}(x) \to \mathsf{f}(\mathsf{a},\mathsf{b}) \tag{13}$$

has a conditional critical pair $\mathsf{f}(x,x) \downarrow \mathsf{g}(x) \Leftarrow \mathsf{g}(x) \to \mathsf{f}(\mathsf{a},\mathsf{b})$. The following structure $\mathcal{A}$ over the finite domain $\{0,1\}$:

$$\mathsf{a}^{\mathcal{A}} = 1 \qquad \mathsf{b}^{\mathcal{A}} = \mathsf{c}^{\mathcal{A}} = 0 \qquad \mathsf{f}^{\mathcal{A}}(x,y) = \begin{cases} x-y+1 & \text{if } x \geq y \\ y-x+1 & \text{otherwise} \end{cases}$$

$$\mathsf{g}^{\mathcal{A}}(x) = 1 \qquad x \to^{\mathcal{A}} y \Leftrightarrow x = y \qquad x\,(\to^*)^{\mathcal{A}}\,y \Leftrightarrow x \geq y$$

is a model $\overline{\mathcal{R}} \cup \{\neg(\exists x)\,\mathsf{g}(x) \to^* \mathsf{f}(\mathsf{a},\mathsf{b})\}$. Thus, the critical pair is infeasible. In [14, Example 23] this is proved by using unification tests together with a transformation. It is discussed that the the alternative tree automata techniques investigated in the paper do not *work for this example.*

Example 9 (Infeasible rule)**.** *Consider the following CTRS $\mathcal{R}$ [14, Example 17]*

$$\mathsf{h}(x) \to \mathsf{a} \tag{14}$$

$$\mathsf{g}(x) \to x \tag{15}$$

$$\mathsf{g}(x) \to \mathsf{a} \Leftarrow \mathsf{h}(x) \to \mathsf{b} \tag{16}$$

$$\mathsf{c} \to \mathsf{c} \tag{17}$$

The following structure $\mathcal{A}$ over $\mathbb{N}$:

$$\begin{array}{cccc} \mathsf{a}^{\mathcal{A}} = 0 & \mathsf{b}^{\mathcal{A}} = \mathsf{c}^{\mathcal{A}} = 1 & \mathsf{g}^{\mathcal{A}}(x) = x+2 & \mathsf{h}^{\mathcal{A}}(x) = 0 \\ x \to^{\mathcal{A}} y \Leftrightarrow x \geq y & x\,(\to^*)^{\mathcal{A}}\,y \Leftrightarrow x \geq y & & \end{array}$$

is a model of $\overline{\mathcal{R}} \cup \{\neg(\exists x)\,\mathsf{h}(x) \to^* \mathsf{b}\}$. Therefore, rule (16) is proved $\mathcal{R}$-infeasible. In [14, Example 17] this is proved by using tree automata techniques. It is also shown that the alternative technique investigated in the paper (the use of unification tests) does not *work in this case.*

Example 10 (Non-joinable terms)**.** *Consider the following CTRS $\mathcal{R}$ [11, Example 7.3.3]:*

$$\mathsf{a} \to \mathsf{b} \tag{18}$$

$$\mathsf{f}(x) \to \mathsf{c} \Leftarrow x \to \mathsf{a} \tag{19}$$

Although there is no critical pair, the system is not (locally) confluent because $\mathsf{f}(\mathsf{a}) \to_{\mathcal{R}} \mathsf{f}(\mathsf{b})$ and $\mathsf{f}(\mathsf{a}) \to_{\mathcal{R}} \mathsf{c}$ but c and $\mathsf{f}(\mathsf{b})$ are not *joinable. The following structure $\mathcal{A}$ over $\mathbb{N} \cup \{-1\}$:*

$$\begin{array}{cccc} \mathsf{a}^{\mathcal{A}} = 0 & \mathsf{b}^{\mathcal{A}} = -1 & \mathsf{c}^{\mathcal{A}} = 1 & \mathsf{f}^{\mathcal{A}}(x) = x+1 \\ x \to^{\mathcal{A}} y \Leftrightarrow x = y & x\,(\to^*)^{\mathcal{A}}\,y \Leftrightarrow x = y & & \end{array}$$

is a model of $\overline{\mathcal{U}(\mathcal{R},\mathsf{f}(\mathsf{b}),\mathsf{c})} \cup \{\neg(\exists x)\,(\mathsf{f}(\mathsf{b}) \to^* x \wedge \mathsf{c} \to^* x)\}$, where $\mathcal{U}(\mathcal{R},\mathsf{f}(\mathsf{b}),\mathsf{c}) = \{(19)\}$. Therefore, $\mathsf{f}(\mathsf{b})$ and c are proved not joinable.

[1] In AGES the universally quantified version $(\forall\vec{x})\neg(s_1 \to^* t_1 \wedge \cdots \wedge s_n \to^* t_n)$ of $\neg(\exists\vec{x})(s_1 \to^* t_1 \wedge \cdots \wedge s_n \to^* t_n)$ is used; actually, only $\neg(s_1 \to^* t_1 \wedge \cdots \wedge s_n \to^* t_n)$ is introduced (universal quantification is assumed).

4 Conclusions

We have presented a semantic approach to prove infeasibility in conditional rewriting. Interestingly, with such a semantic approach we could handle many examples coming from papers developing different specific techniques to deal with these problems. In particular, we could deal with all examples solved in [13, 14] (some of them reported in our examples above; note that these papers explore several *alternative* methods and, as reported by the authors, some of them *fail* in specific examples which require a different approach). We also dealt with all Aoto's examples in [1] in combination with his *usable rules* refinement (see Proposition 4).

We do not have a dedicated, fully automated 'infeasibility' checker yet. Instead we just encode the problem we want to deal with (e.g., infeasibility of a critical pair, or rule; or non-joinability) as an specific infeasibility sequence and then use AGES to apply Proposition 6. The generation of a model is completely automatic, but in order to *succeed* on a given problem, we often require to *manually* change special settings like the generation of piecewise interpretations (like in Example 8), usually 'expensive' and not part of the *default* configuration. Thus, further investigation developing heuristics for an efficient use of the technique, possibly in combination with other existing techniques, would be necessary.

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