Coherence of quasi-terminating decreasing linear polygraphs

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Linear polygraphs

- A 1-**polygraph** is a directed graph (Σ_0, Σ_1)

$$\Sigma_0 \underset{t_0}{\overset{s_0}{\leftleftarrows}} \Sigma_1$$

- A **linear polygraph** is a triple $\Sigma = (\Sigma_0, \Sigma_1, \Sigma_2)$ where
  - (Σ_0, Σ_1) is a 1-polygraph,
  - Σ_2 is a **globular extension** of the free linear 1-category Σ_1^ℓ.
- A **rewriting step** is a 2-cell of the free linear 2-category Σ_2^ℓ over Σ with shape

$$ws_0(\alpha)w' + h \underset{wvw' + h}{\overset{wuw' + h}{\rightrightarrows}} wt_0(\alpha)w' + h \qquad \Downarrow w\alpha w' + h$$

Termination

▶ A linear polygraph Σ **terminates** if it does not generate any infinite reduction sequence

$$u_1 \Longrightarrow u_2 \Longrightarrow \cdots \Longrightarrow u_n \Longrightarrow \cdots$$

▶ A linear polygraph Σ is **exponentiation free** if no monomial m of Σ^ℓ can be rewritten into $\lambda m + f$ for some scalar λ other than 0 or 1 and some non zero 1-cell f which does not contain m in its monomial decomposition.

▶ A linear polygraph Σ is **quasi-terminating** if every infinite reduction sequence

$$u_1 \Longrightarrow u_2 \Longrightarrow \cdots \Longrightarrow u_n \Longrightarrow \cdots$$

cycles, that is the sequence contains an infinite number of occurrences of the same 1-cell.

▶ A 1-cell u of Σ_1^ℓ is called a **quasi-normal form** if for any rewriting step with source u leading to a 1-cell v, there exists a rewriting sequence from v to u.

▶ If Σ is quasi-terminating, any 1-cell u of Σ_1^ℓ admits a quasi-normal form.

- ▷ Note that this quasi-normal form is neither irreducible nor unique in general.

Example

- The linear polygraph

$$\Sigma(\mathcal{A}) = \langle x, y \mid x^2 \overset{\alpha}{\Longrightarrow} xy - y^2,\ y^2 \overset{\beta}{\Longrightarrow} xy - x^2, \rangle$$

presents the algebra $\mathcal{A}$.

  - It is not terminating but it is quasi-terminating.
  - It has two critical branchings.
  - But it has a non empty family of syzygies.

Coherent presentations of categories

▶ A **linear** $(3,1)$-**polygraph** is a data made of

▷ a linear polygraph $(\Sigma_0, \Sigma_1, \Sigma_2)$,

▷ a globular extension Σ_3 of the free linear $(2,1)$-category Σ_2^ℓ.

▶ The linear $(2,1)$-category Σ_2^ℓ corresponds to the 2-category of linear congruences generated by Σ_2.

▶ A **coherent presentation** of an algebra $\mathbf{A}$ is a linear $(3,1)$-polygraph $(\Sigma_0, \Sigma_1, \Sigma_2, \Sigma_3)$ such that

▷ $(\Sigma_0, \Sigma_1, \Sigma_2)$ is a presentation of $\mathbf{A}$:

$$\Sigma_0 = \{\bullet\} \qquad \text{and} \qquad \mathbf{A} \simeq \Sigma_1^\ell / \Sigma_2,$$

▷ the globular extension Σ_3 is a **homotopy basis**.

Labelled two-dimensional polygraphs

▶ A **well-founded labelling** for a linear polygraph Σ is a data $(W, \prec, \psi)$ made of a set W, a well-founded order $\prec$ on W and a map

$$\psi : \Sigma_{stp} \longrightarrow W$$

that associates to a rewriting step f a **label** $\psi(f)$.

▶ Given a rewriting sequence $f = f_1 \cdot \ldots \cdot f_k$, we denote by

$$L^W(f) = \{\psi(f_1), \ldots, \psi(f_k)\}$$

the set of labels of rewriting steps in f.

Labelling to the quasi-normal form

- Let Σ be a confluent and quasi-terminating linear polygraph.
  - By quasi-termination, any 1-cell u admits a (non-unique) quasi-normal form.
  - Given a 1-cell u in Σ_1^ℓ, we fix a quasi-normal form $\widetilde{u}$.
  - By confluence, any two congruent 1-cells of Σ_1^ℓ have the same quasi-normal form.
- The **labelling to the quasi-normal form** associated is the map

$$\psi^{\mathrm{QNF}} : \Sigma_{stp} \longrightarrow \mathbb{N}$$

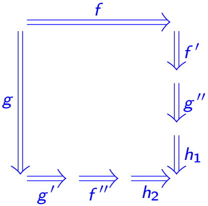
defined, for any rewriting step f of Σ.

$$\psi^{\mathrm{QNF}}(f) = d(t_1(f), \widetilde{t_1(f)}),$$

the length of the shortest rewriting sequence from $t_1(f)$ to its quasi-normal form.

Decreasing two-dimensional polygraphs

- Decreasingness from ARS, (van Oostrom, 1994).
  - Let Σ be a linear polygraph with a well-founded labelling $(W, \psi, \prec)$.
- A local branching (f, g) of Σ is **decreasing** if there is a **decreasing confluence diagram**:



with

i) for each $k \in L^W(f')$, we have $k \prec \psi(f)$,
ii) for each $k \in L^W(g')$, we have $k \prec \psi(g)$,
iii) f'' (resp. g'') is an identity or a rewriting step labelled by $\psi(f)$ (resp. $\psi(g)$),
iv) for each $k \in L^W(h_1) \cup L^W(h_2)$, we have $k \prec \psi(f)$ or $k \prec \psi(g)$.

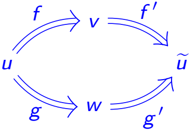
- A linear polygraph Σ is **decreasing** if there exists a well-founded labelling $(W, \prec, \psi)$ of Σ making all its local branching decreasing.

Theorem. Any decreasing linear polygraph is confluent.

Decreasingness from quasi-termination

▶ Any confluent and quasi-terminating linear polygraph Σ is decreasing with respect to any quasi-normal form labelling ψ^{QNF}.

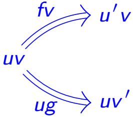
▷ For any local branching $u \Rightarrow (v, w)$ there is a quasi-normal form $\widetilde{u}$ giving a confluence diagram as follows:



▷ We choose the rewriting sequences f' and g' of minimal length, thus making this confluence diagram decreasing with respect ψ^{QNF}.

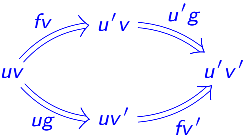
Decreasingness of Peiffer branchings

▶ Given a Peiffer branching



of a linear polygraph Σ.

▶ We will call **Peiffer confluence** the following confluence diagram



▶ If Σ is decreasing,

▷ all its Peiffer branchings can be completed into a decreasing confluence diagram.

▷ However, the Peiffer confluence for this branching is not necessarily decreasing.

▷ it is the case for a QNF labelling when the source uv is already the chosen quasi-normal form.

Peiffer decreasingness

▶ Let Σ be a linear polygraph and let Σ_3 be a globular extension of the linear $(2,1)$-category Σ_2^{ℓ}.

▶ The linear polygraph Σ is **Peiffer decreasing with respect to Σ_3** if there exists a well-founded labelling $(W, \prec, \psi)$ such that the following conditions hold

▷ Σ is decreasing with respect to $(W, \prec, \psi)$,

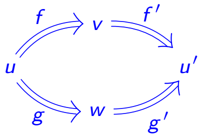
▷ for any Peiffer branching $(fv, ug) : uv \Rightarrow (u'v, uv')$, there exists a decreasing confluence diagram $(fv \cdot f', ug \cdot g')$ such that

$$u'g \star_1 (fv')^- \equiv_{\Sigma_3} f' \star_1 (g')^-.$$

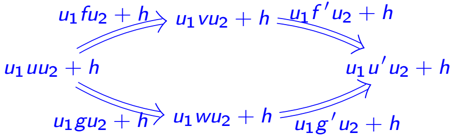
Whisker compatibility

- Let Σ be a linear polygraph with a well-founded labelling $(W, \prec, \psi)$.

- The labelling is **whisker compatible** if for any decreasing confluence diagram



where (f, g) is a local branching, and for any monomials u_1 and u_2 and 1-cell h in Σ_1^{ℓ}, then the following confluence diagram is decreasing:



- Note that a QNF labelling is not whisker compatible in general.

Example

- Consider the linear polygraph

$$\Sigma(\mathcal{A}) = \langle x, y \mid x^2 \overset{\alpha}{\Longrightarrow} xy - y^2,\ y^2 \overset{\beta}{\Longrightarrow} xy - x^2 \rangle$$

- We define the QNF labelling ψ^{QNF} associated to quasi-normal forms coming from the deglex order induced by $y > x$.

- The labelling ψ^{QNF} is whisker compatible.
  - Indeed, for any rewriting steps f and g, have

$$\psi^{\mathrm{QNF}}(g) < \psi^{\mathrm{QNF}}(f) \qquad \text{implies} \qquad \psi^{\mathrm{QNF}}(u_1 f u_2) < \psi^{\mathrm{QNF}}(u_1 g u_2)$$

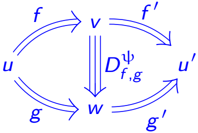
for any 1-cells u_1 and u_2.

Decreasing Squier's completion

▶ Let Σ be a decreasing linear polygraph for a well-founded labelling $(W, \prec, \psi)$.

▶ A **family of generating decreasing confluences** of Σ with respect to ψ is a globular extension of the linear $(2,1)$-category Σ_2^ℓ that contains,

- ▷ for every critical branching (f, g) of Σ, one 3-cell of the form



- ▷ where the confluence diagram $(f \cdot f', g \cdot g')$ is decreasing with respect to ψ.

▶ Any decreasing linear polygraph admits such a family of generating decreasing confluences.

▶ Such a family is not unique in general.

Decreasing Squier's completion

- Let Σ be a decreasing linear polygraph for a well-founded labelling $(W, \prec, \psi)$.

- A **decreasing Squier's completion** of Σ with respect to ψ is a $(3,1)$-polygraph $\mathcal{D}(\Sigma, \psi)$

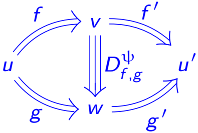
  - that extends the linear polygraph Σ,

  - by a globular extension

$$\mathcal{O}(\Sigma, \psi) \cup \mathcal{L}(\Sigma)$$

where

  - $\mathcal{O}(\Sigma, \psi)$ is a chosen family of generating decreasing confluences with respect to ψ,



  - $\mathcal{L}(\Sigma)$ is a loop extension of Σ, containing exactly one loop for each equivalence classes of elementary loops of Σ_2^{ℓ}.

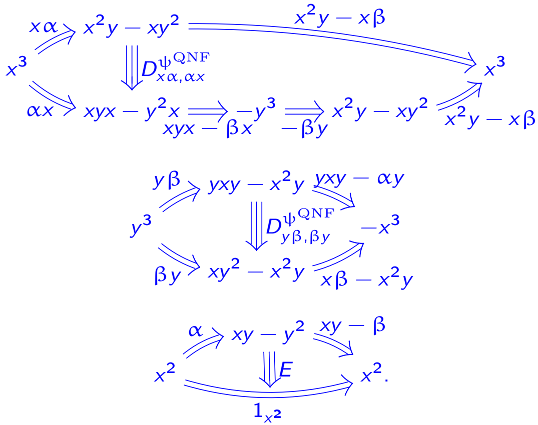
Decreasing Squier's completion

▶ **Example**. The linear polygraph

$$\Sigma(\mathcal{A}) = \langle x, y \mid x^2 \overset{\alpha}{\Longrightarrow} xy - y^2,\ y^2 \overset{\beta}{\Longrightarrow} xy - x^2 \rangle$$

is decreasing for the QNF labelling ψ^{QNF} defined earlier with $y > x$.

▶ A decreasing Squier's completion of the linear polygraph $\Sigma(\mathcal{A})$ is given by



Decreasing Squier's completion

Theorem. (A.)

Let Σ be a linear polygraph and let ψ^{QNF} be a QNF labelling of Σ.

Let $\mathcal{D}(\Sigma, \psi^{\mathrm{QNF}})$ be a decreasing Squier's completion of Σ.

If the three following conditions hold

- Σ is quasi-terminating and exponentiation free,
- ψ^{QNF} is whisker compatible,
- Σ is Peiffer decreasing with respect to ψ^{QNF} and with respect to $\mathcal{D}(\Sigma, \psi^{\mathrm{QNF}})$.

▶ Then Σ is a coherent presentation of the algebra presented by Σ_2.

Example

- Consider the linear polygraph $\Sigma(\mathcal{A})$ with the labelling ψ^{QNF} defined earlier.
  - $\Sigma(\mathcal{A})$ is quasi-terminating and exponentiation free,
  - ψ^{QNF} is whisker compatible,
  - $\Sigma(\mathcal{A})$ admits a coherent presentation by $\Sigma(\mathcal{A})$ and the 3-cells $D^{\psi^{\mathrm{QNF}}}_{x\alpha,\alpha x}$, $D^{\psi^{\mathrm{QNF}}}_{y\beta,\beta y}$ and E.
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