Detecting useless critical pairs

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Plan

I. Motivations

- ▷ The CPC procedure
- ▷ Useless reductions for completion
- ▷ Linear algebra and useless reductions

II. Reduction operators

- ▷ Definition of reduction operators
- ▷ Reduction operators and labelled reductions

III. Lattice criterion for rejecting useless reductions

- ▷ Lattice structure of reduction operators
- ▷ The criterion

Plan

I. Motivations

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  - Find selection strategies:
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  - Find criteria for avoiding useless critical pairs.
    - A first type is based on the structure of branchings.
    - A second type consists in rejecting useless reductions.

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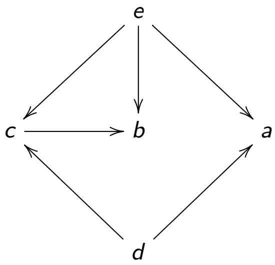

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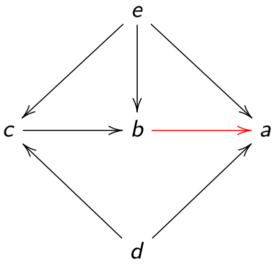

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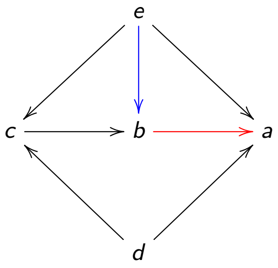

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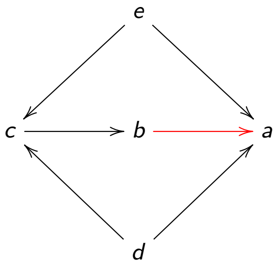

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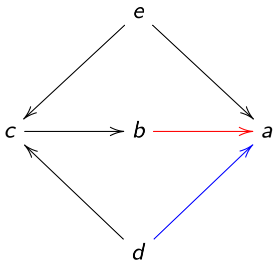

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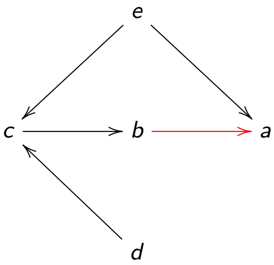

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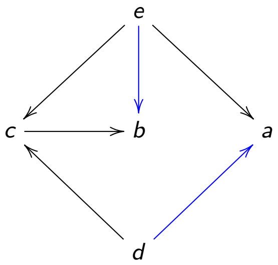

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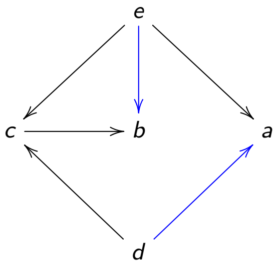

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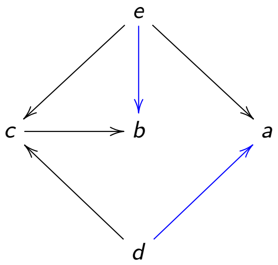

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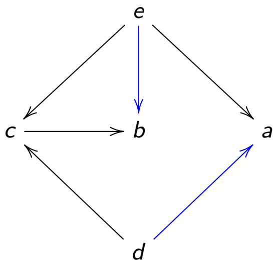

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Remarks.

- Buchberger's algorithm is interpreted as Gaussian elimination [Lazard, 1983].
- A set of terms can be embedded into the vector space it spans.

Linear algebra and useless reductions

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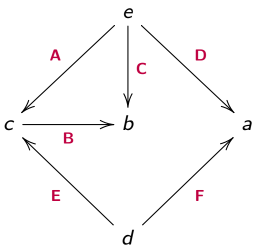

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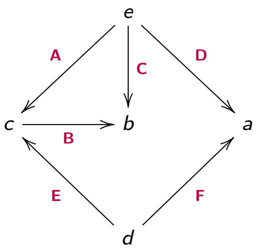

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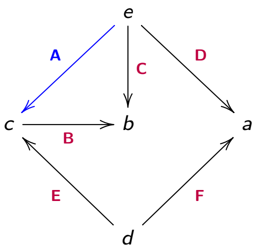

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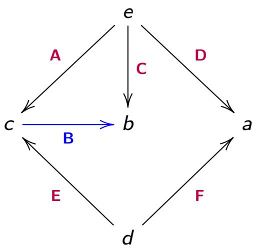

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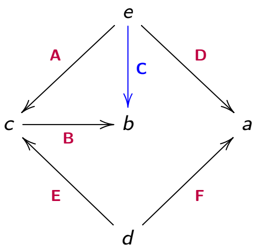

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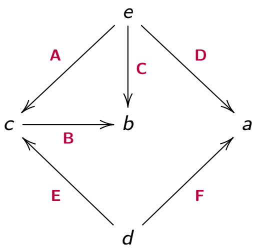

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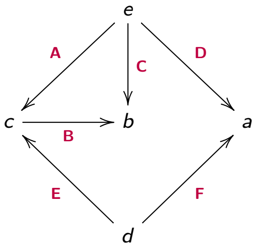

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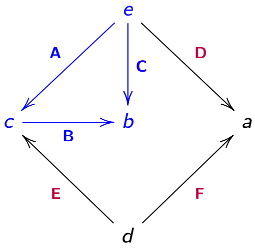

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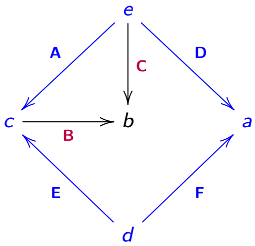

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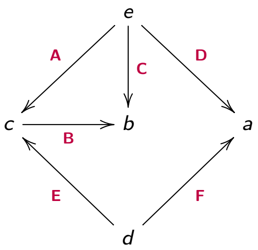

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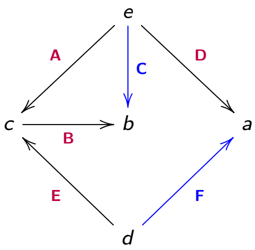

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  - e.g., for **A** $\sqsubset$ **B** $\sqsubset \cdots \sqsubset$ **F** the useless reductions are **C** and **F**.

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II. Reduction operators

Definition and labelled reductions

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- The labels of reductions induced by $F = \{T_1, \cdots, T_n\} \subset \mathbf{RO}(G,<)$ are ordered by:

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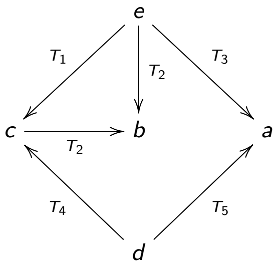

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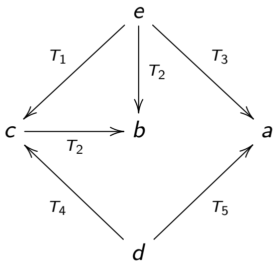

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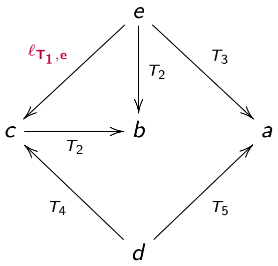

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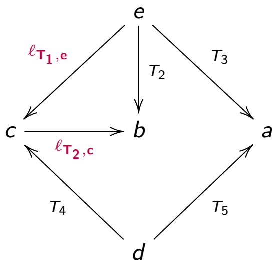

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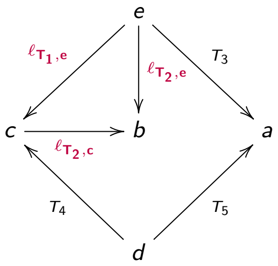

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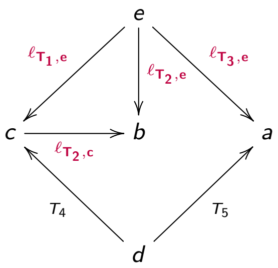

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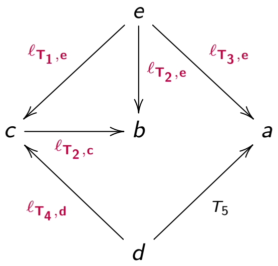

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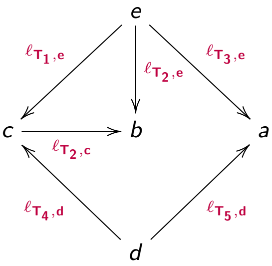

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III. Lattice criterion for rejecting useless reductions

Lattice structure on $\mathbf{RO}(G, <)$ and the criterion

Lattice structure.

- The map $\mathbf{RO}(G,<) \longrightarrow \text{Subspaces}(V),\ T \longmapsto \ker(T)$ is a bijection [C. 2016].

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- A completion of $\{\tilde{T}_1, \cdots, \tilde{T}_n\}$ leads to a completion of $\{T_1, \cdots, T_n\}$.

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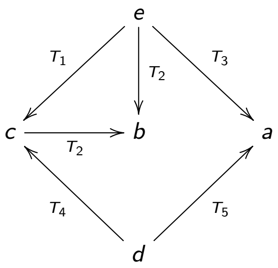
  - $\tilde{T}_i(g) = T_i(g)$, otherwise.
- A completion of $\{\tilde{T}_1, \cdots, \tilde{T}_n\}$ leads to a completion of $\{T_1, \cdots, T_n\}$.
- The rejected reductions have labels $\ell_{T_i,g}$, with $g \notin \mathrm{im}((T_1 \wedge \cdots \wedge T_{i-1}) \vee T_i)$.

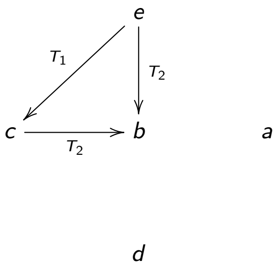
Illustration

- We consider:



Illustration

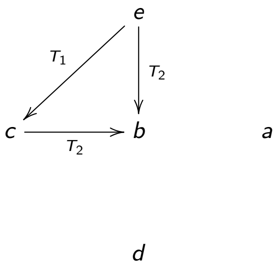
- We consider:



- Step 1: we have $e - c = (e - b) - (c - b)$.

Illustration

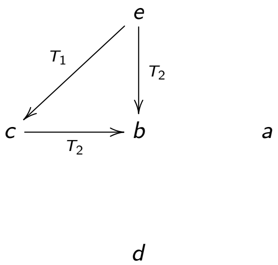
- We consider:



- Step 1: we have $e - c = (e - b) - (c - b)$.
  - $(e - c) \in \ker(T_1) \cap \ker(T_2) = \ker(T_1 \vee T_2)$.

Illustration

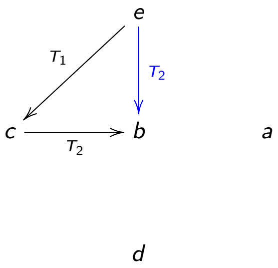
- We consider:



- Step 1: we have $e - c = (e - b) - (c - b)$.
  - $(e - c) \in \ker(T_1) \cap \ker(T_2) = \ker(T_1 \vee T_2)$.
  - $T_1 \vee T_2(e) = T_1 \vee T_2(c)$.
    - We have $\operatorname{lt}(T_1 \vee T_2(c)) \leq c$ and $c < e$.
    - Hence, $T_1 \vee T_2(e) \neq e$, so that $e \notin \operatorname{im}(T_1 \vee T_2)$.

Illustration

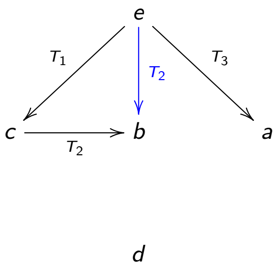
- We consider:



- Step 1: we have $e - c = (e - b) - (c - b)$.
  - $(e - c) \in \ker(T_1) \cap \ker(T_2) = \ker(T_1 \vee T_2)$.
  - $T_1 \vee T_2(e) = T_1 \vee T_2(c)$.
    - We have $\mathrm{lt}(T_1 \vee T_2(c)) \leq c$ and $c < e$.
    - Hence, $T_1 \vee T_2(e) \neq e$, so that $e \notin \mathrm{im}(T_1 \vee T_2)$.
  - The lattice criterion rejects the reduction $e \xrightarrow{T_2} b$.

Illustration

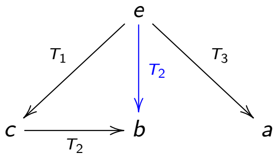
- We consider:



- Step 2: e and a are not connected by T_1 and T_2.

Illustration

- We consider:

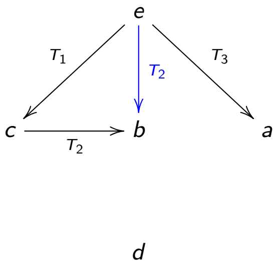


d

- Step 2: e and a are not connected by T_1 and T_2.
  - $T_1 \wedge T_2(e) \neq T_1 \wedge T_2(a)$.

Illustration

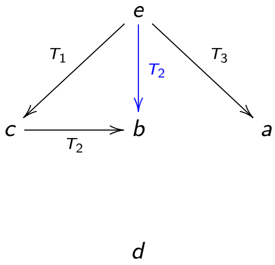
- We consider:



- Step 2: e and a are not connected by T_1 and T_2.
  - $T_1 \wedge T_2(e) \neq T_1 \wedge T_2(a)$.
  - $\ker(T_1 \wedge T_2) \cap \ker(T_3) = \{0\}$.

Illustration

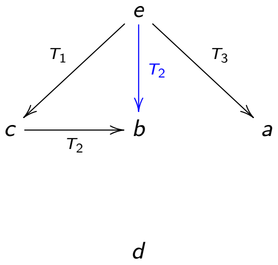
- We consider:



- Step 2: e and a are not connected by T_1 and T_2.
  - $T_1 \wedge T_2(e) \neq T_1 \wedge T_2(a)$.
  - $\ker(T_1 \wedge T_2) \cap \ker(T_3) = \{0\}$.
  - $(T_1 \wedge T_2) \vee T_3 = \mathrm{Id}_V$.

Illustration

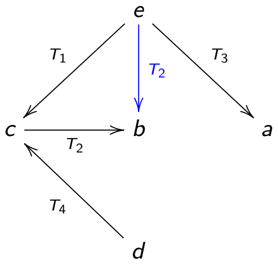
- We consider:



- Step 2: e and a are not connected by T_1 and T_2.
  - $T_1 \wedge T_2(e) \neq T_1 \wedge T_2(a)$.
  - $\ker(T_1 \wedge T_2) \cap \ker(T_3) = \{0\}$.
  - $(T_1 \wedge T_2) \vee T_3 = \mathrm{Id}_V$.
  - The criterion rejects no reduction.

Illustration

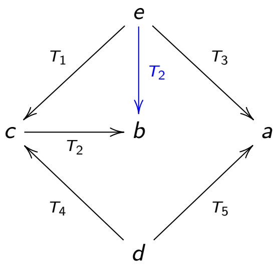
- We consider:



- Step 3: the criterion rejects no reduction.

Illustration

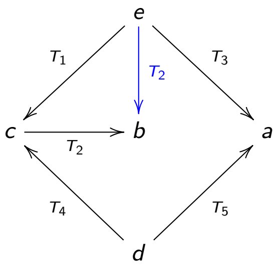
- We consider:



- Step 4: we have $d - a = (d - c) - (e - c) + (e - a)$.

Illustration

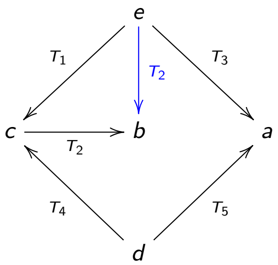
- We consider:



- Step 4: we have $d - a = (d - c) - (e - c) + (e - a)$.
  - $(d - a) \in \ker((T_1 \wedge \cdots \wedge T_4) \vee T_5)$.

Illustration

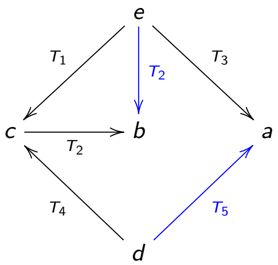
- We consider:



- Step 4: we have $d - a = (d - c) - (e - c) + (e - a)$.
  - $(d - a) \in \ker((T_1 \wedge \cdots \wedge T_4) \vee T_5)$.
  - $d \notin \operatorname{im}((T_1 \wedge \cdots \wedge T_4) \vee T_5)$.

Illustration

- We consider:



- Step 4: we have $d - a = (d - c) - (e - c) + (e - a)$.
  - $(d - a) \in \ker((T_1 \wedge \cdots \wedge T_4) \vee T_5)$.
  - $d \notin \operatorname{im}((T_1 \wedge \cdots \wedge T_4) \vee T_5)$.
  - The criterion rejects the reduction $d \xrightarrow{T_5} a$.

Thank you for listening!