Computational Logic

Aspects of Layer Systems in IsaFoR*

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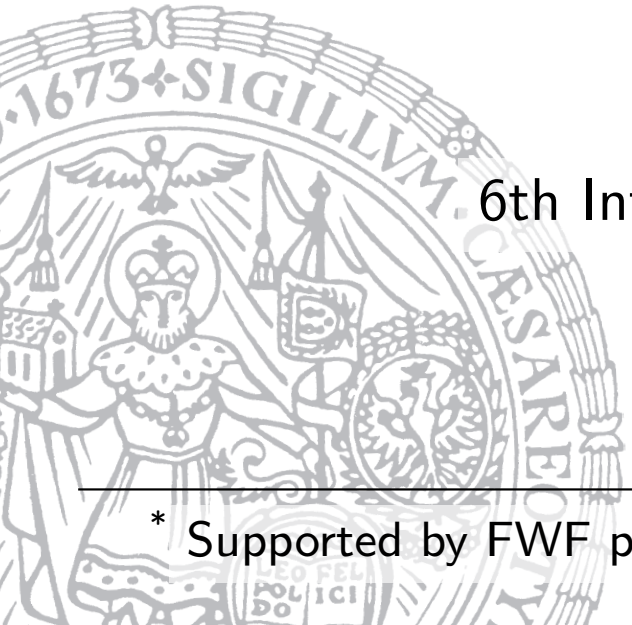

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* Supported by FWF project P27528

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- confluence tools prove confluence, but may be buggy
- solution: certification
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  - development closedness
  - AC techniques
  - advanced labeling techniques
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$\Rightarrow$ goal: formalize persistence & co.

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- Introduction
- Layer Systems
- Formalization

Modularity

Theorem (Toyama 1987, Klop *et al.* 1994, van Oostrom 2008, . . .)

Let $\mathcal{R}_1$, $\mathcal{R}_2$ be TRSs over disjoint signatures. Then

$$\mathsf{CR}(\mathcal{R}_1 \cup \mathcal{R}_2) \iff \mathsf{CR}(\mathcal{R}_1) \wedge \mathsf{CR}(\mathcal{R}_2)$$

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- $\Longrightarrow$ is easy (homogeneous terms are closed under rewriting)

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Example

$$\mathcal{R}_1 = \{\, \mathsf{Ap}(\mathsf{Ap}(\mathsf{K}, x), y) \to x,$$
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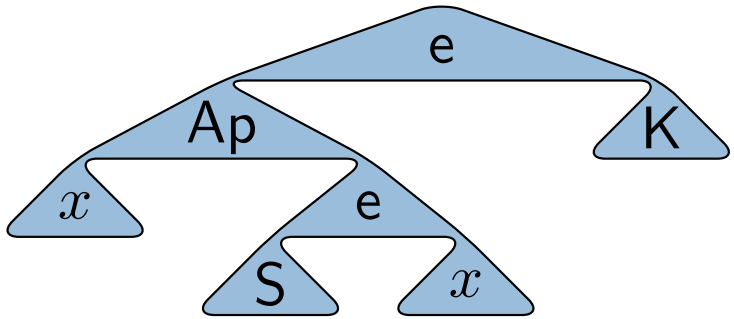

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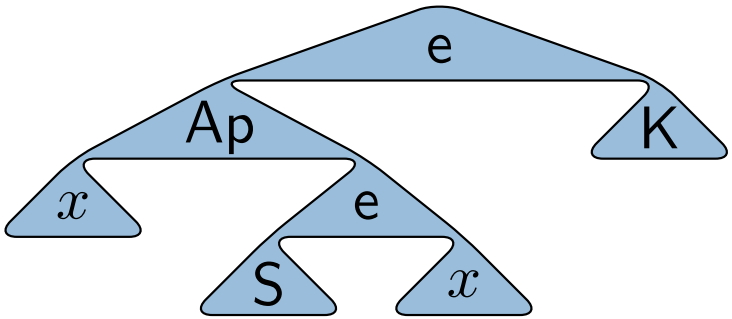

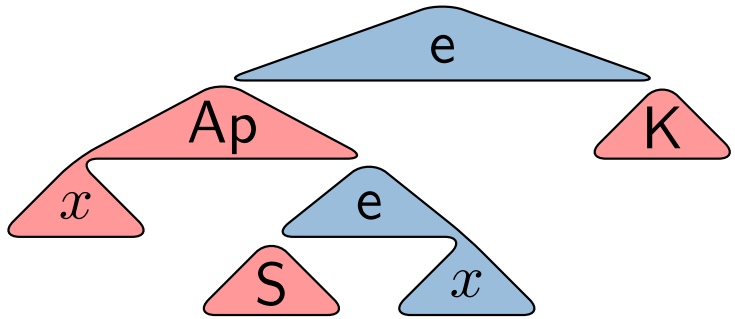

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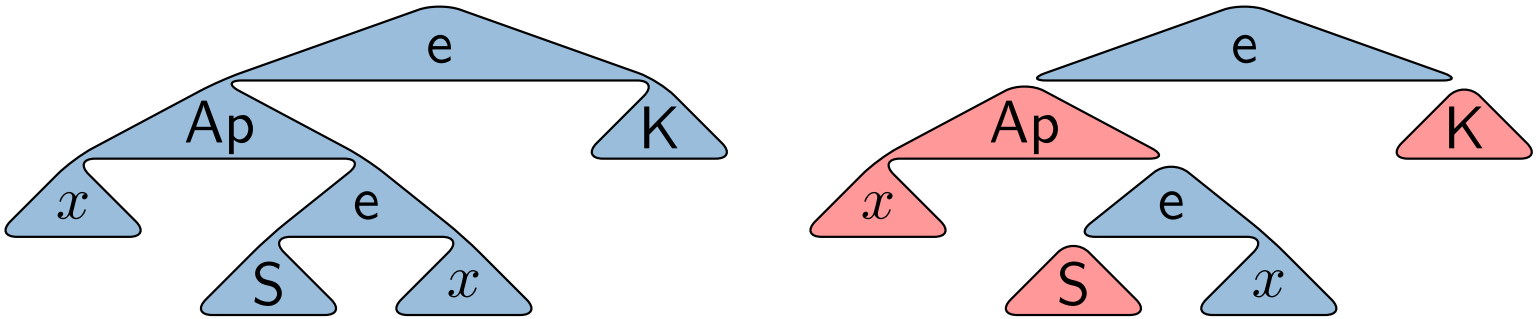

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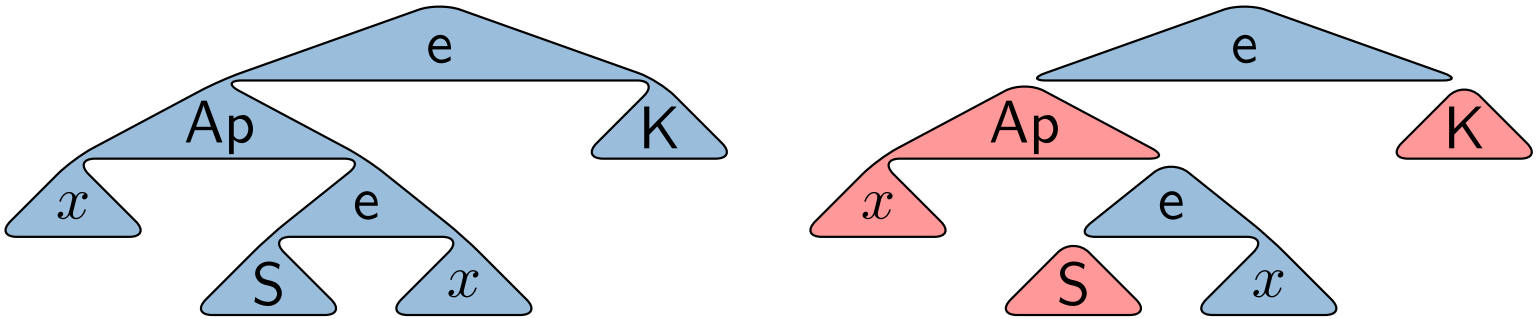

- max-top $\mathsf{e}(\Box, \Box)$, aliens $\mathsf{Ap}(x, \mathsf{e}(\mathsf{S}, x))$ and K, rank 4
- $\mathcal{R}_1$ and $\mathcal{R}_2$ are confluent $\implies$ $\mathcal{R}_1 \cup \mathcal{R}_2$ is confluent

Related Results

Results

- Persistence (Aoto and Toyama 1997)
- Layer preservation (Ohlebusch 1994)
- Quasi-ground systems (Kitahara *et al.* 1995)
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Idea

- layer system $\mathfrak{L}$: set of multi-hole contexts
- top of t: $L \in \mathfrak{L}$ with $L \sqsubseteq t$
- max-top, aliens, rank
- notions: layer system, weakly layered, layered

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Applications

- modularity: $\mathcal{R}_1 \cup \mathcal{R}_2$ is layered by $\mathcal{T}(\mathcal{F}_1, \mathcal{V}) \cup \mathcal{T}(\mathcal{F}_2, \mathcal{V})$.
- ...

Layer Systems Definition

Definition

Under the following conditions, the set of contexts $\mathfrak{L} \subseteq \mathcal{C}$ is layer system:

- L_1 Every term in $\mathcal{T}(\mathcal{F}, \mathcal{V})$ has a non-empty top
- L_2 If $x \in \mathcal{V}$ and $C \in \mathcal{C}$, then $C[x]_p \in \mathfrak{L}$ if and only if $C[\Box]_p \in \mathfrak{L}$
- L_3 If $L, N \in \mathfrak{L}$, $p \in \mathcal{P}\mathrm{os}_{\mathcal{F}}(L)$, and $L|_p \sqcup N$ is defined, then $L[L|_p \sqcup N]_p \in \mathfrak{L}$

Layer Systems Definition

Definition

Under the following conditions, the TRS $\mathcal{R}$ is weakly layered wrt. $\mathfrak{L}$:

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- W If M is the max-top of s, $p \in \mathcal{P}\mathrm{os}_{\mathcal{F}}(M)$, and $s \to_{p,\ell\to r} t$ with $\ell \to r \in \mathcal{R}$, then $M \to_{p,\ell\to r} L$ for some $L \in \mathfrak{L}$

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- (W) respectively (C_1) are most difficult to formalize

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Challenges

Modular proof setup

- separate main results and applications
- avoid boilerplate

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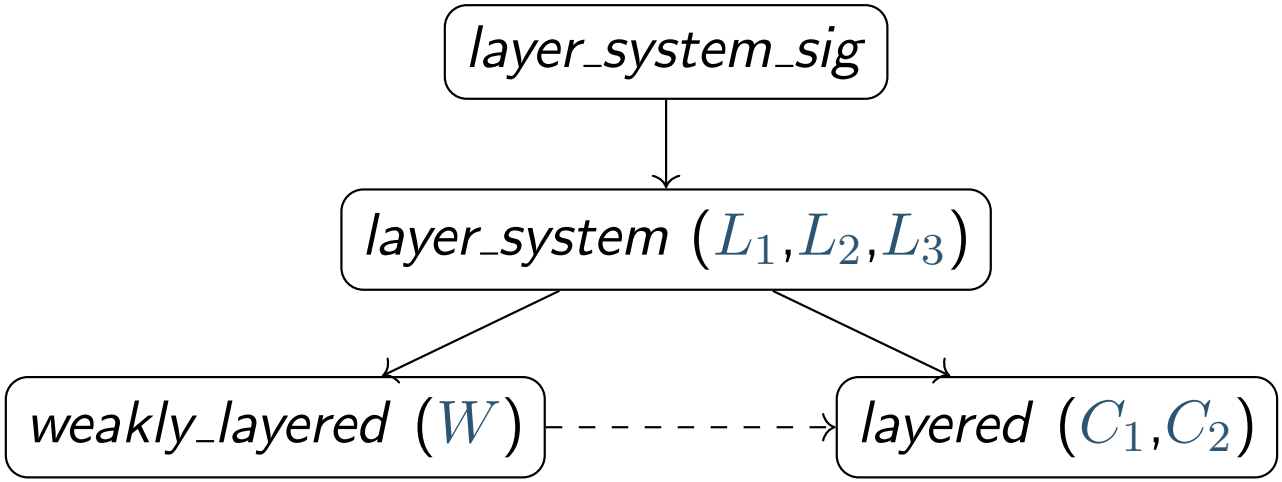
Intuition

- intuition, and pictures of terms, easily convince humans
- for formalization, algebraic properties are often required
- catch phrases: *"it's easy to see ..."*, *"obviously, ..."*

Isabelle locales



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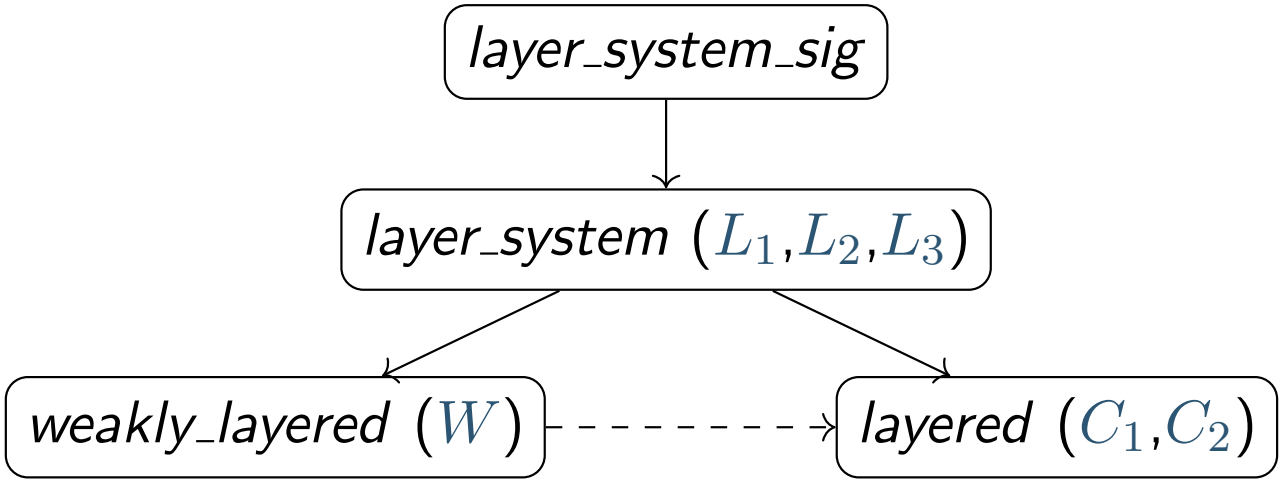

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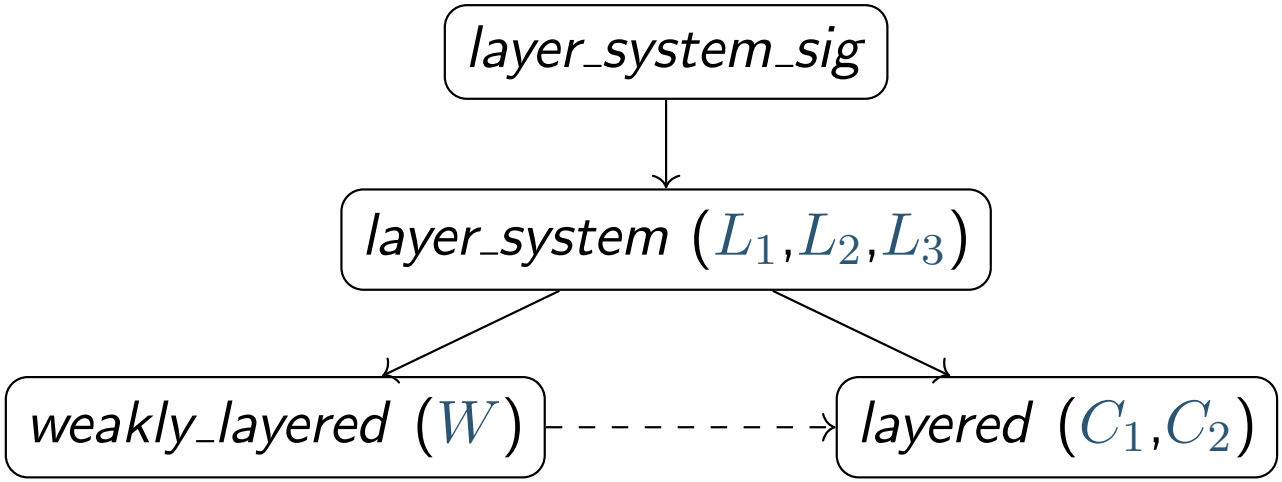

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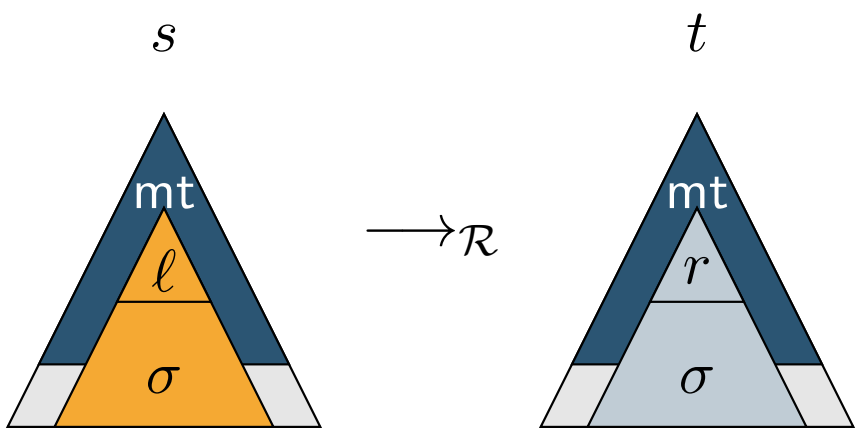
$\Rightarrow$ solves modularization challenge

Example of algebraization

Idea for C_1: (outer) steps can be simulated in max-top



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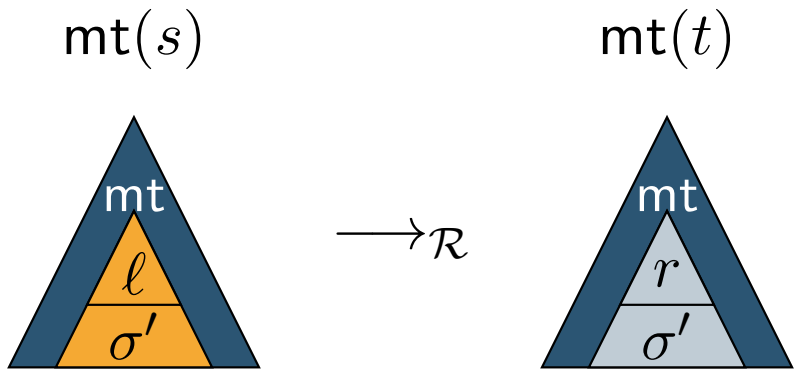

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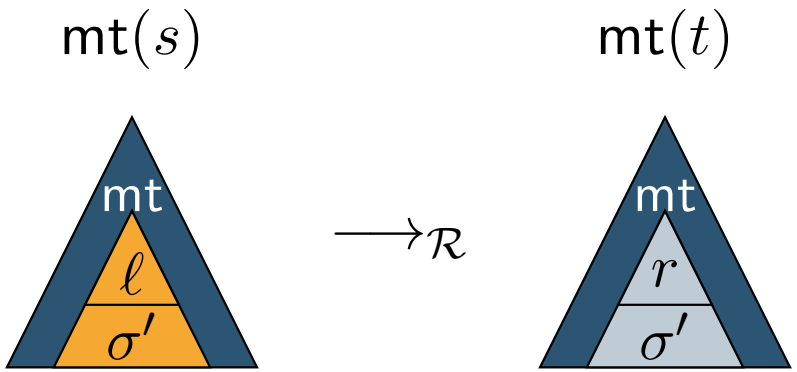

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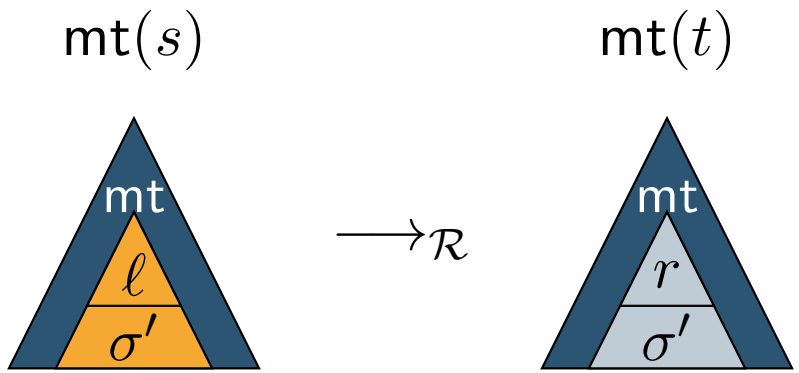

- algebraic computation:

$$\mathsf{mt}(s) = \mathsf{mt}'(C[\ell \cdot \sigma])$$

assuming $\mathsf{mt}(s) = \mathsf{mt}'(s)$, where mt' computes the maxtop with respect to a subset of the layers

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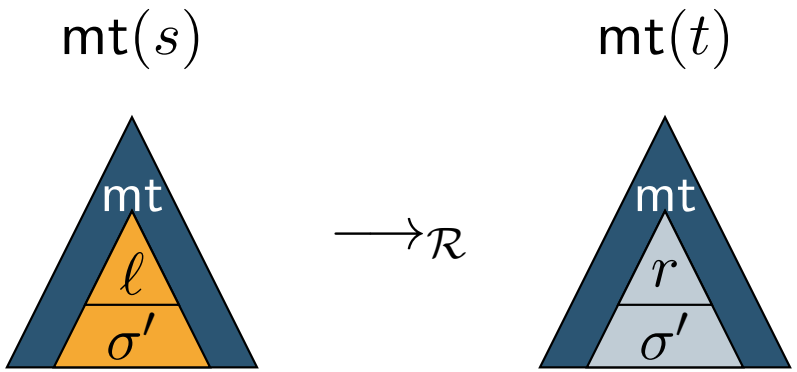

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$$\mathsf{mt}(s) = \mathsf{mt}'(C[\ell \cdot \sigma]) \stackrel{\star}{=} D[\mathsf{mt}'(\ell \cdot \sigma)]$$

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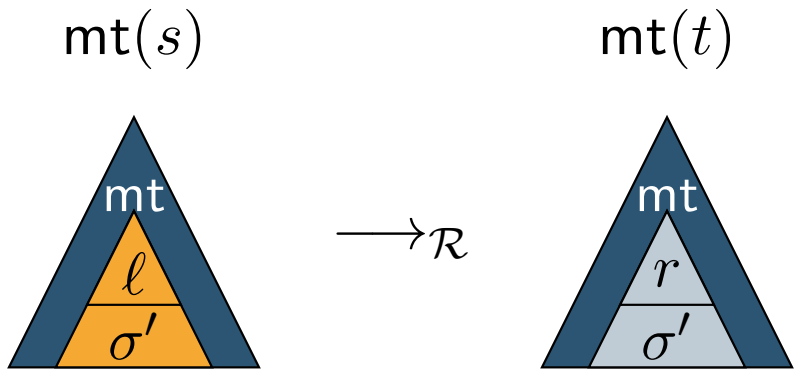

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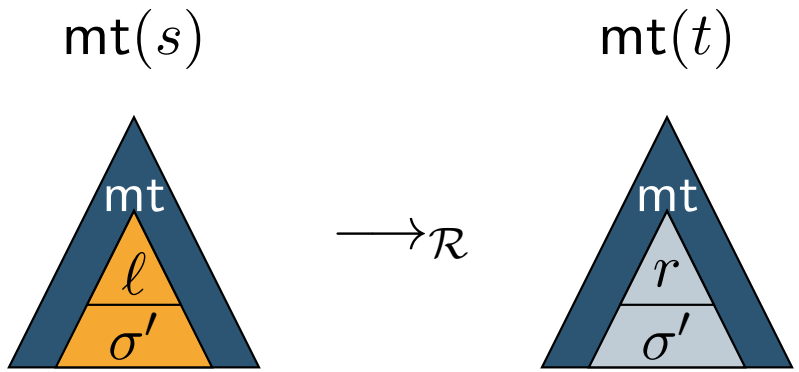

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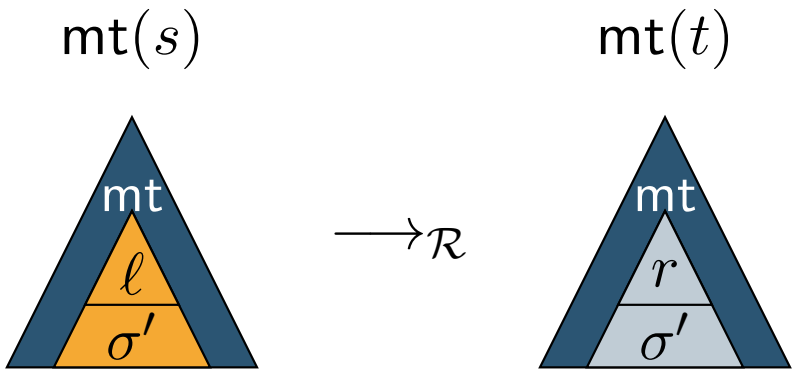

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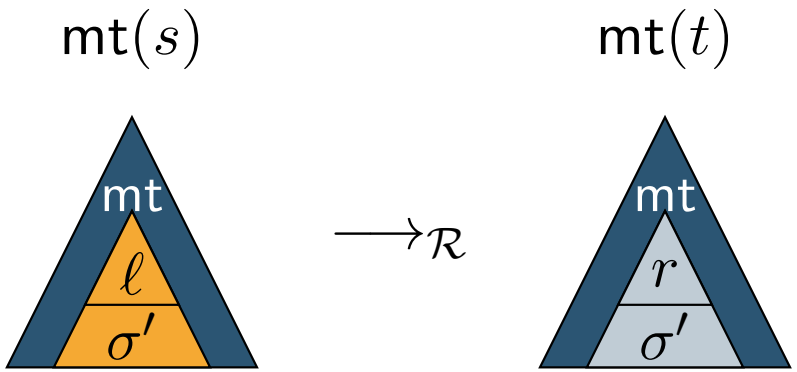

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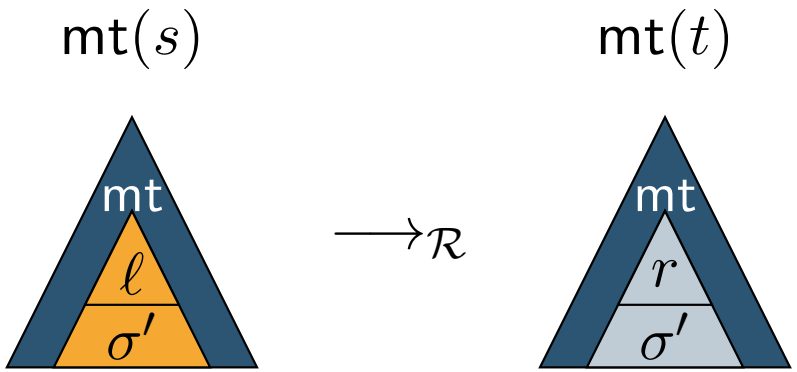

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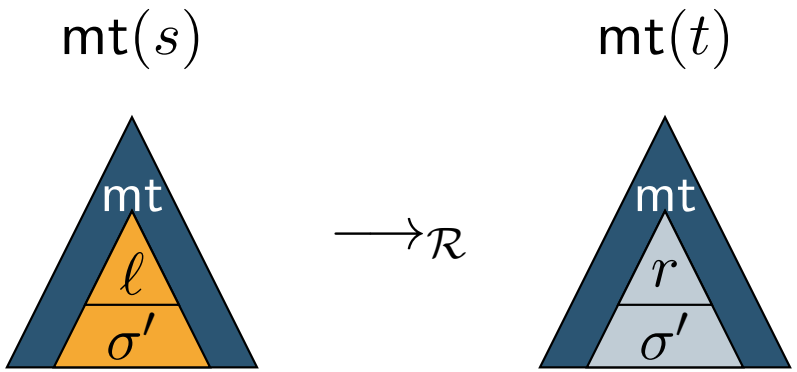

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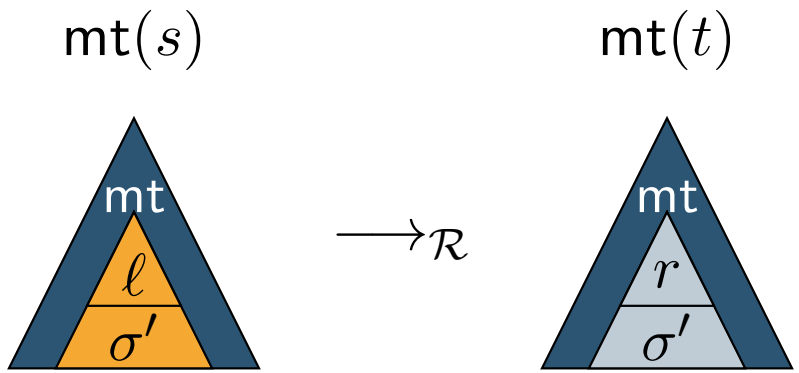

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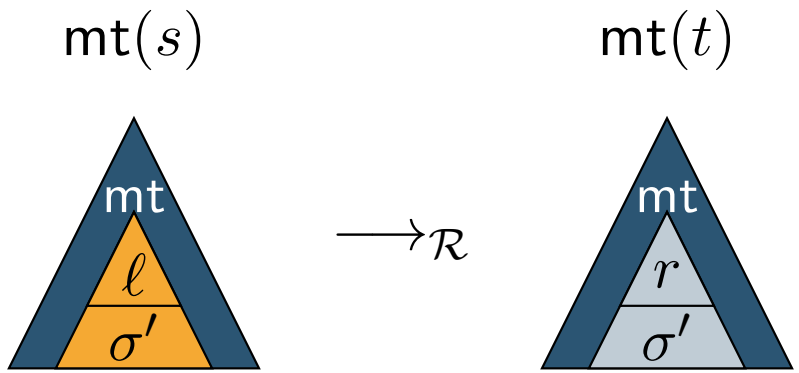
Example of algebraization

Idea for C_1: (outer) steps can be simulated in max-top



- algebraic computation:

$$\begin{aligned}\mathsf{mt}(s) = \mathsf{mt}'(C[\ell\cdot\sigma]) &\stackrel{\star}{=} D[\mathsf{mt}'(\ell\cdot\sigma)] \stackrel{\dagger}{=} D[\mathsf{mt}'(\ell)\cdot\sigma'] \\ &\stackrel{\blacksquare}{=} D[\ell\cdot\sigma'] \to_{p,\ell\to r} D[r\cdot\sigma'] \stackrel{\blacksquare}{=} D[\mathsf{mt}'(r)\cdot\sigma'] \\ &\stackrel{\dagger}{=} D[\mathsf{mt}'(r\cdot\sigma)] \stackrel{\star}{=} \mathsf{mt}'(C[r\cdot\sigma]) = \mathsf{mt}'(t)\end{aligned}$$

- $D = \mathsf{mt}'(C),\ \sigma' = \mathsf{mt}' \circ \sigma$

either $\mathsf{mt}'(t) = \mathsf{mt}(t)$, or $\mathsf{mt}'(t) = \Box$

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  - ◇ quasi-ground systems — future work

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Thanks!