Computational Logic

Critical Peaks Redefined
$\Phi \sqcup \Psi = \top$

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Integrating confluence-by-critical-pair criteria

Theorem (Huet)

term rewrite system is locally confluent if all critical pairs joinable

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abstract rewrite systems: Newman's Lemma and diamond property

integration: decreasing diagrams

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abstract rewrite systems: Newman's Lemma and diamond property

integration: decreasing diagrams

this talk: left-linear first-order term rewrite systems

Critical peak lemma

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a multi–multi peak either

- *is empty or critical; or*
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Assumption

- P set of multi–multi peaks closed under decomposition
- V set of valleys closed under (re)composition

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Theorem

if empty and critical peaks in P are in V, then all peaks in P are.

Proof.

by induction on size, using the assumption in the base case, and closure under decomposition and composition in the step case. □

Derecomposition in action

TRS

$$\begin{array}{rclrclrcl} a & \to & b & g(a) & \to & c & b & \to & d \\ f(g(x),y) & \to & h(x,y,y) & f(c,y) & \to & h(b,y,y) & & & \end{array}$$

Example (types of rewriting)

rewriting from term $t = g(f(g(a),a))$

- empty: $t = t$;
- one$^=$: $t \to g(f(g(b),a))$, $t \to g(f(c,a))$, $t \to g(h(a,a,a))$
- parallel: $t \mathrel{+\!\!\!\!\!\!\to} g(f(g(b),b))$, $t \mathrel{+\!\!\!\!\!\!\to} g(f(c,b))$
- multi: $t \mathrel{-\!\!\!\circ\!\!\!\to} g(h(b,a,a))$, $t \mathrel{-\!\!\!\circ\!\!\!\to} g(h(a,b,b))$
- many: $t \twoheadrightarrow g(f(g(d),a))$

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Example (de/recomposing peaks)

multi–parallel peak $g(h(b,a,a)) \mathrel{\leftarrow\!\circ\!-} g(f(g(a),a)) \mathrel{-\!\!\!\!+\!\!\!\!+\!\!\!\!\to} g(f(c,b))$

- empty peak $g(z) = g(z) = g(z)$; empty joinable
- multi–parallel peak $h(b,a,a) \mathrel{\leftarrow\!\circ\!-} f(g(a),a) \mathrel{-\!\!\!\!+\!\!\!\!+\!\!\!\!\to} f(c,b)$
  - empty–one peak $a = a \to b$; one–empty joinable
  - critical multi–one peak $h(b,u,u) \mathrel{\leftarrow\!\circ\!-} f(g(a),u) \to f(c,u)$; empty–one joinable (by rule $f(c,y) \to h(b,y,y)$)

  parallel–one joinable $h(b,a,a) \mathrel{-\!\!\!\!+\!\!\!\!+\!\!\!\!\to} h(b,b,b) \leftarrow f(c,b)$

parallel–one joinable $g(h(b,a,a)) \mathrel{-\!\!\!\!+\!\!\!\!+\!\!\!\!\to} g(h(b,b,b)) \leftarrow g(f(c,b))$

Corollaries to critical peak lemma

Corollary (Huet)

term rewrite system is locally confluent if all critical pairs joinable

Proof.

- P = set of all one$^=$–one$^=$ peaks
- V = set of all valleys

base case empty or ordinary (one–one) critical peak □

Corollaries to critical peak lemma

Corollary (Rosen)

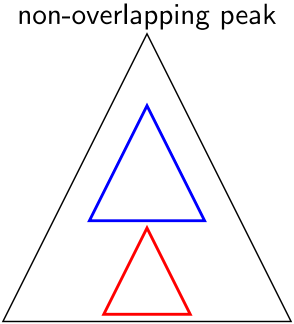
left-linear term rewrite system is confluent if it has no critical pairs

Proof.

- P = set of all multi–multi peaks
- V = set of all multi–multi valleys

only empty base case by assumption □

Pattern overlap intuition

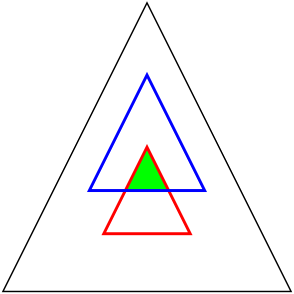


Example

$a \leftarrow f(g(g(b))) \rightarrow f(g(c))$ for $f(g(x)) \rightarrow a$ and $g(b) \rightarrow c$

Pattern overlap intuition

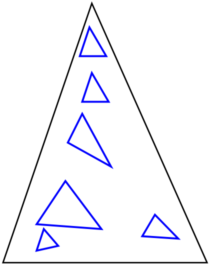
encompasses critical peak



Example

$h(a) \leftarrow h(f(g(b))) \rightarrow h(f(c))$ for $f(g(x)) \rightarrow a$ and $g(b) \rightarrow c$

Multiple patterns



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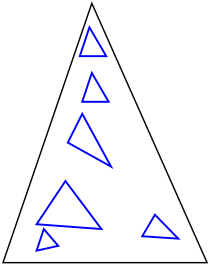
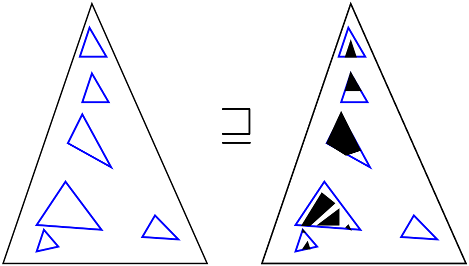


Definition (cluster)

term with multiple occurrences of patterns $t = M^{[\![\vec{X} := \vec{\ell}]\!]}$

- M is the skeleton; term linear in $\vec{X}$
- $\vec{X}$ is list of second-order variables; gaps
- $\vec{\ell}$ is list of patterns; non-var, linear first-order terms

Coarsening/refining clusters



coarser than order $\sqsupseteq$ (finer than $\sqsubseteq$) intuition: split and forget

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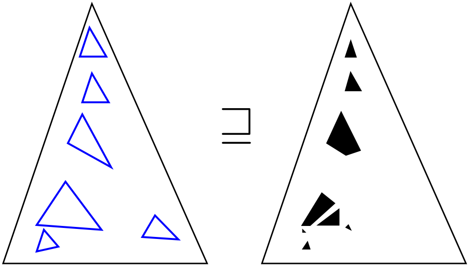

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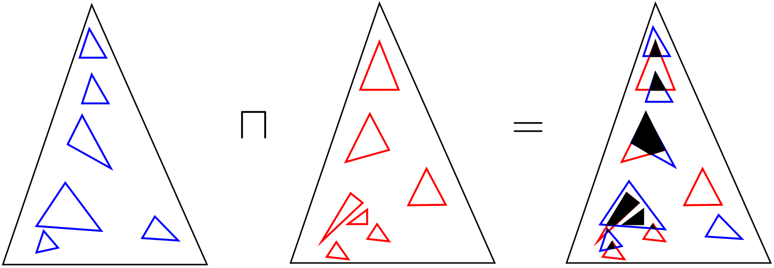
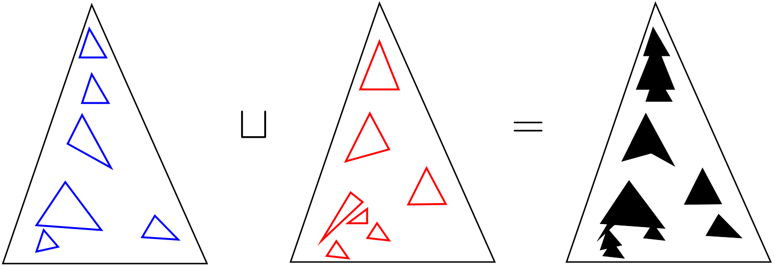
Meet of clusters



refinement order: $\varsigma \sqsubseteq \zeta$ iff $\varsigma = \varsigma \sqcap \zeta$

Join of clusters



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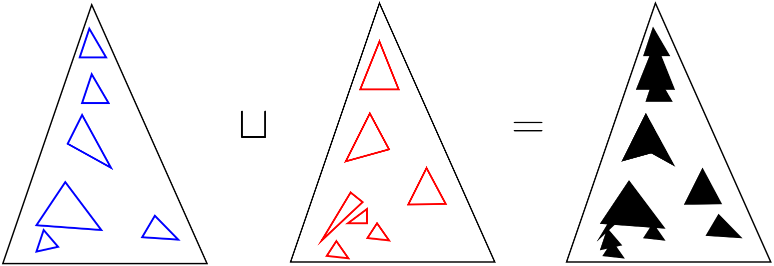

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$\bot$: term without patterns

$\top$: term one big pattern (except for root-edge, vars)

Definition

$(N,\beta) \sqsupseteq (M,\alpha)$ if $N^\gamma = M$ and $\beta = \alpha \circ \gamma$ for meta-substitution γ

Coarsening finite distributive lattice

Birkhoff's Fundamental Theorem for Distributive Lattices

a finite distributive lattice $\sqsubseteq$ is isomorphic to the $\subseteq$-lattice of downward closed sets of its join-irreducible elements

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node and edge positions are join-irreducible w.r.t. $\sqsubseteq$

Theorem

clusters are sets of positions that are downward-closed
(edge is larger than its endpoints/nodes)
$\sqsubseteq$ is finite distributive lattice isomorphic to $\subseteq$ (on sets of positions)

Redefining critical peaks via refinement

Lemma (Multisteps as clusters)

$t \multimap\!\!\to s$ iff $t = M^{[\![\vec{X}:=\vec{\ell}]\!]}$ and $M^{[\![\vec{X}:=\vec{r}]\!]} = s$, for rules $\overrightarrow{\ell \to r}$

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refinement extended to multisteps via left-hand side (t)

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Critical peak lemma

if $s \;{}_{\Phi}\!\leftarrow\!\!\multimap\; t \multimap\!\!\to_{\Psi} u$ then

- $\Phi \sqcup \Psi = \top$: empty or variable-instance of critical peak; or
- $\Phi \sqcup \Psi \neq \top$: $\Phi = \Phi_0^{[x:=\Phi_1]}$ and $\Psi = \Psi_0^{[x:=\Psi_1]}$, both smaller

More consequences of critical peak lemma

Corollary (Okui)

if multi–one critical peaks are many–multi joinable then confluent

Proof.

- P = set of all multi–one$^=$ peaks
- V = set of all many–multi valleys □

More consequences of critical peak lemma

Corollary (Gramlich, Toyama, Felgenhauer)

confluent if parallel–one critical peaks are many–parallel joinable

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parallel not composition closed; [Toyama,Gramlich] conditions □

More consequences of critical peak lemma

Corollary (Huet, Toyama, vO)

confluent if every inner–outer critical peak multi–empty joinable

Proof.

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multi–multi critical peaks split into such with less overlap
induction on amount of overlap; based on distributive lattice

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- critical peak redefinition as $\Phi \sqcup \Psi = \top$
- critical peak definitions in literature covered
  - one–one: Knuth–Bendix, Huet
  - parallel–one: Toyama, Gramlich
  - multi–one: Okui
  - multi–multi: Felgenhauer

Current and future work

- integrate with decreasing diagrams into HOT-criterion
  (same authors; work done at moment of FSCD deadline . . .)

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- investigate when finitely many critical multi–multi peaks
- investigate closure under (re)composition of decreasingness