Formalized Ground Completion

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Outline

- Introduction
- Correctness
- Completeness
- Formalization
- Conclusion

Definition (Abstract Completion)

set of equations $\mathcal{E}$ set of rewrite rules $\mathcal{R}$ reduction order $>$

inference system KB consists of eight rules

delete $$\frac{\mathcal{E} \uplus \{ s \approx s \}, \mathcal{R}}{\mathcal{E}, \mathcal{R}}$$

compose $$\frac{\mathcal{E}, \mathcal{R} \uplus \{ s \to t \}}{\mathcal{E}, \mathcal{R} \cup \{ s \to u \}}$$ if $t \to_{\mathcal{R}} u$

simplify $$\frac{\mathcal{E} \uplus \{ s \approx t \}, \mathcal{R}}{\mathcal{E} \cup \{ s \approx u \}, \mathcal{R}}$$ if $t \to_{\mathcal{R}} u$ $$\frac{\mathcal{E} \uplus \{ t \approx s \}, \mathcal{R}}{\mathcal{E} \cup \{ u \approx s \}, \mathcal{R}}$$

orient $$\frac{\mathcal{E} \uplus \{ s \approx t \}, \mathcal{R}}{\mathcal{E}, \mathcal{R} \cup \{ s \to t \}}$$ if $s > t$ $$\frac{\mathcal{E} \uplus \{ t \approx s \}, \mathcal{R}}{\mathcal{E}, \mathcal{R} \cup \{ s \to t \}}$$

collapse $$\frac{\mathcal{E}, \mathcal{R} \uplus \{ t \to s \}}{\mathcal{E} \cup \{ u \approx s \}, \mathcal{R}}$$ if $t \to_{\mathcal{R}} u$

deduce $$\frac{\mathcal{E}, \mathcal{R}}{\mathcal{E} \cup \{ s \approx t \}, \mathcal{R}}$$ if $s \; {}_{\mathcal{R}}{\leftarrow} \cdot \to_{\mathcal{R}} t$

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2. *every reduced ground rewrite system is canonical and can be obtained by* KB^- *completion from any equivalent set of ground equations*

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Our Contribution

first formalized proofs of these results in Isabelle/HOL, on top of IsaFoR

Example

$\mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{a}))) \approx \mathsf{f}(\mathsf{b}) \qquad \mathsf{f}(\mathsf{f}(\mathsf{b})) \approx \mathsf{c} \qquad \mathsf{f}(\mathsf{c}) \approx \mathsf{a} \qquad \mathsf{f}(\mathsf{a}) \approx \mathsf{f}(\mathsf{f}(\mathsf{b}))$

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$f(f(f(a))) \approx f(b) \qquad f(f(b)) \approx c \qquad f(c) \approx a \qquad f(a) \approx f(f(b))$

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orient $\quad \dfrac{\mathcal{E} \uplus \{ s \approx t \}, \mathcal{R}}{\mathcal{E}, \mathcal{R} \cup \{ s \to t \}} \qquad \dfrac{\mathcal{E} \uplus \{ t \approx s \}, \mathcal{R}}{\mathcal{E}, \mathcal{R} \cup \{ s \to t \}} \qquad$ if $s > t$

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- collapse applies

Outline

Lemma

for finite ground ESs $\mathcal{E}_0$ *there are no infinite derivations*

$$(\mathcal{E}_0, \varnothing) \vdash_{\mathsf{KB}^-} (\mathcal{E}_1, \mathcal{R}_1) \vdash_{\mathsf{KB}^-} \cdots$$

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Proof

- measure $P(\mathcal{E}, \mathcal{R}) = (M(\mathcal{E}, \mathcal{R}), |\mathcal{E}|)$ with

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Lemma

if $(\mathcal{E}, \mathcal{R}) \vdash^*_{\mathsf{KB}} (\mathcal{E}', \mathcal{R}')$ *and* $\mathcal{R} \subseteq >$ *then* $\xleftrightarrow[\mathcal{E} \cup \mathcal{R}]{*} \;=\; \xleftrightarrow[\mathcal{E}' \cup \mathcal{R}']{*}$ *and* $\mathcal{R}' \subseteq >$

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if reduction order $>$ *is total on* $\mathcal{E}$*-equivalent ground terms then every maximal* KB^- *run produces equivalent canonical presentation for ground* ES $\mathcal{E}$

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LPO and KBO based on total precedence are total on ground terms

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Theorem (van Oostrom, RTA 2007)

if TRS $\mathcal{R}$ *has random descent and* $s \xleftrightarrow{*} t$ *with normal form* t *then* s *is complete and all rewrite sequences from* s *to* t *have same length*

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TRS $\mathcal{R}$ consisting of single rule $\mathsf{f}(x) \to \mathsf{b}$ is complete presentation of $\mathcal{E}$

Outline

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- Completeness
- Formalization
- Conclusion

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