A Ground Joinability Criterion for Ordered Completion

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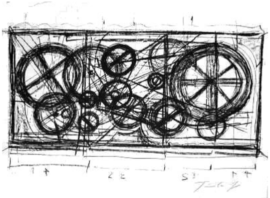
Computational Logic Group @ University of Innsbruck

6th International Workshop on Confluence, Oxford
September 8, 2017

Ordered Completion

$E_0 \longrightarrow$

$\succ \longrightarrow$

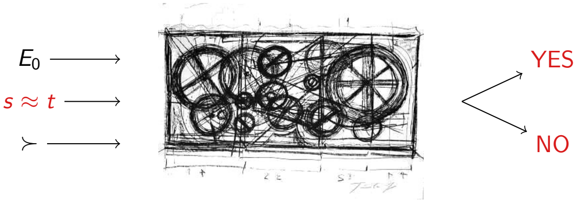


$\longrightarrow E$

ground-complete

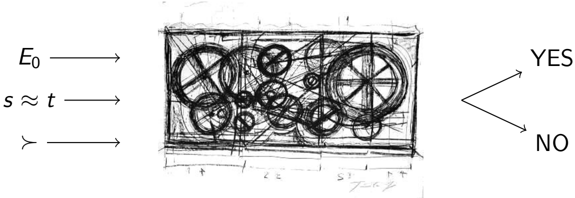
- $s \leftrightarrow^*_{E_0} t$ iff $s \downarrow_{E_\succ} t$ for all ground terms s, t

Ordered Completion



- if $s \leftrightarrow^*_{E_0} t$ then YES
- if $s \not\leftrightarrow^*_{E_0} t$ (and procedure terminates) then NO

Ordered Completion



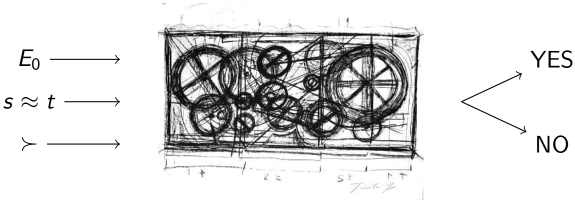
Example (Robbins Algebras are Boolean)

question raised in 1930s, proved by McCune in 1997:

$$x \vee y \approx y \vee x$$
$$(x \vee y) \vee z \approx x \vee (y \vee z)$$
$$\neg(\neg x \vee y) \vee \neg(\neg x \vee \neg y) \approx x$$

implies $\neg(\neg(x \vee y) \vee \neg(x \vee \neg y)) \approx x$

Ordered Completion



Example (Automatic Disproofs in Isabelle)

Definitions

- $E_{\succ} = \{s\sigma \rightarrow t\sigma \mid s \approx t \in E^{\pm} \text{ and } s\sigma \succ t\sigma\}$

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- Newman's Lemma + Extended Critical Pair Lemma:
  $E_\succ$ is $\mathcal{F}$-ground complete iff $\mathrm{CP}_\succ(E)$ are $\mathcal{F}$-ground joinable

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input signature $\mathcal{F}_0$ + infinite set of constants $\mathcal{C}$

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Example (Signature Matters)

$$\mathsf{f}(\mathsf{g}(\mathsf{f}(x))) \to \mathsf{g}(\mathsf{f}(\mathsf{g}(x))) \qquad \mathsf{f}(\mathsf{a}) \to \mathsf{a} \qquad \mathsf{g}(\mathsf{a}) \to \mathsf{a}$$

- terminating and ground confluent over $\mathcal{F}_0 = \{\mathsf{f}, \mathsf{g}, \mathsf{a}\}$

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- terminating and ground confluent over $\mathcal{F}_0 = \{\mathsf{f}, \mathsf{g}, \mathsf{a}\}$
- not ground confluent over $\mathcal{F}_\mathcal{C}$:

$$\mathsf{g}(\mathsf{f}(\mathsf{g}(\mathsf{g}(\mathsf{f}(\mathsf{c}))))) \leftarrow \mathsf{f}(\mathsf{g}(\mathsf{f}(\mathsf{g}(\mathsf{f}(\mathsf{c}))))) \to \mathsf{f}(\mathsf{g}(\mathsf{g}(\mathsf{f}(\mathsf{g}(\mathsf{c})))))$$

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Restriction

$\succ$ is ground-total reduction order

Ground Completeness Criteria

U. Martin and T. Nipkow
Ordered Rewriting and Confluence.
Proc. 10th CADE, 1990.

Ground Completeness Criteria

Example: Group of Exponent 2

Let $\succ$ be LPO with precedence $\mathsf{f} > 1$.
For following equations E:

$$\begin{array}{llll} \mathsf{f}(x,1) \to x & \mathsf{f}(1,x) \to x & \mathsf{f}(\mathsf{f}(x,y),z) \to \mathsf{f}(x,\mathsf{f}(y,z)) & \mathsf{f}(x,y) \approx \mathsf{f}(y,x) \\ \mathsf{f}(x,x) \to 1 & \mathsf{f}(x,\mathsf{f}(x,y)) \to y & \mathsf{f}(x,\mathsf{f}(y,z)) \approx \mathsf{f}(y,\mathsf{f}(x,z)) & \end{array}$$

$\mathsf{f}(x,\mathsf{f}(y,\mathsf{f}(x,z))) \approx \mathsf{f}(y,z)$ in $\mathrm{CP}_{\succ}(E)$

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- $x > y$

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- $x > y$ ✓

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$\mathsf{f}(x,\mathsf{f}(y,\mathsf{f}(x,z))) \approx \mathsf{f}(y,z)$ in $\mathrm{CP}_{\succ}(E)$ is ground joinable

- $x > y$ ✓
- $y > x$ ✓
- $x = y$ ✓

Ground Completeness Criteria

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Let $\succ$ be LPO with precedence $\mathsf{f} > 1$.
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Criterion (Martin, Nipkow 1990)

$E_\succ$ is ground-complete if $\mathrm{CP}_\succ(E) \subseteq \leftrightarrow_E \cup \downarrow_{E_{\succcurlyeq}}$
for all well-founded $\succcurlyeq\ \supseteq\ \succ$ such that $x \prec\!\!\!\prec y$, $x \succ\!\!\!\succ y$, or $x = y \quad \forall x, y \in \mathcal{V}$

Example: Abelian Groups

equations E with LPO $\succ$ for precedence $\mathsf{i} > \mathsf{f} > 1$

$$\mathsf{f}(\mathsf{i}(x), x) \to 1 \qquad \mathsf{f}(y, \mathsf{f}(\mathsf{i}(y), x)) \to x \qquad \mathsf{i}(1) \to 1$$

$$\mathsf{f}(x, \mathsf{i}(x)) \to 1 \qquad \mathsf{f}(\mathsf{f}(x, y), z) \to \mathsf{f}(x, \mathsf{f}(y, z)) \qquad \mathsf{i}(\mathsf{i}(x)) \to x$$

$$\mathsf{f}(\mathsf{i}(y), \mathsf{f}(y, x)) \to x \qquad \mathsf{i}(\mathsf{f}(x, y)) \to \mathsf{f}(\mathsf{i}(y), \mathsf{i}(x)) \qquad \mathsf{f}(x, 1) \to x$$

$$\mathsf{f}(\mathsf{i}(y), \mathsf{f}(x, y)) \to x \qquad \mathsf{f}(x, y) \approx \mathsf{f}(y, x) \qquad \mathsf{f}(1, x) \to x$$

$$\mathsf{f}(y, \mathsf{f}(x, \mathsf{i}(y))) \to x \qquad \mathsf{f}(x, \mathsf{f}(y, z)) \approx \mathsf{f}(y, \mathsf{f}(x, z))$$

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$$
\begin{array}{rrr}
\mathsf{f}(\mathsf{i}(x), x) \to \mathsf{1} & \mathsf{f}(y, \mathsf{f}(\mathsf{i}(y), x)) \to x & \mathsf{i}(\mathsf{1}) \to \mathsf{1} \\
\mathsf{f}(x, \mathsf{i}(x)) \to \mathsf{1} & \mathsf{f}(\mathsf{f}(x, y), z) \to \mathsf{f}(x, \mathsf{f}(y, z)) & \mathsf{i}(\mathsf{i}(x)) \to x \\
\mathsf{f}(\mathsf{i}(y), \mathsf{f}(y, x)) \to x & \mathsf{i}(\mathsf{f}(x, y)) \to \mathsf{f}(\mathsf{i}(y), \mathsf{i}(x)) & \mathsf{f}(x, \mathsf{1}) \to x \\
\mathsf{f}(\mathsf{i}(y), \mathsf{f}(x, y)) \to x & \mathsf{f}(x, y) \approx \mathsf{f}(y, x) & \mathsf{f}(\mathsf{1}, x) \to x \\
\mathsf{f}(y, \mathsf{f}(x, \mathsf{i}(y))) \to x & \mathsf{f}(x, \mathsf{f}(y, z)) \approx \mathsf{f}(y, \mathsf{f}(x, z)) &
\end{array}
$$

$\mathsf{f}(x, y) \approx \mathsf{f}(z, \mathsf{f}(x, \mathsf{f}(y, \mathsf{i}(z))))$ in $\mathrm{CP}_{\succ}(E)$

Example: Abelian Groups

equations E with LPO $\succ$ for precedence $\mathsf{i} > \mathsf{f} > 1$

$$\begin{array}{rcc}
\mathsf{f}(\mathsf{i}(x), x) \to 1 & \mathsf{f}(y, \mathsf{f}(\mathsf{i}(y), x)) \to x & \mathsf{i}(1) \to 1 \\
\mathsf{f}(x, \mathsf{i}(x)) \to 1 & \mathsf{f}(\mathsf{f}(x, y), z) \to \mathsf{f}(x, \mathsf{f}(y, z)) & \mathsf{i}(\mathsf{i}(x)) \to x \\
\mathsf{f}(\mathsf{i}(y), \mathsf{f}(y, x)) \to x & \mathsf{i}(\mathsf{f}(x, y)) \to \mathsf{f}(\mathsf{i}(y), \mathsf{i}(x)) & \mathsf{f}(x, 1) \to x \\
\mathsf{f}(\mathsf{i}(y), \mathsf{f}(x, y)) \to x & \mathsf{f}(x, y) \approx \mathsf{f}(y, x) & \mathsf{f}(1, x) \to x \\
\mathsf{f}(y, \mathsf{f}(x, \mathsf{i}(y))) \to x & \mathsf{f}(x, \mathsf{f}(y, z)) \approx \mathsf{f}(y, \mathsf{f}(x, z)) &
\end{array}$$

$\mathsf{f}(x, y) \approx \mathsf{f}(z, \mathsf{f}(x, \mathsf{f}(y, \mathsf{i}(z))))$ in $\mathrm{CP}_{\succ}(E)$

- $x > y > z$

Example: Abelian Groups

equations E with LPO $\succ$ for precedence $\mathsf{i} > \mathsf{f} > 1$

$$\begin{array}{rrr}
\mathsf{f}(\mathsf{i}(x), x) \to 1 & \mathsf{f}(y, \mathsf{f}(\mathsf{i}(y), x)) \to x & \mathsf{i}(1) \to 1 \\
\mathsf{f}(x, \mathsf{i}(x)) \to 1 & \mathsf{f}(\mathsf{f}(x, y), z) \to \mathsf{f}(x, \mathsf{f}(y, z)) & \mathsf{i}(\mathsf{i}(x)) \to x \\
\mathsf{f}(\mathsf{i}(y), \mathsf{f}(y, x)) \to x & \mathsf{i}(\mathsf{f}(x, y)) \to \mathsf{f}(\mathsf{i}(y), \mathsf{i}(x)) & \mathsf{f}(x, 1) \to x \\
\mathsf{f}(\mathsf{i}(y), \mathsf{f}(x, y)) \to x & \mathsf{f}(x, y) \approx \mathsf{f}(y, x) & \mathsf{f}(1, x) \to x \\
\mathsf{f}(y, \mathsf{f}(x, \mathsf{i}(y))) \to x & \mathsf{f}(x, \mathsf{f}(y, z)) \approx \mathsf{f}(y, \mathsf{f}(x, z)) &
\end{array}$$

$\mathsf{f}(x, y) \approx \mathsf{f}(z, \mathsf{f}(x, \mathsf{f}(y, \mathsf{i}(z))))$ in $\mathsf{CP}_{\succ}(E)$

- $x > y > z$

$$\mathsf{f}(z, \mathsf{f}(x, \mathsf{f}(y, \mathsf{i}(z)))) \to \mathsf{f}(z, \mathsf{f}(y, \mathsf{f}(x, \mathsf{i}(z))))$$

Example: Abelian Groups

equations E with LPO $\succ$ for precedence $\mathsf{i} > \mathsf{f} > 1$

$$\begin{array}{rcr} \mathsf{f}(\mathsf{i}(x),x) \to 1 & \mathsf{f}(y,\mathsf{f}(\mathsf{i}(y),x)) \to x & \mathsf{i}(1) \to 1 \\ \mathsf{f}(x,\mathsf{i}(x)) \to 1 & \mathsf{f}(\mathsf{f}(x,y),z) \to \mathsf{f}(x,\mathsf{f}(y,z)) & \mathsf{i}(\mathsf{i}(x)) \to x \\ \mathsf{f}(\mathsf{i}(y),\mathsf{f}(y,x)) \to x & \mathsf{i}(\mathsf{f}(x,y)) \to \mathsf{f}(\mathsf{i}(y),\mathsf{i}(x)) & \mathsf{f}(x,1) \to x \\ \mathsf{f}(\mathsf{i}(y),\mathsf{f}(x,y)) \to x & \mathsf{f}(x,y) \approx \mathsf{f}(y,x) & \mathsf{f}(1,x) \to x \\ \mathsf{f}(y,\mathsf{f}(x,\mathsf{i}(y))) \to x & \mathsf{f}(x,\mathsf{f}(y,z)) \approx \mathsf{f}(y,\mathsf{f}(x,z)) & \end{array}$$

$\mathsf{f}(x,y) \approx \mathsf{f}(z,\mathsf{f}(x,\mathsf{f}(y,\mathsf{i}(z))))$ in $\mathrm{CP}_{\succ}(E)$

- $x > y > z$ ✗

Example: Abelian Groups

equations E with LPO $\succ$ for precedence $\mathsf{i} > \mathsf{f} > 1$

$$
\begin{array}{rcr}
\mathsf{f}(\mathsf{i}(x),x) \to 1 & \mathsf{f}(y,\mathsf{f}(\mathsf{i}(y),x)) \to x & \mathsf{i}(1) \to 1 \\
\mathsf{f}(x,\mathsf{i}(x)) \to 1 & \mathsf{f}(\mathsf{f}(x,y),z) \to \mathsf{f}(x,\mathsf{f}(y,z)) & \mathsf{i}(\mathsf{i}(x)) \to x \\
\mathsf{f}(\mathsf{i}(y),\mathsf{f}(y,x)) \to x & \mathsf{i}(\mathsf{f}(x,y)) \to \mathsf{f}(\mathsf{i}(y),\mathsf{i}(x)) & \mathsf{f}(x,1) \to x \\
\mathsf{f}(\mathsf{i}(y),\mathsf{f}(x,y)) \to x & \mathsf{f}(x,y) \approx \mathsf{f}(y,x) & \mathsf{f}(1,x) \to x \\
\mathsf{f}(y,\mathsf{f}(x,\mathsf{i}(y))) \to x & \mathsf{f}(x,\mathsf{f}(y,z)) \approx \mathsf{f}(y,\mathsf{f}(x,z)) &
\end{array}
$$

$\mathsf{f}(x,y) \approx \mathsf{f}(z,\mathsf{f}(x,\mathsf{f}(y,\mathsf{i}(z))))$ in $\mathrm{CP}_{\succ}(E)$

- $x > y > z$ ✗

▶ $E_{\succ}$ ground complete, even for signature extended by constants $a_1, \ldots, a_n$

Example: Abelian Groups, Revisited

system $(E, \succ)$

$$
\begin{array}{rrr}
\mathsf{f}(\mathsf{i}(x), x) \to 1 & \mathsf{f}(y, \mathsf{f}(\mathsf{i}(y), x)) \to x & \mathsf{i}(1) \to 1 \\
\mathsf{f}(x, \mathsf{i}(x)) \to 1 & \mathsf{f}(\mathsf{f}(x, y), z) \to \mathsf{f}(x, \mathsf{f}(y, z)) & \mathsf{i}(\mathsf{i}(x)) \to x \\
\mathsf{f}(\mathsf{i}(y), \mathsf{f}(y, x)) \to x & \mathsf{i}(\mathsf{f}(x, y)) \to \mathsf{f}(\mathsf{i}(y), \mathsf{i}(x)) & \mathsf{f}(x, 1) \to x \\
\mathsf{f}(\mathsf{i}(y), \mathsf{f}(x, y)) \to x & \mathsf{f}(x, y) \approx \mathsf{f}(y, x) & \mathsf{f}(1, x) \to x \\
\mathsf{f}(y, \mathsf{f}(x, \mathsf{i}(y))) \to x & \mathsf{f}(x, \mathsf{f}(y, z)) \approx \mathsf{f}(y, \mathsf{f}(x, z)) &
\end{array}
$$

$\mathsf{f}(x, y) \approx \mathsf{f}(z, \mathsf{f}(x, \mathsf{f}(y, \mathsf{i}(z))))$ $(\star)$ in $\mathsf{CP}_{\succ}(E)$

Example: Abelian Groups, Revisited

system $(E, \succ)$

$$
\begin{array}{rrr}
f(i(x), x) \to 1 & f(y, f(i(y), x)) \to x & i(1) \to 1 \\
f(x, i(x)) \to 1 & f(f(x, y), z) \to f(x, f(y, z)) & i(i(x)) \to x \\
f(i(y), f(y, x)) \to x & i(f(x, y)) \to f(i(y), i(x)) & f(x, 1) \to x \\
f(i(y), f(x, y)) \to x & f(x, y) \approx f(y, x) & f(1, x) \to x \\
f(y, f(x, i(y))) \to x & f(x, f(y, z)) \approx f(y, f(x, z)) &
\end{array}
$$

$f(x, y) \approx f(z, f(x, f(y, i(z))))$ $(\star)$ in $CP_{\succ}(E)$

- $x > y > z$

Example: Abelian Groups, Revisited

system $(E, \succ)$

$$\begin{array}{rcl} \mathsf{f}(\mathsf{i}(x),x) \to 1 & \mathsf{f}(y,\mathsf{f}(\mathsf{i}(y),x)) \to x & \mathsf{i}(1) \to 1 \\ \mathsf{f}(x,\mathsf{i}(x)) \to 1 & \mathsf{f}(\mathsf{f}(x,y),z) \to \mathsf{f}(x,\mathsf{f}(y,z)) & \mathsf{i}(\mathsf{i}(x)) \to x \\ \mathsf{f}(\mathsf{i}(y),\mathsf{f}(y,x)) \to x & \mathsf{i}(\mathsf{f}(x,y)) \to \mathsf{f}(\mathsf{i}(y),\mathsf{i}(x)) & \mathsf{f}(x,1) \to x \\ \mathsf{f}(\mathsf{i}(y),\mathsf{f}(x,y)) \to x & \mathsf{f}(x,y) \approx \mathsf{f}(y,x) & \mathsf{f}(1,x) \to x \\ \mathsf{f}(y,\mathsf{f}(x,\mathsf{i}(y))) \to x & \mathsf{f}(x,\mathsf{f}(y,z)) \approx \mathsf{f}(y,\mathsf{f}(x,z)) & \end{array}$$

$\mathsf{f}(x,y) \approx \mathsf{f}(z,\mathsf{f}(x,\mathsf{f}(y,\mathsf{i}(z))))$ $(\star)$ in $\mathrm{CP}_{\succ}(E)$

- $x > y > \mathsf{f}(z_1, z_2)$

  $\mathsf{f}(\mathsf{f}(z_1,z_2),\mathsf{f}(x,\mathsf{f}(y,\mathsf{i}(\mathsf{f}(z_1,z_2)))))$

Example: Abelian Groups, Revisited

system $(E, \succ)$

$$
\begin{array}{rcr}
\mathsf{f}(\mathsf{i}(x), x) \to 1 & \mathsf{f}(y, \mathsf{f}(\mathsf{i}(y), x)) \to x & \mathsf{i}(1) \to 1 \\
\mathsf{f}(x, \mathsf{i}(x)) \to 1 & \mathsf{f}(\mathsf{f}(x, y), z) \to \mathsf{f}(x, \mathsf{f}(y, z)) & \mathsf{i}(\mathsf{i}(x)) \to x \\
\mathsf{f}(\mathsf{i}(y), \mathsf{f}(y, x)) \to x & \mathsf{i}(\mathsf{f}(x, y)) \to \mathsf{f}(\mathsf{i}(y), \mathsf{i}(x)) & \mathsf{f}(x, 1) \to x \\
\mathsf{f}(\mathsf{i}(y), \mathsf{f}(x, y)) \to x & \mathsf{f}(x, y) \approx \mathsf{f}(y, x) & \mathsf{f}(1, x) \to x \\
\mathsf{f}(y, \mathsf{f}(x, \mathsf{i}(y))) \to x & \mathsf{f}(x, \mathsf{f}(y, z)) \approx \mathsf{f}(y, \mathsf{f}(x, z)) &
\end{array}
$$

$\mathsf{f}(x, y) \approx \mathsf{f}(z, \mathsf{f}(x, \mathsf{f}(y, \mathsf{i}(z))))$ $(\star)$ in $\mathrm{CP}_{\succ}(E)$

- $x > y > \mathsf{f}(z_1, z_2)$

$$\mathsf{f}(\mathsf{f}(z_1, z_2), \mathsf{f}(x, \mathsf{f}(y, \mathsf{i}(\mathsf{f}(z_1, z_2))))) \to \mathsf{f}(z_1, \mathsf{f}(z_2, \mathsf{f}(x, \mathsf{f}(y, \mathsf{i}(\mathsf{f}(z_1, z_2))))))$$

Example: Abelian Groups, Revisited

system $(E, \succ)$

$$\begin{array}{rrr}
\mathsf{f}(\mathsf{i}(x), x) \to 1 & \mathsf{f}(y, \mathsf{f}(\mathsf{i}(y), x)) \to x & \mathsf{i}(1) \to 1 \\
\mathsf{f}(x, \mathsf{i}(x)) \to 1 & \mathsf{f}(\mathsf{f}(x, y), z) \to \mathsf{f}(x, \mathsf{f}(y, z)) & \mathsf{i}(\mathsf{i}(x)) \to x \\
\mathsf{f}(\mathsf{i}(y), \mathsf{f}(y, x)) \to x & \mathsf{i}(\mathsf{f}(x, y)) \to \mathsf{f}(\mathsf{i}(y), \mathsf{i}(x)) & \mathsf{f}(x, 1) \to x \\
\mathsf{f}(\mathsf{i}(y), \mathsf{f}(x, y)) \to x & \mathsf{f}(x, y) \approx \mathsf{f}(y, x) & \mathsf{f}(1, x) \to x \\
\mathsf{f}(y, \mathsf{f}(x, \mathsf{i}(y))) \to x & \mathsf{f}(x, \mathsf{f}(y, z)) \approx \mathsf{f}(y, \mathsf{f}(x, z)) &
\end{array}$$

suppose these constraints are satisfiable

$\mathsf{f}(x, y) \approx \mathsf{f}(z, \mathsf{f}(x, \mathsf{f}(y, \mathsf{i}(z))))$ $(\star)$ in $\mathrm{CP}_{\succ}(E)$

- $x > y > \mathsf{f}(z_1, z_2) > \mathsf{i}(z_1), \mathsf{i}(z_2)$

$$\begin{aligned}
\mathsf{f}(\mathsf{f}(z_1, z_2), \mathsf{f}(x, \mathsf{f}(y, \mathsf{i}(\mathsf{f}(z_1, z_2))))) &\to \mathsf{f}(z_1, \mathsf{f}(z_2, \mathsf{f}(x, \mathsf{f}(y, \mathsf{i}(\mathsf{f}(z_1, z_2)))))) \\
&\to \mathsf{f}(z_1, \mathsf{f}(z_2, \mathsf{f}(x, \mathsf{f}(y, \mathsf{f}(\mathsf{i}(z_1), \mathsf{i}(z_2))))))
\end{aligned}$$

Example: Abelian Groups, Revisited

system $(E, \succ)$

$$\begin{array}{rcc}
\mathsf{f}(\mathsf{i}(x), x) \to 1 & \mathsf{f}(y, \mathsf{f}(\mathsf{i}(y), x)) \to x & \mathsf{i}(1) \to 1 \\
\mathsf{f}(x, \mathsf{i}(x)) \to 1 & \mathsf{f}(\mathsf{f}(x, y), z) \to \mathsf{f}(x, \mathsf{f}(y, z)) & \mathsf{i}(\mathsf{i}(x)) \to x \\
\mathsf{f}(\mathsf{i}(y), \mathsf{f}(y, x)) \to x & \mathsf{i}(\mathsf{f}(x, y)) \to \mathsf{f}(\mathsf{i}(y), \mathsf{i}(x)) & \mathsf{f}(x, 1) \to x \\
\mathsf{f}(\mathsf{i}(y), \mathsf{f}(x, y)) \to x & \mathsf{f}(x, y) \approx \mathsf{f}(y, x) & \mathsf{f}(1, x) \to x \\
\mathsf{f}(y, \mathsf{f}(x, \mathsf{i}(y))) \to x & \mathsf{f}(x, \mathsf{f}(y, z)) \approx \mathsf{f}(y, \mathsf{f}(x, z)) &
\end{array}$$

$\mathsf{f}(x, y) \approx \mathsf{f}(z, \mathsf{f}(x, \mathsf{f}(y, \mathsf{i}(z))))$ $(\star)$ in $\mathrm{CP}_{\succ}(E)$

- $x > y > \mathsf{f}(z_1, z_2) > \mathsf{i}(z_1), \mathsf{i}(z_2)$

$$\begin{aligned}
\mathsf{f}(\mathsf{f}(z_1, z_2), \mathsf{f}(x, \mathsf{f}(y, \mathsf{i}(\mathsf{f}(z_1, z_2))))) &\to \mathsf{f}(z_1, \mathsf{f}(z_2, \mathsf{f}(x, \mathsf{f}(y, \mathsf{i}(\mathsf{f}(z_1, z_2)))))) \\
&\to \mathsf{f}(z_1, \mathsf{f}(z_2, \mathsf{f}(x, \mathsf{f}(y, \mathsf{f}(\mathsf{i}(z_1), \mathsf{i}(z_2)))))) \\
&\to^{+} \mathsf{f}(z_1, \mathsf{f}(z_2, \mathsf{f}(\mathsf{i}(z_2), \mathsf{f}(\mathsf{i}(z_1), \mathsf{f}(x, y)))))
\end{aligned}$$

Example: Abelian Groups, Revisited

system $(E, \succ)$

$$\begin{array}{rcr} \mathsf{f}(\mathsf{i}(x), x) \to 1 & \mathsf{f}(y, \mathsf{f}(\mathsf{i}(y), x)) \to x & \mathsf{i}(1) \to 1 \\ \mathsf{f}(x, \mathsf{i}(x)) \to 1 & \mathsf{f}(\mathsf{f}(x, y), z) \to \mathsf{f}(x, \mathsf{f}(y, z)) & \mathsf{i}(\mathsf{i}(x)) \to x \\ \mathsf{f}(\mathsf{i}(y), \mathsf{f}(y, x)) \to x & \mathsf{i}(\mathsf{f}(x, y)) \to \mathsf{f}(\mathsf{i}(y), \mathsf{i}(x)) & \mathsf{f}(x, 1) \to x \\ \mathsf{f}(\mathsf{i}(y), \mathsf{f}(x, y)) \to x & \mathsf{f}(x, y) \approx \mathsf{f}(y, x) & \mathsf{f}(1, x) \to x \\ \mathsf{f}(y, \mathsf{f}(x, \mathsf{i}(y))) \to x & \mathsf{f}(x, \mathsf{f}(y, z)) \approx \mathsf{f}(y, \mathsf{f}(x, z)) & \end{array}$$

$\mathsf{f}(x, y) \approx \mathsf{f}(z, \mathsf{f}(x, \mathsf{f}(y, \mathsf{i}(z))))$ $(\star)$ in $\mathrm{CP}_\succ(E)$

- $x > y > \mathsf{f}(z_1, z_2) > \mathsf{i}(z_1), \mathsf{i}(z_2)$

$$\begin{aligned} \mathsf{f}(\mathsf{f}(z_1, z_2), \mathsf{f}(x, \mathsf{f}(y, \mathsf{i}(\mathsf{f}(z_1, z_2))))) &\to \mathsf{f}(z_1, \mathsf{f}(z_2, \mathsf{f}(x, \mathsf{f}(y, \mathsf{i}(\mathsf{f}(z_1, z_2)))))) \\ &\to \mathsf{f}(z_1, \mathsf{f}(z_2, \mathsf{f}(x, \mathsf{f}(y, \mathsf{f}(\mathsf{i}(z_1), \mathsf{i}(z_2)))))) \\ &\to^+ \mathsf{f}(z_1, \mathsf{f}(z_2, \mathsf{f}(\mathsf{i}(z_2), \mathsf{f}(\mathsf{i}(z_1), \mathsf{f}(x, y))))) \\ &\to \mathsf{f}(z_1, \mathsf{f}(\mathsf{i}(z_1), \mathsf{f}(x, y))) \end{aligned}$$

Example: Abelian Groups, Revisited

system $(E, \succ)$

$$
\begin{array}{rcr}
\mathsf{f}(\mathsf{i}(x), x) \to 1 & \mathsf{f}(y, \mathsf{f}(\mathsf{i}(y), x)) \to x & \mathsf{i}(1) \to 1 \\
\mathsf{f}(x, \mathsf{i}(x)) \to 1 & \mathsf{f}(\mathsf{f}(x, y), z) \to \mathsf{f}(x, \mathsf{f}(y, z)) & \mathsf{i}(\mathsf{i}(x)) \to x \\
\mathsf{f}(\mathsf{i}(y), \mathsf{f}(y, x)) \to x & \mathsf{i}(\mathsf{f}(x, y)) \to \mathsf{f}(\mathsf{i}(y), \mathsf{i}(x)) & \mathsf{f}(x, 1) \to x \\
\mathsf{f}(\mathsf{i}(y), \mathsf{f}(x, y)) \to x & \mathsf{f}(x, y) \approx \mathsf{f}(y, x) & \mathsf{f}(1, x) \to x \\
\mathsf{f}(y, \mathsf{f}(x, \mathsf{i}(y))) \to x & \mathsf{f}(x, \mathsf{f}(y, z)) \approx \mathsf{f}(y, \mathsf{f}(x, z)) &
\end{array}
$$

$\mathsf{f}(x, y) \approx \mathsf{f}(z, \mathsf{f}(x, \mathsf{f}(y, \mathsf{i}(z))))$ $(\star)$ in $\mathrm{CP}_{\succ}(E)$

- $x > y > \mathsf{f}(z_1, z_2) > \mathsf{i}(z_1), \mathsf{i}(z_2)$ ✓

$$
\begin{aligned}
\mathsf{f}(\mathsf{f}(z_1, z_2), \mathsf{f}(x, \mathsf{f}(y, \mathsf{i}(\mathsf{f}(z_1, z_2))))) &\to \mathsf{f}(z_1, \mathsf{f}(z_2, \mathsf{f}(x, \mathsf{f}(y, \mathsf{i}(\mathsf{f}(z_1, z_2)))))) \\
&\to \mathsf{f}(z_1, \mathsf{f}(z_2, \mathsf{f}(x, \mathsf{f}(y, \mathsf{f}(\mathsf{i}(z_1), \mathsf{i}(z_2)))))) \\
&\to^{+} \mathsf{f}(z_1, \mathsf{f}(z_2, \mathsf{f}(\mathsf{i}(z_2), \mathsf{f}(\mathsf{i}(z_1), \mathsf{f}(x, y))))) \\
&\to \mathsf{f}(z_1, \mathsf{f}(\mathsf{i}(z_1), \mathsf{f}(x, y))) \\
&\to \mathsf{f}(x, y)
\end{aligned}
$$

Example: Abelian Groups, Revisited

system $(E, \succ)$

$$
\begin{array}{rcr}
\mathsf{f}(\mathsf{i}(x), x) \to 1 & \mathsf{f}(y, \mathsf{f}(\mathsf{i}(y), x)) \to x & \mathsf{i}(1) \to 1 \\
\mathsf{f}(x, \mathsf{i}(x)) \to 1 & \mathsf{f}(\mathsf{f}(x, y), z) \to \mathsf{f}(x, \mathsf{f}(y, z)) & \mathsf{i}(\mathsf{i}(x)) \to x \\
\mathsf{f}(\mathsf{i}(y), \mathsf{f}(y, x)) \to x & \mathsf{i}(\mathsf{f}(x, y)) \to \mathsf{f}(\mathsf{i}(y), \mathsf{i}(x)) & \mathsf{f}(x, 1) \to x \\
\mathsf{f}(\mathsf{i}(y), \mathsf{f}(x, y)) \to x & \mathsf{f}(x, y) \approx \mathsf{f}(y, x) & \mathsf{f}(1, x) \to x \\
\mathsf{f}(y, \mathsf{f}(x, \mathsf{i}(y))) \to x & \mathsf{f}(x, \mathsf{f}(y, z)) \approx \mathsf{f}(y, \mathsf{f}(x, z)) &
\end{array}
$$

$\mathsf{f}(x, y) \approx \mathsf{f}(z, \mathsf{f}(x, \mathsf{f}(y, \mathsf{i}(z))))$ $(\star)$ in $\mathrm{CP}_{\succ}(E)$

- $x > y > \mathsf{f}(z_1, z_2) > \mathsf{i}(z_1), \mathsf{i}(z_2)$ ✓
- $x > y > \mathsf{i}(z_1)$

  $$\mathsf{f}(\mathsf{i}(z_1), \mathsf{f}(x, \mathsf{f}(y, \mathsf{i}(\mathsf{i}(z_1)))))$$

Example: Abelian Groups, Revisited

system $(E, \succ)$

$$\begin{array}{rcr} \mathsf{f}(\mathsf{i}(x), x) \to 1 & \mathsf{f}(y, \mathsf{f}(\mathsf{i}(y), x)) \to x & \mathsf{i}(1) \to 1 \\ \mathsf{f}(x, \mathsf{i}(x)) \to 1 & \mathsf{f}(\mathsf{f}(x, y), z) \to \mathsf{f}(x, \mathsf{f}(y, z)) & \mathsf{i}(\mathsf{i}(x)) \to x \\ \mathsf{f}(\mathsf{i}(y), \mathsf{f}(y, x)) \to x & \mathsf{i}(\mathsf{f}(x, y)) \to \mathsf{f}(\mathsf{i}(y), \mathsf{i}(x)) & \mathsf{f}(x, 1) \to x \\ \mathsf{f}(\mathsf{i}(y), \mathsf{f}(x, y)) \to x & \mathsf{f}(x, y) \approx \mathsf{f}(y, x) & \mathsf{f}(1, x) \to x \\ \mathsf{f}(y, \mathsf{f}(x, \mathsf{i}(y))) \to x & \mathsf{f}(x, \mathsf{f}(y, z)) \approx \mathsf{f}(y, \mathsf{f}(x, z)) & \end{array}$$

$\mathsf{f}(x, y) \approx \mathsf{f}(z, \mathsf{f}(x, \mathsf{f}(y, \mathsf{i}(z))))$ $(\star)$ in $\mathrm{CP}_{\succ}(E)$

- $x > y > \mathsf{f}(z_1, z_2) > \mathsf{i}(z_1), \mathsf{i}(z_2)$ ✓
- $x > y > \mathsf{i}(z_1) > z_1$

$$\mathsf{f}(\mathsf{i}(z_1), \mathsf{f}(x, \mathsf{f}(y, \mathsf{i}(\mathsf{i}(z_1))))) \to \mathsf{f}(\mathsf{i}(z_1), \mathsf{f}(x, \mathsf{f}(y, z_1)))$$

Example: Abelian Groups, Revisited

system $(E, \succ)$

$$\begin{array}{rcr}
\mathsf{f}(\mathsf{i}(x), x) \to 1 & \mathsf{f}(y, \mathsf{f}(\mathsf{i}(y), x)) \to x & \mathsf{i}(1) \to 1 \\
\mathsf{f}(x, \mathsf{i}(x)) \to 1 & \mathsf{f}(\mathsf{f}(x, y), z) \to \mathsf{f}(x, \mathsf{f}(y, z)) & \mathsf{i}(\mathsf{i}(x)) \to x \\
\mathsf{f}(\mathsf{i}(y), \mathsf{f}(y, x)) \to x & \mathsf{i}(\mathsf{f}(x, y)) \to \mathsf{f}(\mathsf{i}(y), \mathsf{i}(x)) & \mathsf{f}(x, 1) \to x \\
\mathsf{f}(\mathsf{i}(y), \mathsf{f}(x, y)) \to x & \mathsf{f}(x, y) \approx \mathsf{f}(y, x) & \mathsf{f}(1, x) \to x \\
\mathsf{f}(y, \mathsf{f}(x, \mathsf{i}(y))) \to x & \mathsf{f}(x, \mathsf{f}(y, z)) \approx \mathsf{f}(y, \mathsf{f}(x, z)) &
\end{array}$$

$\mathsf{f}(x, y) \approx \mathsf{f}(z, \mathsf{f}(x, \mathsf{f}(y, \mathsf{i}(z))))$ $(\star)$ in $\mathrm{CP}_{\succ}(E)$

- $x > y > \mathsf{f}(z_1, z_2) > \mathsf{i}(z_1), \mathsf{i}(z_2)$ ✓
- $x > y > \mathsf{i}(z_1) > z_1$

$$\begin{aligned}
\mathsf{f}(\mathsf{i}(z_1), \mathsf{f}(x, \mathsf{f}(y, \mathsf{i}(\mathsf{i}(z_1))))) &\to \mathsf{f}(\mathsf{i}(z_1), \mathsf{f}(x, \mathsf{f}(y, z_1))) \\
&\to \mathsf{f}(\mathsf{i}(z_1), \mathsf{f}(x, \mathsf{f}(z_1, y)))
\end{aligned}$$

Example: Abelian Groups, Revisited

system $(E, \succ)$

$$
\begin{array}{ccc}
\mathsf{f}(\mathsf{i}(x), x) \to 1 & \mathsf{f}(y, \mathsf{f}(\mathsf{i}(y), x)) \to x & \mathsf{i}(1) \to 1 \\
\mathsf{f}(x, \mathsf{i}(x)) \to 1 & \mathsf{f}(\mathsf{f}(x, y), z) \to \mathsf{f}(x, \mathsf{f}(y, z)) & \mathsf{i}(\mathsf{i}(x)) \to x \\
\mathsf{f}(\mathsf{i}(y), \mathsf{f}(y, x)) \to x & \mathsf{i}(\mathsf{f}(x, y)) \to \mathsf{f}(\mathsf{i}(y), \mathsf{i}(x)) & \mathsf{f}(x, 1) \to x \\
\mathsf{f}(\mathsf{i}(y), \mathsf{f}(x, y)) \to x & \mathsf{f}(x, y) \approx \mathsf{f}(y, x) & \mathsf{f}(1, x) \to x \\
\mathsf{f}(y, \mathsf{f}(x, \mathsf{i}(y))) \to x & \mathsf{f}(x, \mathsf{f}(y, z)) \approx \mathsf{f}(y, \mathsf{f}(x, z)) &
\end{array}
$$

$\mathsf{f}(x, y) \approx \mathsf{f}(z, \mathsf{f}(x, \mathsf{f}(y, \mathsf{i}(z))))$ $(\star)$ in $\mathrm{CP}_{\succ}(E)$

- $x > y > \mathsf{f}(z_1, z_2) > \mathsf{i}(z_1), \mathsf{i}(z_2)$ ✓
- $x > y > \mathsf{i}(z_1) > z_1$

$$
\begin{aligned}
\mathsf{f}(\mathsf{i}(z_1), \mathsf{f}(x, \mathsf{f}(y, \mathsf{i}(\mathsf{i}(z_1))))) &\to \mathsf{f}(\mathsf{i}(z_1), \mathsf{f}(x, \mathsf{f}(y, z_1))) \\
&\to \mathsf{f}(\mathsf{i}(z_1), \mathsf{f}(x, \mathsf{f}(z_1, y))) \\
&\to \mathsf{f}(\mathsf{i}(z_1), \mathsf{f}(z_1, \mathsf{f}(x, y)))
\end{aligned}
$$

Example: Abelian Groups, Revisited

system $(E, \succ)$

$$
\begin{array}{rrr}
\mathsf{f}(\mathsf{i}(x), x) \to 1 & \mathsf{f}(y, \mathsf{f}(\mathsf{i}(y), x)) \to x & \mathsf{i}(1) \to 1 \\
\mathsf{f}(x, \mathsf{i}(x)) \to 1 & \mathsf{f}(\mathsf{f}(x, y), z) \to \mathsf{f}(x, \mathsf{f}(y, z)) & \mathsf{i}(\mathsf{i}(x)) \to x \\
\mathsf{f}(\mathsf{i}(y), \mathsf{f}(y, x)) \to x & \mathsf{i}(\mathsf{f}(x, y)) \to \mathsf{f}(\mathsf{i}(y), \mathsf{i}(x)) & \mathsf{f}(x, 1) \to x \\
\mathsf{f}(\mathsf{i}(y), \mathsf{f}(x, y)) \to x & \mathsf{f}(x, y) \approx \mathsf{f}(y, x) & \mathsf{f}(1, x) \to x \\
\mathsf{f}(y, \mathsf{f}(x, \mathsf{i}(y))) \to x & \mathsf{f}(x, \mathsf{f}(y, z)) \approx \mathsf{f}(y, \mathsf{f}(x, z)) &
\end{array}
$$

$\mathsf{f}(x, y) \approx \mathsf{f}(z, \mathsf{f}(x, \mathsf{f}(y, \mathsf{i}(z))))$ $(\star)$ in $\mathrm{CP}_{\succ}(E)$

- $x > y > \mathsf{f}(z_1, z_2) > \mathsf{i}(z_1), \mathsf{i}(z_2)$ ✓
- $x > y > \mathsf{i}(z_1) > z_1$ ✓

$$
\begin{aligned}
\mathsf{f}(\mathsf{i}(z_1), \mathsf{f}(x, \mathsf{f}(y, \mathsf{i}(\mathsf{i}(z_1))))) &\to \mathsf{f}(\mathsf{i}(z_1), \mathsf{f}(x, \mathsf{f}(y, z_1))) \\
&\to \mathsf{f}(\mathsf{i}(z_1), \mathsf{f}(x, \mathsf{f}(z_1, y))) \\
&\to \mathsf{f}(\mathsf{i}(z_1), \mathsf{f}(z_1, \mathsf{f}(x, y))) \\
&\to \mathsf{f}(x, y)
\end{aligned}
$$

Example: Abelian Groups, Revisited

system $(E, \succ)$

$$
\begin{array}{rcr}
f(i(x), x) \to 1 & f(y, f(i(y), x)) \to x & i(1) \to 1 \\
f(x, i(x)) \to 1 & f(f(x, y), z) \to f(x, f(y, z)) & i(i(x)) \to x \\
f(i(y), f(y, x)) \to x & i(f(x, y)) \to f(i(y), i(x)) & f(x, 1) \to x \\
f(i(y), f(x, y)) \to x & f(x, y) \approx f(y, x) & f(1, x) \to x \\
f(y, f(x, i(y))) \to x & f(x, f(y, z)) \approx f(y, f(x, z)) &
\end{array}
$$

$f(x, y) \approx f(z, f(x, f(y, i(z))))$ $(\star)$ in $CP_{\succ}(E)$

- $x > y > f(z_1, z_2) > i(z_1), i(z_2)$ ✓
- $x > y > i(z_1) > z_1$ ✓
- $x > y > 1$

$$f(1, f(x, f(y, i(1))))$$

Example: Abelian Groups, Revisited

system $(E, \succ)$

$$\begin{array}{rrr}
\mathsf{f}(\mathsf{i}(x), x) \to 1 & \mathsf{f}(y, \mathsf{f}(\mathsf{i}(y), x)) \to x & \mathsf{i}(1) \to 1 \\
\mathsf{f}(x, \mathsf{i}(x)) \to 1 & \mathsf{f}(\mathsf{f}(x, y), z) \to \mathsf{f}(x, \mathsf{f}(y, z)) & \mathsf{i}(\mathsf{i}(x)) \to x \\
\mathsf{f}(\mathsf{i}(y), \mathsf{f}(y, x)) \to x & \mathsf{i}(\mathsf{f}(x, y)) \to \mathsf{f}(\mathsf{i}(y), \mathsf{i}(x)) & \mathsf{f}(x, 1) \to x \\
\mathsf{f}(\mathsf{i}(y), \mathsf{f}(x, y)) \to x & \mathsf{f}(x, y) \approx \mathsf{f}(y, x) & \mathsf{f}(1, x) \to x \\
\mathsf{f}(y, \mathsf{f}(x, \mathsf{i}(y))) \to x & \mathsf{f}(x, \mathsf{f}(y, z)) \approx \mathsf{f}(y, \mathsf{f}(x, z)) &
\end{array}$$

$\mathsf{f}(x, y) \approx \mathsf{f}(z, \mathsf{f}(x, \mathsf{f}(y, \mathsf{i}(z))))$ $(\star)$ in $\mathsf{CP}_\succ(E)$

- $x > y > \mathsf{f}(z_1, z_2) > \mathsf{i}(z_1), \mathsf{i}(z_2)$ ✓
- $x > y > \mathsf{i}(z_1) > z_1$ ✓
- $x > y > 1$

$$\mathsf{f}(1, \mathsf{f}(x, \mathsf{f}(y, \mathsf{i}(1)))) \to \mathsf{f}(x, \mathsf{f}(y, \mathsf{i}(1)))$$

Example: Abelian Groups, Revisited

system $(E, \succ)$

$$\begin{array}{rrr}
\mathsf{f}(\mathsf{i}(x), x) \to 1 & \mathsf{f}(y, \mathsf{f}(\mathsf{i}(y), x)) \to x & \mathsf{i}(1) \to 1 \\
\mathsf{f}(x, \mathsf{i}(x)) \to 1 & \mathsf{f}(\mathsf{f}(x, y), z) \to \mathsf{f}(x, \mathsf{f}(y, z)) & \mathsf{i}(\mathsf{i}(x)) \to x \\
\mathsf{f}(\mathsf{i}(y), \mathsf{f}(y, x)) \to x & \mathsf{i}(\mathsf{f}(x, y)) \to \mathsf{f}(\mathsf{i}(y), \mathsf{i}(x)) & \mathsf{f}(x, 1) \to x \\
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\end{array}$$

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- $x > y > \mathsf{f}(z_1, z_2) > \mathsf{i}(z_1), \mathsf{i}(z_2)$ ✓
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- $x > y > 1$

$$\begin{aligned}
\mathsf{f}(1, \mathsf{f}(x, \mathsf{f}(y, \mathsf{i}(1)))) &\to \mathsf{f}(x, \mathsf{f}(y, \mathsf{i}(1))) \\
&\to \mathsf{f}(x, \mathsf{f}(y, 1))
\end{aligned}$$

Example: Abelian Groups, Revisited

system $(E, \succ)$

$$
\begin{array}{rcr}
\mathsf{f}(\mathsf{i}(x), x) \to 1 & \mathsf{f}(y, \mathsf{f}(\mathsf{i}(y), x)) \to x & \mathsf{i}(1) \to 1 \\
\mathsf{f}(x, \mathsf{i}(x)) \to 1 & \mathsf{f}(\mathsf{f}(x, y), z) \to \mathsf{f}(x, \mathsf{f}(y, z)) & \mathsf{i}(\mathsf{i}(x)) \to x \\
\mathsf{f}(\mathsf{i}(y), \mathsf{f}(y, x)) \to x & \mathsf{i}(\mathsf{f}(x, y)) \to \mathsf{f}(\mathsf{i}(y), \mathsf{i}(x)) & \mathsf{f}(x, 1) \to x \\
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$$
\begin{aligned}
\mathsf{f}(1, \mathsf{f}(x, \mathsf{f}(y, \mathsf{i}(1)))) &\to \mathsf{f}(x, \mathsf{f}(y, \mathsf{i}(1))) \\
&\to \mathsf{f}(x, \mathsf{f}(y, 1)) \\
&\to \mathsf{f}(x, y)
\end{aligned}
$$

Example: Abelian Groups, Revisited

system $(E, \succ)$ where $\succ$ is KBO with $\mathsf{i} > \mathsf{f} > 1$, $w_0 = w(1) = 1$, $w(\mathsf{i}) = w(\mathsf{f}) = 0$

$$\mathsf{f}(\mathsf{i}(x), x) \to 1 \qquad \mathsf{f}(y, \mathsf{f}(\mathsf{i}(y), x)) \to x \qquad \mathsf{i}(1) \to 1$$

$$\mathsf{f}(x, \mathsf{i}(x)) \to 1 \qquad \mathsf{f}(\mathsf{f}(x, y), z) \to \mathsf{f}(x, \mathsf{f}(y, z)) \qquad \mathsf{i}(\mathsf{i}(x)) \to x$$

$$\mathsf{f}(\mathsf{i}(y), \mathsf{f}(y, x)) \to x \qquad \mathsf{i}(\mathsf{f}(x, y)) \to \mathsf{f}(\mathsf{i}(y), \mathsf{i}(x)) \qquad \mathsf{f}(x, 1) \to x$$

$$\mathsf{f}(\mathsf{i}(y), \mathsf{f}(x, y)) \to x \qquad \mathsf{f}(x, y) \approx \mathsf{f}(y, x) \qquad \mathsf{f}(1, x) \to x$$

$$\mathsf{f}(y, \mathsf{f}(x, \mathsf{i}(y))) \to x \qquad \mathsf{f}(x, \mathsf{f}(y, z)) \approx \mathsf{f}(y, \mathsf{f}(x, z))$$

$\mathsf{f}(x, y) \approx \mathsf{f}(z, \mathsf{f}(x, \mathsf{f}(y, \mathsf{i}(z))))$ $(\star)$ in $\mathrm{CP}_{\succ}(E)$

- $x > y > \mathsf{f}(z_1, z_2) > \mathsf{i}(z_1), \mathsf{i}(z_2)$ ✓
- $x > y > \mathsf{i}(z_1) > z_1$ ✓
- $x > y > 1$ ✓

Example: Abelian Groups, Revisited

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$$\begin{array}{rrr}
\mathsf{f}(\mathsf{i}(x), x) \to 1 & \mathsf{f}(y, \mathsf{f}(\mathsf{i}(y), x)) \to x & \mathsf{i}(1) \to 1 \\
\mathsf{f}(x, \mathsf{i}(x)) \to 1 & \mathsf{f}(\mathsf{f}(x, y), z) \to \mathsf{f}(x, \mathsf{f}(y, z)) & \mathsf{i}(\mathsf{i}(x)) \to x \\
\mathsf{f}(\mathsf{i}(y), \mathsf{f}(y, x)) \to x & \mathsf{i}(\mathsf{f}(x, y)) \to \mathsf{f}(\mathsf{i}(y), \mathsf{i}(x)) & \mathsf{f}(x, 1) \to x \\
\mathsf{f}(\mathsf{i}(y), \mathsf{f}(x, y)) \to x & \mathsf{f}(x, y) \approx \mathsf{f}(y, x) & \mathsf{f}(1, x) \to x \\
\mathsf{f}(y, \mathsf{f}(x, \mathsf{i}(y))) \to x & \mathsf{f}(x, \mathsf{f}(y, z)) \approx \mathsf{f}(y, \mathsf{f}(x, z)) &
\end{array}$$

$\mathsf{f}(x, y) \approx \mathsf{f}(z, \mathsf{f}(x, \mathsf{f}(y, \mathsf{i}(z))))$ $(\star)$ in $\mathrm{CP}_{\succ}(E)$

- $x > y > \mathsf{f}(z_1, z_2) > \mathsf{i}(z_1), \mathsf{i}(z_2)$ ✓
- $x > y > \mathsf{i}(z_1) > z_1$ ✓
- $x > y > 1$ ✓

$E_{\succ}$ is ground-complete

Definition

- ordering constraint is set O of variable pairs $s > t$

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- ground joinability problem is set $\mathcal{G}$ of tuples $(s \approx t, O, \sigma)$ for equation $s \approx t$, ordering constraint O, and substitution σ

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Definition ($\Rightarrow$)

Given ES E,

delete $\quad \mathcal{G} \uplus \{(s \approx s, O, \sigma)\} \Rightarrow \mathcal{G}$

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rewrite $\quad \mathcal{G} \uplus \{(s \approx t, O, \sigma)\} \Rightarrow \mathcal{G} \cup \{(u \approx t, O, \sigma)\}$
if $s \rightarrow_{E_{\succcurlyeq^{O}}} u$

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Definition ($\Rightarrow$)

Given ES E,

delete	$\mathcal{G} \uplus \{(s \approx s, O, \sigma)\} \Rightarrow \mathcal{G}$
rewrite	$\mathcal{G} \uplus \{(s \approx t, O, \sigma)\} \Rightarrow \mathcal{G} \cup \{(u \approx t, O, \sigma)\}$ if $s \to_{E_{\succcurlyeq^O}} u$
equation	$\mathcal{G} \uplus \{(s \approx t, O, \sigma)\} \Rightarrow \mathcal{G}$ if $s \leftrightarrow_E t$

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if $s \rightarrow_{E_{\ggcurly^O}} u$

equation $\quad \mathcal{G} \uplus \{(s \approx t, O, \sigma)\} \Rightarrow \mathcal{G}$
if $s \leftrightarrow_{E} t$

orient $\quad \mathcal{G} \uplus \{(s \approx t, O, \sigma)\} \Rightarrow \mathcal{G} \cup \{(s \approx t, O \cup \{x > y\}, \sigma), (s \approx t, O \cup \{y > x\}, \sigma), (s\rho \approx t\rho, O\rho, \sigma\rho)\}$
if $x, y \in \mathcal{V}\mathrm{ar}(s \approx t)$ and $\rho = \{x \mapsto y\}$

Definition ($\Rightarrow$, cont'd)

instantiate $\quad \mathcal{G} \uplus \{(s \approx t, O, \sigma)\} \Rightarrow \mathcal{G} \cup \{(s\sigma_f \approx t\sigma_f, O\sigma_f, \sigma\sigma_f\} \mid f \in \mathcal{F}\}$
where $\sigma_f = \{x \mapsto f(\vec{z})\}$ for some $x \in \mathcal{V}\text{ar}(s \approx t)$ and $\vec{z}$ fresh

Definition ($\Rightarrow$, cont'd)

instantiate $\mathcal{G} \uplus \{(s \approx t, O, \sigma)\} \Rightarrow \mathcal{G} \cup \{(s\sigma_f \approx t\sigma_f, O\sigma_f, \sigma\sigma_f\} \mid f \in \mathcal{F}\}$
where $\sigma_f = \{x \mapsto f(\vec{z})\}$ for some $x \in \mathcal{V}\mathrm{ar}(s \approx t)$ and $\vec{z}$ fresh

unsat $\mathcal{G} \uplus \{(s \approx t, O, \sigma)\} \Rightarrow \mathcal{G}$
if $\succ \cup\, O$ is not satisfiable by any substitution

Definition (⇒, cont'd)

instantiate $\mathcal{G} \uplus \{(s \approx t, O, \sigma)\} \Rightarrow$ … $\} \mid f \in \mathcal{F}\}$
where $\sigma_f = \{x \mapsto f($ … and $\vec{z}$ fresh

decidable for LPO and KBO
and encodable in SAT

unsat $\mathcal{G} \uplus \{(s \approx t, O, \sigma)\} \Rightarrow \mathcal{G}$
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Definition ($\Rightarrow$, cont'd)

instantiate $\mathcal{G} \uplus \{(s \approx t, O, \sigma)\} \Rightarrow \mathcal{G} \cup \{(s\sigma_f \approx t\sigma_f, O\sigma_f, \sigma\sigma_f\} \mid f \in \mathcal{F}\}$
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subreducible $\mathcal{G} \uplus \{(s \approx t, O, \sigma)\} \Rightarrow \mathcal{G}$
if σ is $E_{\succ\!\!\succ_O}$-reducible

Definition (⇒, cont'd)

instantiate $\mathcal{G} \uplus \{(s \approx t, O, \sigma)\} \Rightarrow \mathcal{G} \cup \{(s\sigma_f \approx t\sigma_f, O\sigma_f, \sigma\sigma_f\} \mid f \in \mathcal{F}\}$
where $\sigma_f = \{x \mapsto f(\vec{z})\}$ for some $x \in \mathcal{V}\mathrm{ar}(s \approx t)$ and $\vec{z}$ fresh

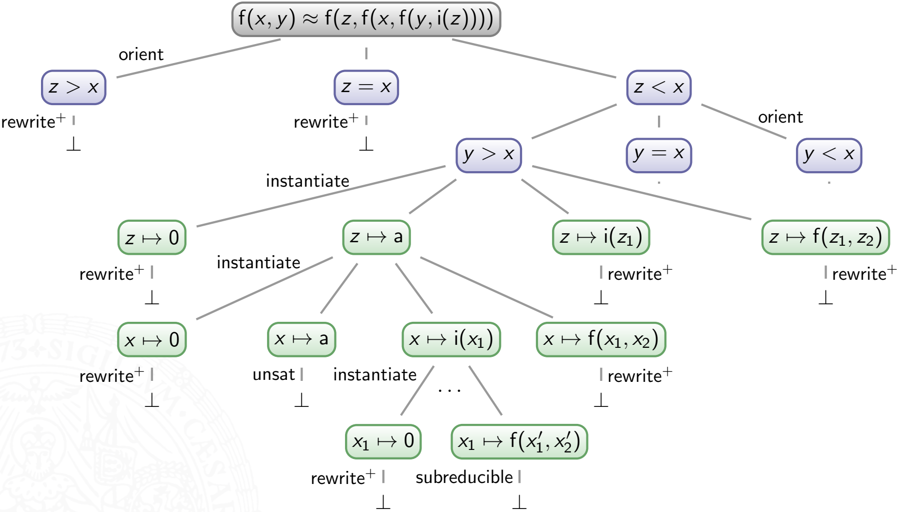
unsat $\mathcal{G} \uplus \{(s \approx t, O, \sigma)\} \Rightarrow \mathcal{G}$
if $\succ \cup\, O$ is not satisfiable by any substitution

subreducible $\mathcal{G} \uplus \{(s \approx t, O, \sigma)\} \Rightarrow \mathcal{G}$
if σ is $E_{\succcurlyeq O}$-reducible

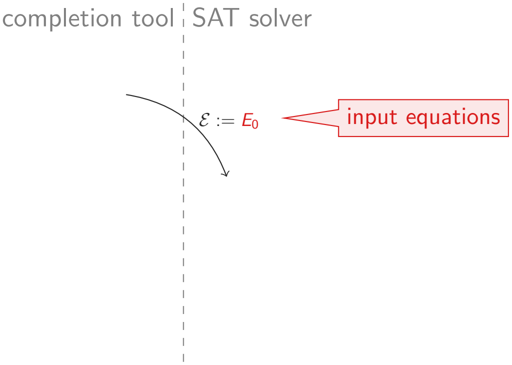
Lemma (Soundness)

Suppose $\{(s \approx t, \varnothing, \varnothing)\} \Rightarrow^+ \varnothing$ *for all* $s \approx t$ *in* $\mathrm{CP}_\succ(E)$.
Then $E_\succ$ *is ground-complete.*

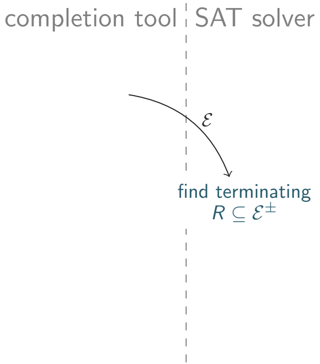
proof


f(x, y) ≈ f(z, f(x, f(y, i(z))))
orient
z > x
z = x
z < x
rewrite+
⊥
rewrite+
⊥
orient
y > x
y = x
y < x
instantiate
z ↦ 0
z ↦ a
z ↦ i(z1)
z ↦ f(z1, z2)
rewrite+
⊥
instantiate
rewrite+
⊥
rewrite+
⊥
x ↦ 0
x ↦ a
x ↦ i(x1)
x ↦ f(x1, x2)
rewrite+
⊥
unsat
⊥
instantiate
. . .
rewrite+
⊥
x1 ↦ 0
x1 ↦ f(x′1, x′2)
rewrite+
⊥
subreducible
⊥

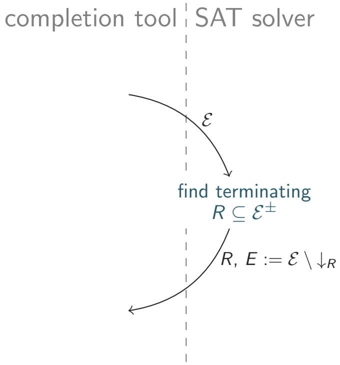
Maximal Ordered Completion



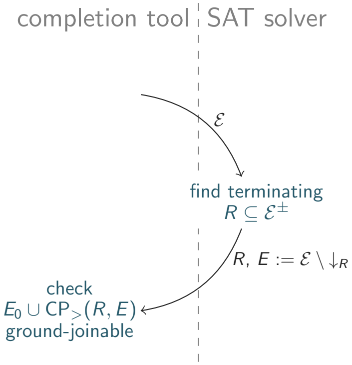
Maximal Ordered Completion



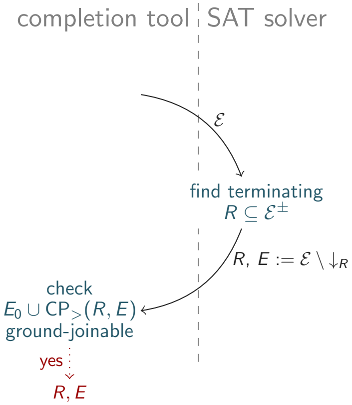
Maximal Ordered Completion



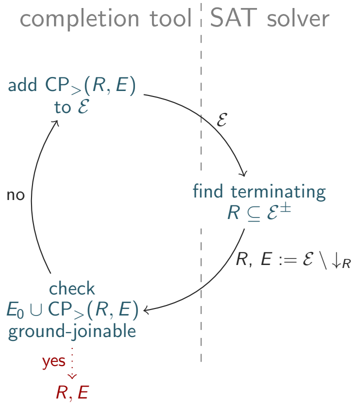
Maximal Ordered Completion



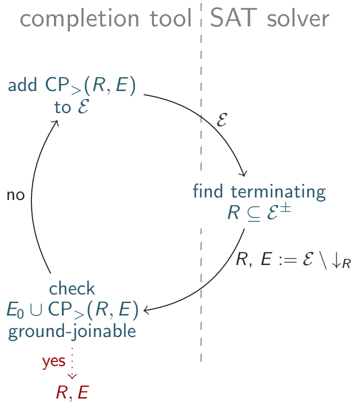
Maximal Ordered Completion



Maximal Ordered Completion

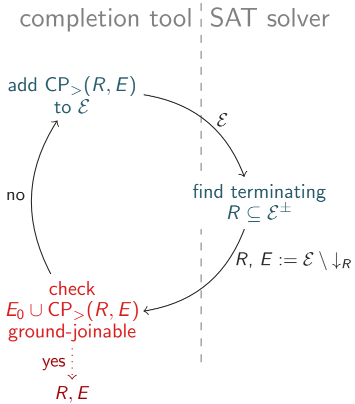


Maximal Ordered Completion



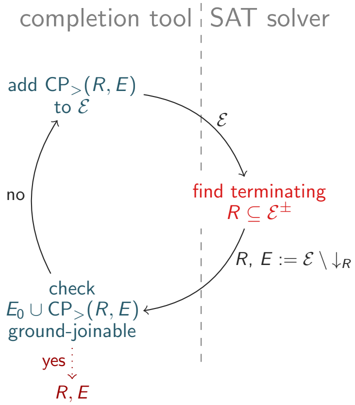
D. Klein and N. Hirokawa.
Maximal Completion.
Proc. 22nd RTA, 2011.

Maximal Ordered Completion



- using SAT encodings of LPO and KBO: unsat rule exploits that LPO and KBO constraints are decidable

Maximal Ordered Completion



- using SAT encodings of LPO and KBO: unsat rule exploits that LPO and KBO constraints are decidable
- ground joinability expressible as a SAT/SMT formula: when searching $R \subseteq \mathcal{E}$ one can maximize number of joined equations

Experiments

	TPTP UEQ/SAT	Examples from MN90
E	12	4
Waldmeister	9	4
mædmax	13	7

http://cl-informatik.uibk.ac.at/software/maedmax

Experiments

	TPTP UEQ/SAT	Examples from MN90
E	12	4
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Summary

- new ground joinability/ground confluence test
- ground joinability of equation can be expressed as constraints on reduction order
- implemented in maximal ordered completion tool mædmax

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$\exists$ *projections* $D_1, D_2, D_3, \ldots$ *to* σ*:*
for all i *have* $D_i \in \mathcal{G}_i \cup \{\bot\}$ *such that if* $D_i = (u \approx v, O, \tau)$ *there is* δ*:*
(1) $\sigma = \tau\delta$ *(2)* $\forall\, x > y \in O$, $\sigma(x) \succ \sigma(y)$ *(3)* $\{D_i\} \Rightarrow \{D_{i+1}, \ldots\}$ *or* $\{D_i\} \Rightarrow \varnothing$

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If $E \subseteq \leftrightarrow^*_{E_0}$ *and* $\{(s \approx t, \varnothing, \varnothing)\} \Rightarrow^+ \varnothing$ *for all* $s \approx t$ *in* $\mathrm{CP}_\succ(E)$ *then* $E_\succ$ *is ground-complete.*

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call $D\delta$ joinable if $u\delta \downarrow_{E_\succ} v\delta$ for all $(u \approx v, \ldots) \in D$

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- by induction, $s\sigma \approx t\sigma$ is joinable: contradiction

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- in particular, all extended critical pairs are ground joinable

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- by Extended Critical Pair Lemma, $(E, \succ)$ is ground-confluent ∎

For sequence $D_1, \ldots, D_k$ set $D_i = \bot$ for all $i > k$.

Proof of Claim $(\star)$: Joinability of $D_{i+1}\delta_{i+1}$ implies joinability of $D_i\delta_i$.

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- orient: D_{i+1} reflects $x\delta > y\delta$, $x\delta < y\delta$, or $x\delta = y\delta$
- instantiate joinability is preserved as $\sigma = \tau\sigma_f\delta_{i+1}$ for $f = \mathrm{root}(x\delta)$

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- unsat $O \cup \succ$ is not unsatisfiable as it is satisfied by σ

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- **delete:** $u\delta \downarrow_{E_\succ} u\delta$ always holds
- **rewrite:** suppose $u \to_{E_{\not\succ O}} w$: if $w\delta \downarrow_{E_\succ} v\delta$ then $u\delta \downarrow_{E_\succ} v\delta$
- **equation:** if $u \leftrightarrow_{E,p} v$ then $u\delta \to_{E_\succ} v\delta$ or $v\delta \to_{E_\succ} u\delta$ since either $u\delta \succ v\delta$ or $v\delta \succ u\delta$ must hold, so $u\delta \downarrow_{E_\succ} v\delta$
- **orient:** D_{i+1} reflects $x\delta > y\delta$, $x\delta < y\delta$, or $x\delta = y\delta$
- **instantiate** joinability is preserved as $\sigma = \tau\sigma_f\delta_{i+1}$ for $f = \mathrm{root}(x\delta)$
- **unsat** $O \cup \succ$ is not unsatisfiable as it is satisfied by σ
- **subreducible** if τ is reducible then $\sigma = \tau\delta$ is reducible to, say, σ'.
  Have $s\sigma \succ s\sigma'$ or $t\sigma \succ t\sigma'$, and $s\sigma' \leftrightarrow^*_{E_0} t\sigma'$ so by minimality $s\sigma' \approx t\sigma'$ must be joinable