Critical pairs for Gray categories

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Abstract

Higher categories are a generalization of standard categories where there are not only 1-cells between 0-cells but more generally n+1-cells between n-cells. Semi-strict categories, such as Gray categories in dimension 3, is a flavour of higher categories suited for rewriting and used in this work. Here, we are interested in proving *coherence* of certain algebraic structures in dimension 3 using rewriting, where "coherence" is the property that there is at most one 3-cell between two 2-cells. Checking coherence then amounts to compute critical pairs of a rewriting system and use a variant of Newmann's lemma. In this setting, an algorithm exists to compute these critical pairs.

Introduction

It is well-known that rewriting can be used to manipulate algebraic theories. In this setting, the terms are the algebraic terms that arise from the *signature* of the theory and rewrite rules come from an orientation of the equations of the theory. In the context of higher categories, these techniques need to be adapted. Take monoids as an example. A monoid is given by a set M, an operation $m : M \times M \to M$ and an element $e \in M$ such that $m(m(x,y),z) = m(x,m(y,z))$, $m(x,e) = x = m(e,x)$. More generally, there is a notion of monoid in 2-category where the elements m and e are 2-generators in a 2-category: $m : M *_0 M \Rightarrow M$ and $e : 1 \Rightarrow M$ and such that equalities of 2-cells similar to the previous ones hold. The term rewriting system (or *TRS*) associated to the theory of monoids is then given by the signature $\mathsf{S} = \{m : M \times M \to M, e : 1 \to M\}$ and the following rewrite rules on formal compositions obtained by orienting the equations: $m \circ (m \times 1_M) \to m \circ (1_M \times m)$, $m \circ (e \times 1_M) \to 1_M$ and $m \circ (1_M \times e) \to 1_M$. The standard tools of rewriting i.e., termination, critical pair lemma and Newman's lemma entails uniqueness of normal forms. In order to go from interpretations in n-categories to interpretations in n+1-categories, the usual recipe is to replace equations on n-cells by n+1-isomorphisms and by adding equations on the new n+1-cells, called *coherence cells*, in order to entail the property of *coherence*, which states that, modulo the equations, there is at most one 3-cell between two 2-cells. For monoids, by going from dimension 2 to 3, we obtain the theory of pseudomonoids, which is important in category theory since the notion of monoidal category can be seen as a pseudomonoid in the category of categories.

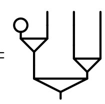
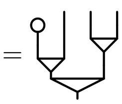
Several variants of 3-categories exist with different levels of expressivity and ease to manipulate. On the one end of the spectrum, weak 3-categories are the most general but are complex since they have a lot of coherence cells. On the other end, strict 3-categories have no coherence cells, only simple equations. But they are less expressive. Gray categories [4, 5] are a middle ground between the two. In this work, we will study interpretations of algebraic structures inside Gray categories. As a previous work[3] has shown, in order to have coherence, it is sufficient to enforce equations on coherence cells that come from the *critical branchings* (or critical pairs) of an adequate rewriting system. So there is a strong need for a tool that can automate the computation of these critical branchings.

1 Signatures and rewriting system

A *graph* $(\mathsf{S}_0, \mathsf{S}_1, \mathsf{s}_0, \mathsf{t}_0)$ is given by a set S_0 of *points* and a set S_1 of *arrows* and source and target functions $\mathsf{s}_0, \mathsf{t}_0 : \mathsf{S}_1 \to \mathsf{S}_0$. We denote S_1^* the set of paths in the graph and $\mathsf{s}_0^*, \mathsf{t}_0^* : \mathsf{S}_1^* \to \mathsf{S}_0$ the source and the target functions on paths, and $*_0$ the composition operation on paths. A *signature* S is given by a graph $(\mathsf{S}_0, \mathsf{S}_1, \mathsf{s}_0, \mathsf{t}_0)$, by a set of 2-generators S_2 with source and target functions $\mathsf{s}_1, \mathsf{t}_1 : \mathsf{S}_2 \to \mathsf{S}_1^*$ such that $\mathsf{s}_0^* \circ \mathsf{s}_1 = \mathsf{s}_0^* \circ \mathsf{t}_1$ and $\mathsf{t}_0^* \circ \mathsf{s}_1 = \mathsf{t}_0^* \circ \mathsf{t}_1$. An example of signature is the *monoid* signature P, where:

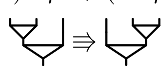
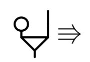
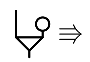
$$\mathsf{P}_0 = \{\star\} \qquad \mathsf{P}_1 = \{1 : \star \to \star\} \qquad \mathsf{P}_2 = \{\mu : 2 \Rightarrow 1, \eta : 0 \Rightarrow 1\}$$

Note that we write n for the path $\underbrace{\star \xrightarrow{1} \star ... \star \xrightarrow{1} \star}_{n}$. A *whisker* w is then given by two paths $u, v \in \mathsf{S}_1^*$ and a 2-generator $\alpha \in \mathsf{S}_2$ and is denoted $u * \alpha * v$. The 1-*source* and the 1-*target* are defined as $u *_0 \mathsf{s}_1\alpha *_0 v$ and $v *_0 \mathsf{t}_1\alpha *_0 v$ and are respectively denoted $\mathsf{s}_1 w$ and $\mathsf{t}_1 w$. A 2-*cell* α is given by a sequence of whiskers $w_1, ..., w_p$ that are 1-*composable*, i.e., $\mathsf{t}_1 w_i = \mathsf{s}_1 w_{i+1}$. We denote α as $w_1 * ... * w_p$. The 1-*source* and the 1-*target* of α are defined as $\mathsf{s}_1 w_1$ and $\mathsf{t}_1 w_p$ and are denoted $\mathsf{s}_1\alpha$ and $\mathsf{t}_1\alpha$ respectively. We denote S_2^* the set of 2-cells. For two 1-composable cells $\alpha = w_1 * ... * w_p$ and $\beta = w_1' * ... * w_q'$, we define the 1-composition $\alpha *_1 \beta$ as the 2-cell $w_1 * ... * w_p * w_1' * ... * w_q'$. Note that 2-cells can easily be represented as string diagrams. For example, in the case of monoids, if we picture η with and μ with , the following two 2-cells can be defined:

$(0 * \eta * 3) * (0 * \mu * 2) * (1 * \mu * 0) * \mu =$  $(0 * \eta * 3) * (2 * \mu * 0) * (0 * \mu * 1) * \mu =$ 

Note that in these pictures, there can be only one generator at a given height, and the relative heights matter, so that the two 2-cells are not considered to be equal (contrarily to 2-categories). A *rewriting system* consists of a signature S together with a set S_3 of 3-*generators*, or *rewriting rules*, equipped with source and target functions $\mathsf{s}_2, \mathsf{t}_2 : \mathsf{S}_3 \to \mathsf{S}_2^*$. For example, the rewriting system of monoids, which extends the associated signature, has the following rewrite rules:

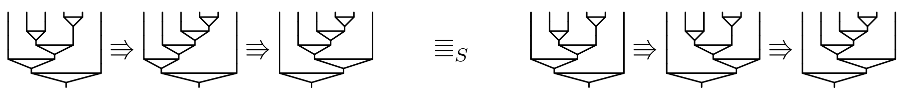
$$A : (\mu * 1) * \mu \Rrightarrow (1 * \mu) * \mu \qquad L : (\eta * 1) * \mu \Rrightarrow \mu \qquad R : (1 * \eta) * \mu \Rrightarrow \mu$$

A *context* $E = \phi * (u * \_ * v) * \psi$ is given by $u, v \in \mathsf{S}_1^*$ and $\phi, \psi \in \mathsf{S}_2^*$. For A a rewrite rule, E is *compatible with* θ when $E[A] = \phi *_1 (u *_0 A *_0 v) *_1 \psi$ exists. A *rewriting step* R is then given by a rewrite rule $A \in \mathsf{S}_3$ and a compatible context E. It can be seen as an elementary 3-cell of type $\phi *_1 (u *_0 \mathsf{s}_2 A *_0 v) *_1 \psi \Rrightarrow \phi *_1 (u *_0 \mathsf{t}_2 A *_0 v) *_1 \psi$. A *rewriting path* is a finite sequence of composable rewriting steps $R_1, ..., R_n$ with $R_i : \theta_i \Rrightarrow \theta_{i+1}$. We denote such a rewriting path as $R_1 * ... * R_n$ or 1_θ for the empty path starting from the 2-cell θ. We write S_3^* for the set of rewriting paths, and $\mathsf{s}_2^*, \mathsf{t}_2^* : \mathsf{S}_3^* \to \mathsf{S}_2^*$ for the associated source and target functions. If $R : \theta_1 \Rrightarrow \theta_2$ is a rewrite step, we define the *reverted rewrite step* $R^{-1} : \theta_2 \Rrightarrow \theta_1$ as a formal inverse of R. Then, a *rewriting zigzag* is a sequence $Z_1, ..., Z_n$ where Z_i is either a rewrite step or a reverted rewrite step and such that $\mathsf{t}_2 Z_i = \mathsf{s}_2 Z_{i+1}$. We denote such a rewriting zigzag as $Z_1 * ... * Z_n$. We denote $\mathsf{S}_3^\top$ the set of rewriting zigzags. If $Z = Z_1 * ... * Z_n$, we define Z^{-1} as the zigzag $Z' = Z_n' * ... * Z_1'$ with $Z_i' = R^{-1}$ if $Z_i = R$ and $Z_i' = R$ if $Z_i = R^{-1}$. There is also a composition operation of zigzags given by the concatenation of sequences. A *coherated rewriting system* is given by a rewrite system S and a congruence $\equiv$ on the rewriting zigzags $\mathsf{S}_3^\top$. By "congruence", we mean that $\equiv$ is an equivalence relation compatible with the difference compositions and the inverse operations. For example, if $Z_1 \equiv Z_2$ then $U * Z_1 * V \equiv U * Z_2 * V$ and ${Z_1}^{-1} \equiv {Z_2}^{-1}$. The *standard congruence* $\equiv_S$ on a rewriting system S is the smallest congruence such that:

1. if $E_1[A_1]$ and $E_2[A_2]$ are two rewrite steps of same source θ and "acting on independent zones of θ" (notion that will be precised later) then $E_1[A_1] * E_2'[A_2] \equiv_S E_2[A_2] * E_1'[A_1]$ where E_1' is the "residual context" of E_1 after the rewrite step $E_2[A_2]$ and similarly for E_2'.
2. if R is a rewrite step, then $R * R^{-1} \equiv_S 1_{\mathsf{s}_2 R}$ and $R^{-1} * R \equiv_S 1_{\mathsf{t}_2 R}$

For instance, in the rewriting system of monoids, there is the following instance of 1:

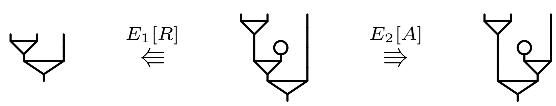


We say that a congruence $\equiv$ is *standard* when $\equiv_S \subset \equiv$. Note that signatures and rewriting systems are simplified definitions of *prepolygraphs*, or *polygraphs on precategories* [6, 3].

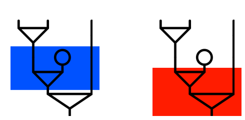
In what follows, we supposed a fixed rewriting system S.

2 Rewriting

Branchings. A *branching* $\mathcal{B}$ is a pair of rewriting steps $(R_1, R_2) = (E_1[A_1], E_2[A_2])$ with $\mathsf{s}_2 R_1 = \mathsf{s}_2 R_2$. We call the *source of the branching* $\mathcal{B}$, denoted $\mathsf{s}_2\mathcal{B}$ to be $\mathsf{s}_2 R_1$. Recall that a 2-cell θ is of the form $w_1 * ... * w_n$. We then define the *size* of θ, denoted $|\theta|$, as n and the *interval* of θ, denoted I_θ, as the set $\{1, ..., n\}$. If A is a rewrite rule and $E = \phi * (u * \_ * v) * \psi$ is a compatible context, we define the *action interval* of $E[A] : \theta \Rrightarrow \theta'$ on θ, denoted $I_{E[A]}$ to be the subsegment $\{|\phi| + 1, ...|\phi| + |\mathsf{s}_2 A|\}$ of I_θ and the *action index* to be $|\phi|$. Note that the size of $I_{E[A]}$ is given by $|\mathsf{s}_2 A|$. In what follows, we will suppose that for all rewrite rules $A \in \mathsf{S}_3$, we have $|\mathsf{s}_2 A| \geq 1$. For a branching $\mathcal{B} = (E_1[A_1], E_2[A_2])$ with $E_i = \phi_i * (u_i * \_ * v_i) * \psi_i$, define the *relative offset* of $\mathcal{B}$ to be $|\phi_2| - |\phi_1|$. Also, we say that that the actions of $E_1[A_1]$ and $E_2[A_2]$ are *overlapping* if $I_{E_1[A_1]} \cap I_{E_2[A_2]} \neq \emptyset$. For example, in the theory of monoids, there is the following branching:



whose action indexes are respectively 1 and 2 and whose action intervals are respectively $\{2, 3\}$ and $\{3, 4\}$, are overlapping and can be depicted as follows:



Let $\mathcal{B} = (E_1[A_1], E_2[A_2])$ a branching with $E_i = \phi_i * (u_i * \_ * v_i) * \psi_i$. We say that $\mathcal{B}$ is *trivial* when $E_1[A_1] = E_2[A_2]$, *non-minimal* when there is another branching $(F_1[A_1], F_2[A_2])$ with $F_i = \alpha_i * (r_i * \_ * s_i) * \beta_i$ such that there exists r, s, α, β not all identities such that $\phi_i = \alpha * \alpha_i$, $\psi_i = \beta_i * \beta$, $u_i = r * r_i$ and $v_i = s_i * s$, *independent* when the actions $E_1[A_1]$ and $E_2[A_2]$ are not overlapping and *critical* when it is of none of the above. We adapt the notion of confluence in the setting of a cohérated rewriting system $(\mathsf{S},\equiv)$: a branching $\mathcal{B}=(R_1,R_2)$ with $R_i : \alpha \Rrightarrow \beta_i$ is said to be *confluent* when there exists a pair of rewriting paths (S_1,S_2) with $S_i : \beta_i \Rrightarrow \gamma$ such that $R_1 * S_1 \equiv R_2 * S_2$. We also define a straight-forward notion of termination: we say that a rewriting system S is *terminating* if there is no infinite sequence $(R_i)_{i\in\mathbb{N}}$ of rewriting steps with $R_i : \phi_i \Rrightarrow \phi_{i+1}$. In another work, we have the following result that motivates the computing of critical branchings:

Theorem 1 (FSCD18, Newman's lemma)**.** *Let* $(\mathsf{S},\equiv)$ *be a cohérated rewriting system such that the rewriting system* S *is terminating,* $\equiv$ *is standard and all critical branchings are confluent. Then* $(\mathsf{S},\equiv)$ *is coherent.*

Computation of critical branchings. Let $\mathcal{B}=(E_1[A_1],E_2[A_2])$ be a critical branching. Because $\mathcal{B}$ is in particular non-independent, it holds that $I_{E_1[A_1]} \cap I_{E_2[A_2]} \neq \emptyset$. But since $I_{E_i[A_i]} = \{|\phi_i|+1,...,|\phi_i|+|\mathsf{s}_2 A_i|\}$, the relative offset is bounded: $0 \leq |\phi_2|-|\phi_1| < |\mathsf{s}_2 A_1|$. Moreover, for a given offset, we have a uniqueness property:

Proposition 1. *Let* A_1 *and* A_2 *be two rewrite rules and* p *such that* $0 \leq p < |\mathsf{s}_2 A_1|$. *Then there is at most one critical branching* $\mathcal{B}=(E_1[A_1],E_2[A_2])$ *such that the actions of* $E_1[A_1]$ *and* $E_2[A_2]$ *have* p *as relative offset.*

Let A_1 and A_2 and n_1,n_2 such that $n_i = |\mathsf{s}_2 A_i|$ and

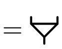
$$\mathsf{s}_2 A_i = (r_{i,1} * \alpha_{i,1} * s_{i,1}) * ... * (r_{i,n_1} * \alpha_{i,n_1} *_0 s_{i,n_1})$$

The proof of the last property gives us a procedure to compute all the context E_1,E_2 such that $(E_1[A_1],E_2[A_2])$ is a critical branching. See figure 2 for the procedure.

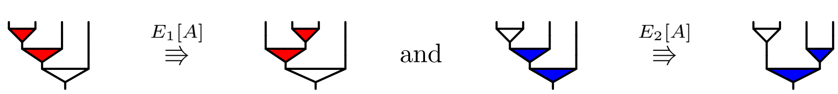
Example. We apply this procedure for the computation of critical branchings between the rewrite rules A and A in the theory of monoids. There are only two possible relative offset p to test: 0 and 1. When $p=0$, the procedure produces no context because it is the case of the trivial branching. So we focus on the case $p=1$. In this case, the two whiskers to unify are the following:

 and 

It is easy to unify them using $u_1=0$, $u_2=0$, $v_1=1$ and $v_2=0$. Using the formulas of the procedure, we then define

$\phi_1 = 1$ $\phi_2 =$  $\psi_1 =$  $\psi_2 = 1$

These elements define contexts E_1,E_2 with $E_i = \phi_i * (u_i * \_ * v_i) * \psi_i$ and they define a branching $\mathcal{B}=(E_1[A],E_2[A])$ where $E_1[A]$ and $E_2[A]$ are rewrite step as follows:



$\mathcal{B}$ is non-independent since the action intervals of $E_1[A]$ and $E_2[A]$ are respectively $\{1,2\}$ and $\{2,3\}$ so they overlap. Moreover, this branching is minimal. So $\mathcal{B}$ is critical.

procedure CriticalBranching(A_1, A_2)
    let $P = \emptyset$
    for $p = 0$ **to** $n_1 - 1$ **do** (MainFor) ▷ All possible relative offset are tested
        if ($p = 0$ **and** $A_1 = A_2$)
         or ($r_{1,p+1}$ is not a suffix of $r_{2,1}$ **and** $r_{2,1}$ is not a suffix of $r_{1,p+1}$)
         or ($s_{1,p+1}$ is not a prefix of $s_{2,1}$ **and** $s_{2,1}$ is not a prefix of $s_{1,p+1}$) **then**
            continue MainFor
        end if
        let u_1, u_2 **be** the smallest such that $u_1 * r_{1,p+1} = u_2 * r_{2,1}$
        let v_1, v_2 **be** the smallest such that $s_{1,p+1} * v_1 = s_{2,1} * v_2$
        for $i = p + 1$ **to** n_1 **do**
            if $u_1 * r_{1,i} \neq u_2 * r_{2,i}$ **or** $\alpha_{1,i} \neq \alpha_{2,i-p}$ **then**
                continue MainFor
            end if
        end for
        let $\phi_1 = 1$ **and** $\phi_2 = *_{i=1}^{p}((u_1 * r_{1,i}) * \alpha_{1,i} * (s_{1,i} * v_1))$
        let $\psi_1 = *_{i=n_1-p+1}^{n_2}((u_2 * r_{2,i}) * \alpha_{2,i} * (s_{2,i} * v_2))$
        and $\psi_2 = *_{i=n_2+p+1}^{n_1}((u_1 * r_{1,i}) * \alpha_{1,i} * (s_{1,i} * v_1))$
        $P \leftarrow P \cup \{(\phi_1 * (u_1 * {}_- * v_2) * \psi_1, \phi_2 * (u_2 * {}_- * v_2) * \psi_2)\}$
    end for
    return P
end procedure

Conclusion

In this work, we showed how rewriting formalism can be used for the interpretation of algebraic structures in a 3-dimensional categorical setting. In particular, we defined the notion of signatures, rewriting systems, rewrite rules and rewrite paths in this setting and stated an adaptation of Newman's lemma for coherence, which relates the coherence property to the critical branchings of the rewriting system. Then we gave an algorithm to compute the critical branchings, and gave an example for the theory of pseudomonoids. Even though we restricted ourselves to dimension 3, the formalism and the algorithm can be readily used with higher dimensions.

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