The diamond lemma for free modules

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Plan

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I. Motivations

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  - Example: $\mathbf{R} = C^\infty(\mathbb{R}^n)$ and $X = \left\{ \partial_1^{k_1} \cdots \partial_n^{k_n} \mid k_i \in \mathbb{N} \right\}$.

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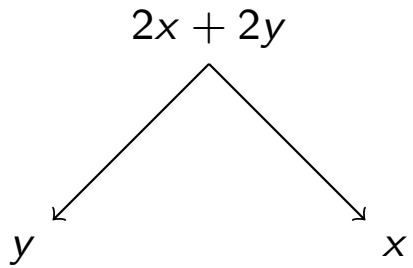

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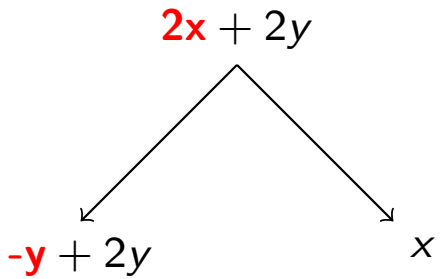

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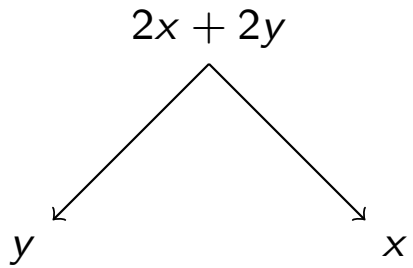

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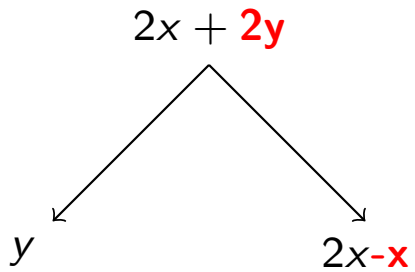

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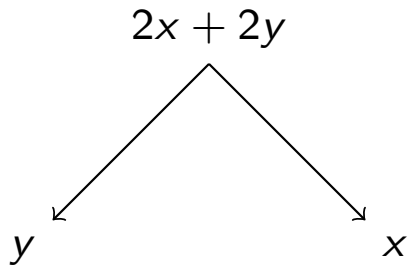

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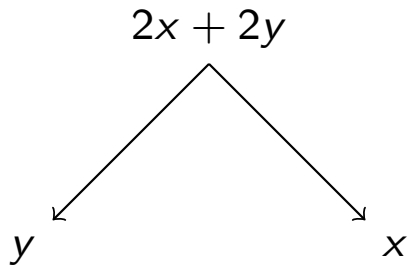

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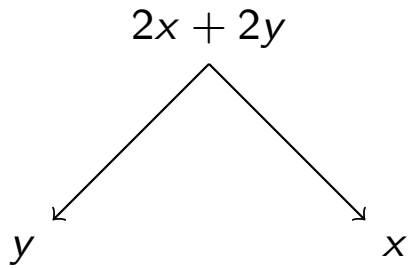

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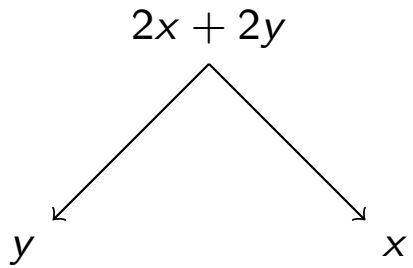

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II. Rewriting over free modules

Rewrite rules over free modules

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Proposition. *The equational theory of $\mathscr{R}$ is the module generated by the elements $rx - f$.*

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  - the right product of $r \in \mathbf{R}$ on ∂ is given by: $\partial r = \sigma(r)\partial + \delta(r)$.

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  - Is the rewriting system induced by $\mathscr{R}$ confluent?

Plan

III. The diamond lemma

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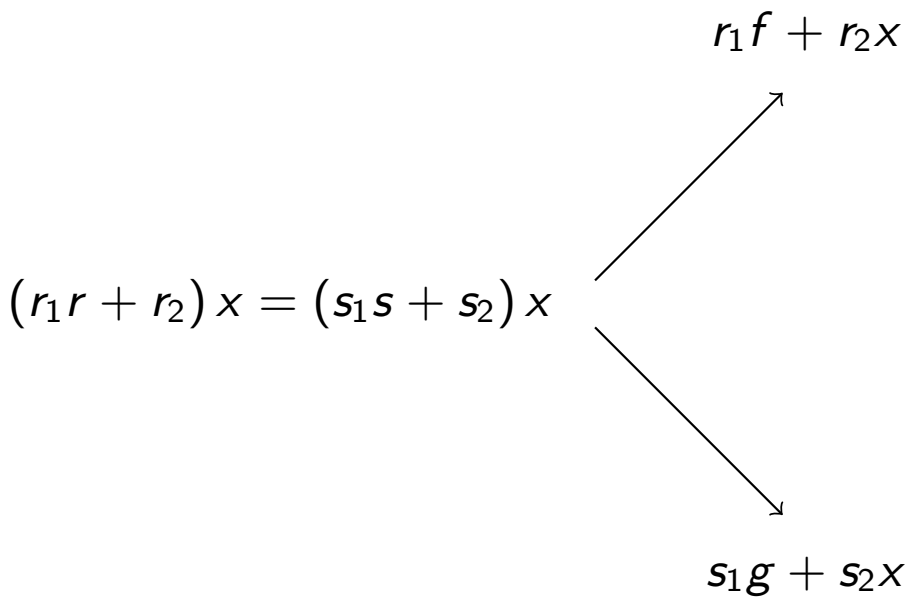

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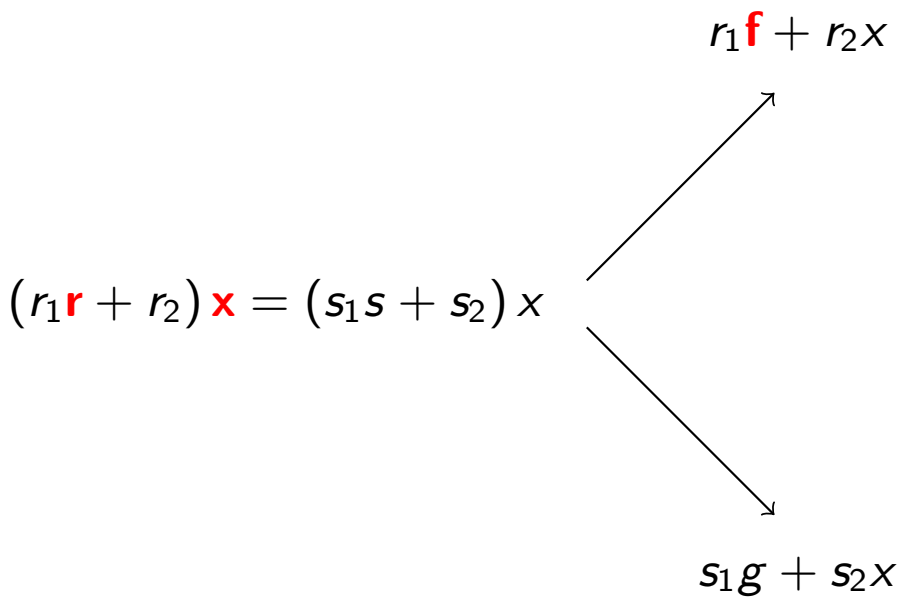

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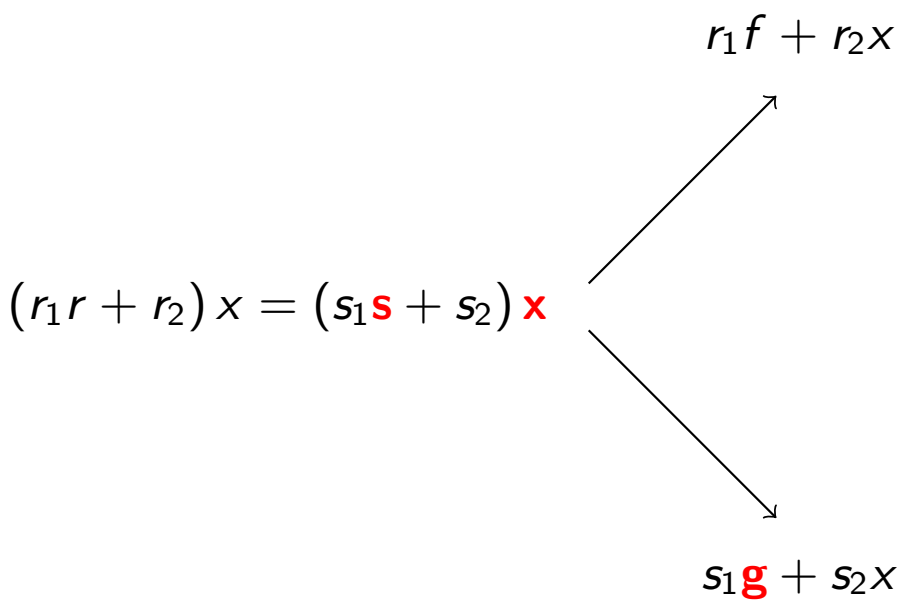

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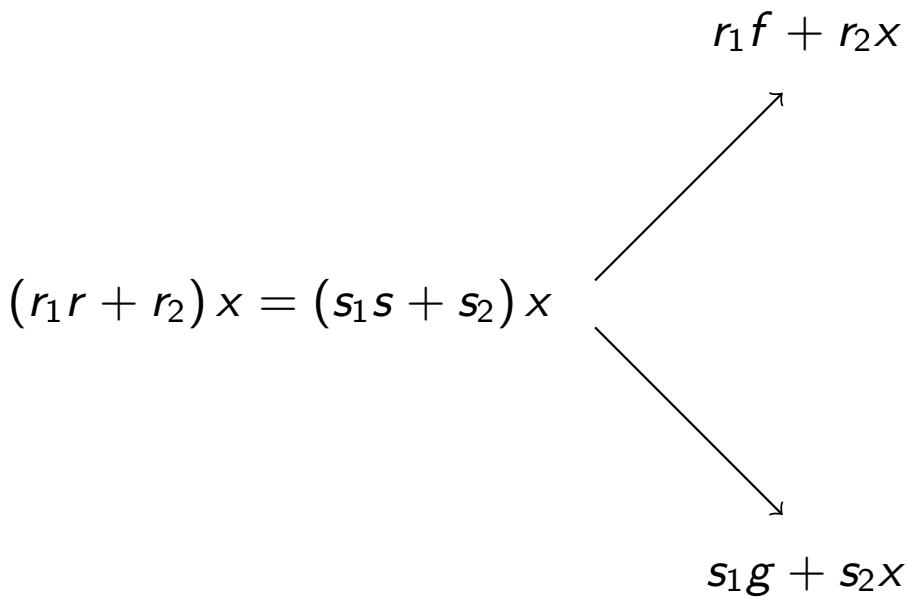

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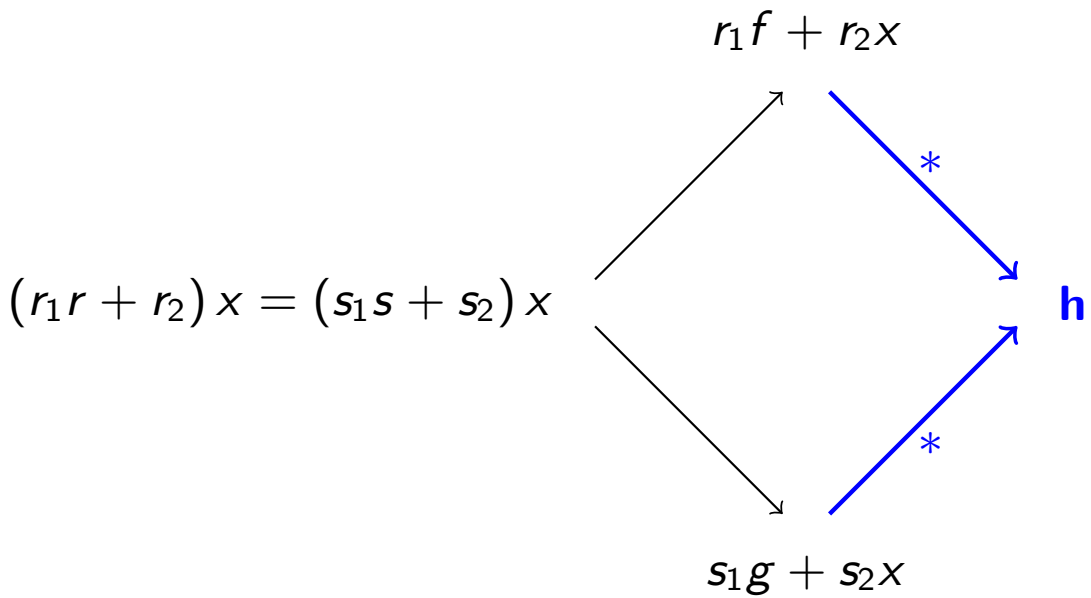

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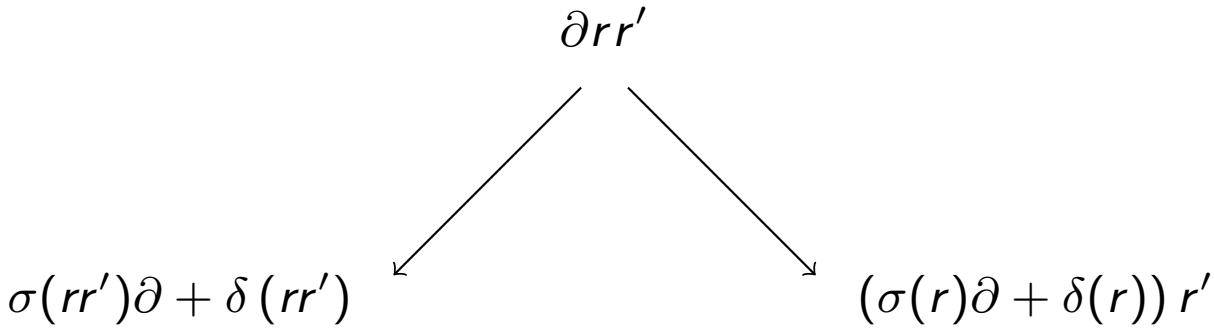

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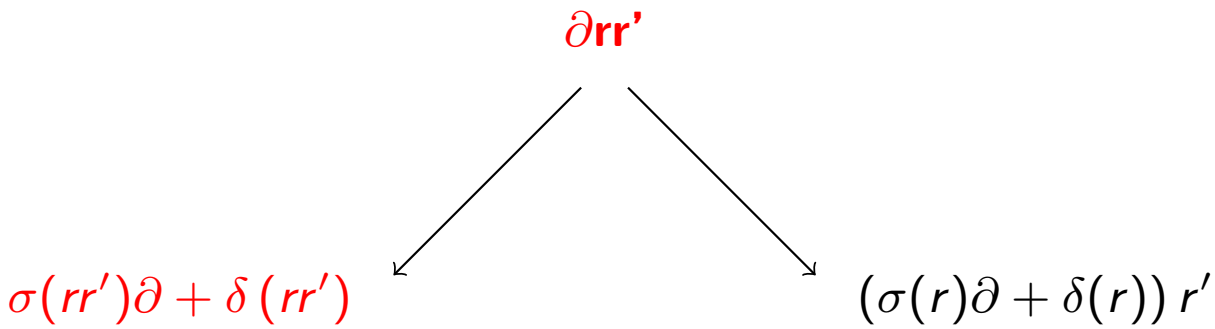

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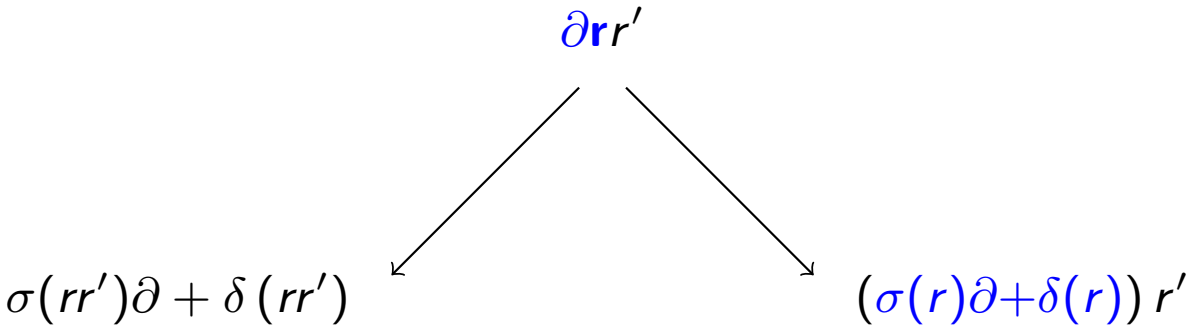

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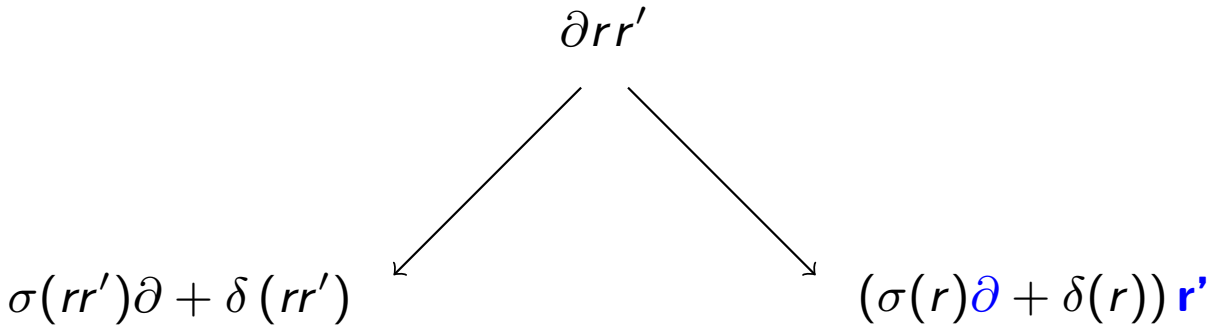

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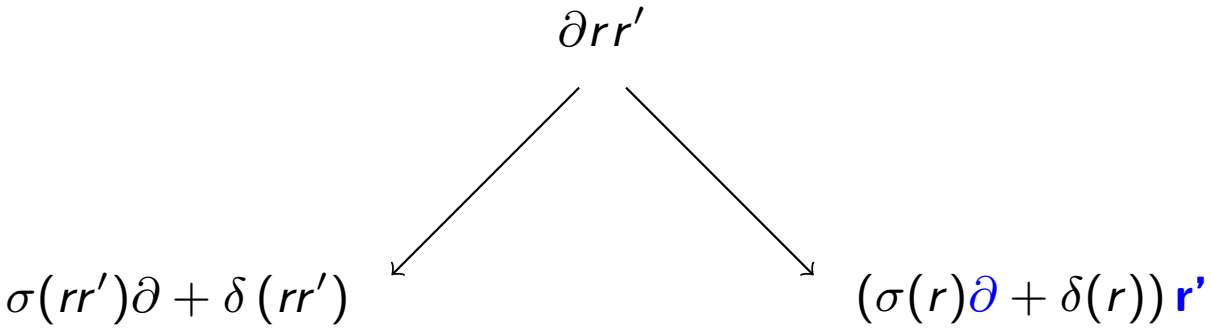

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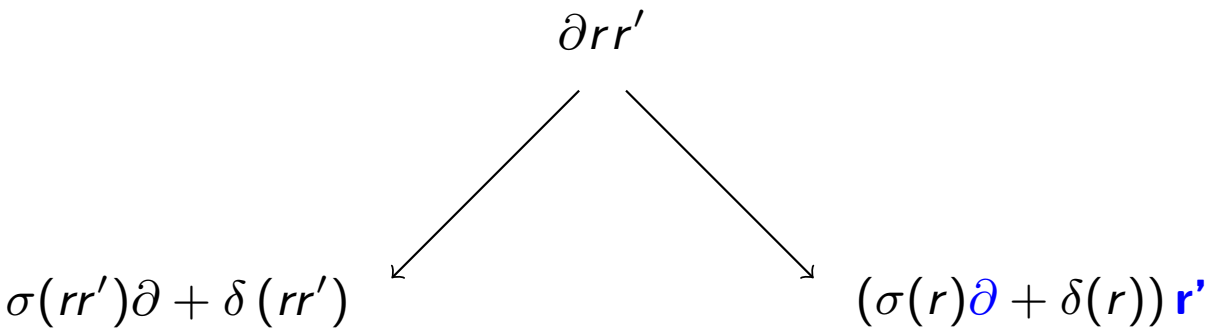

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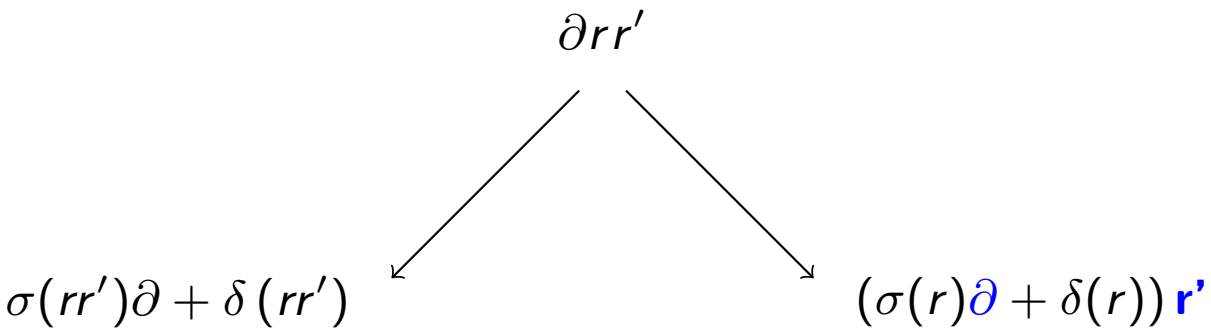

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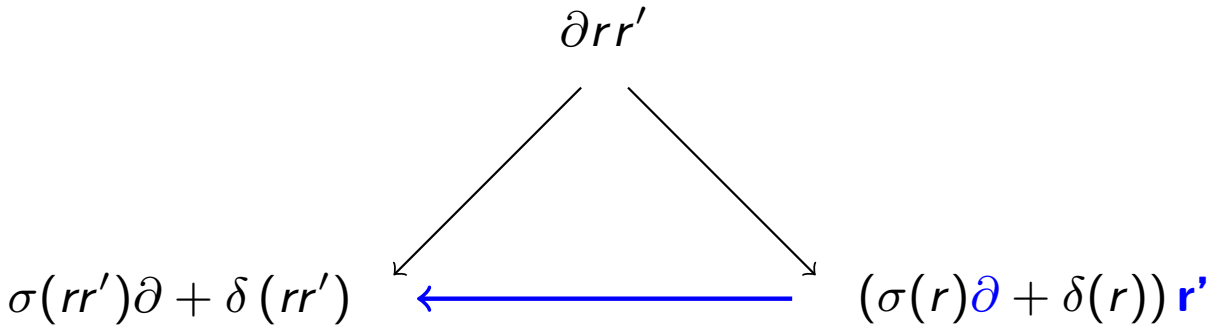

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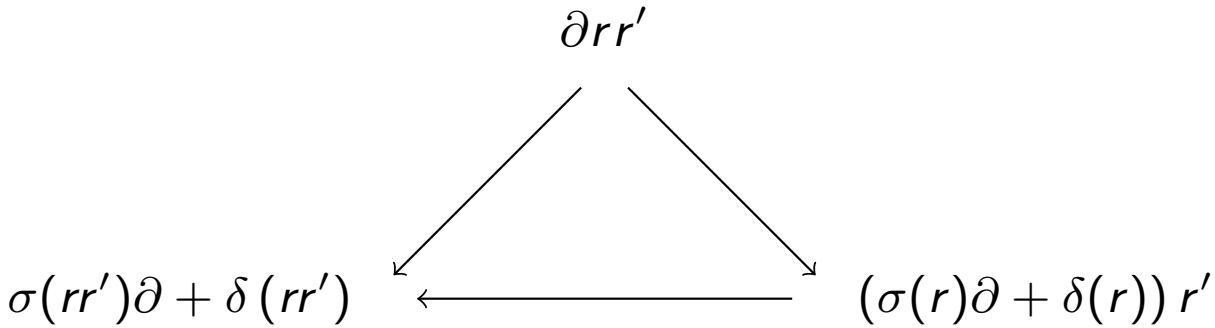

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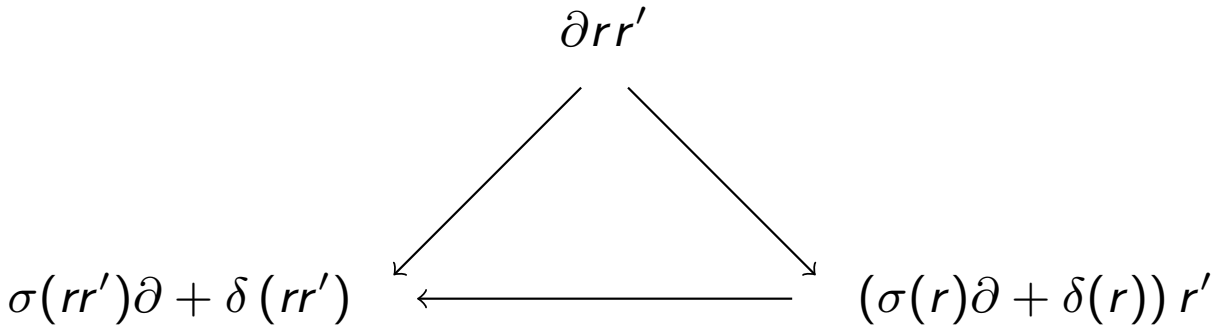

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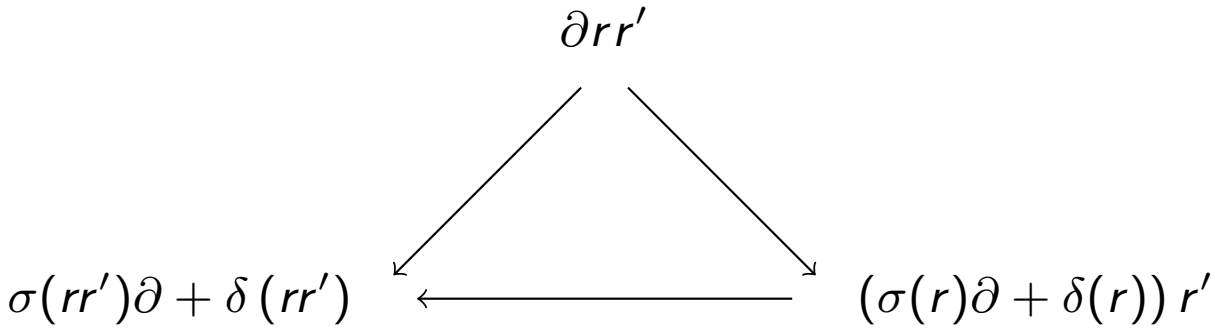

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Plan

IV. Conclusion and perspectives

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Thank you for listening!