Coherence of monoids by insertions

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7th International Workshop on Confluence, IWC 2018
July 7th 2018, Oxford, United Kingdom
Plan:

1. Motivations
2. String Data Structures
3. Coherent presentations and String data structures
4. Conclusion
Motivations

- **Data structures** describe a way to organize and to store a collection of structured data.
  - appear in *combinatorial algebra*, *combinatorics* and *fundamental computer science*,
  - describe combinatorial structures: *arrays*, *tableaux*, *staircases*, *binary search trees*...
  - array structures can be used to study *plactic*, *Chinese*, *hypoplactic* and *sylvester monoids*...

- Study **string rewriting systems** (SRS):
  - normal forms can be described using a data structure,
  - rewriting rules are induced by insertion algorithms.

- Introduce the notion of **string data structure** (SDS):
  - the data are constructed using an insertion algorithm,
  - they are described by words through a reading map.
Motivations

Plactic monoids:

- (Young, 1900): Young tableaux

- (Schensted, 1961): left and right insertions on Young tableaux

- (Knuth, 1970): $u, v \in \{1, \ldots, n\}^*$, $u \sim v \iff P(u) = P(v)$.

  - $\sim$ coincides with the congruence generated by Knuth relations:
    
    $zxy = xzy$ for $x \leq y < z$ and $yzx = yxz$ for $x < y \leq z$.

- (Lascoux, Schützenberger, 1981): plactic monoid $= \{1, \ldots, n\}^*/\sim$.

- Applications on combinatorics, representation theory, rewriting theory...
Motivations

Plactic monoids:

- (Kubat, Okninski, 2014): $n > 4 \Rightarrow$ no finite completion of the Knuth presentation w.r.t the lexicographic order.

- (Cain, Gray, Malheiro, 2015), (Bokut, Chen, Chen, Li, 2015): finite completions obtained by adding new generators:
  - column or row generators $\Rightarrow$ convergent presentation of plactic monoids.

- (Hage, Malbos, 2017): coherent presentations of the plactic monoid giving all the relations among the relations of its presentations.
  - Confluence property is essential to obtain such coherence results.
  - Commutation of right and left insertions $\Rightarrow$ confluence of the presentation.

- Study these confluence results in a general algebraic framework using the notion of string data structure.

- Explicit coherent presentation of the monoid presented by an SDS:
  - a presentation of the monoid (generators + rewriting rules describing the insertion algorithms)
  - extended by homotopy generators of all the relations among the insertion algorithms.
Plan:

1. Motivations

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String Data Structures

- **String data structure** \( (\mathcal{S}, \ell, I, R) \) on an alphabet \( A \):
  - a set \( D_A \),
  - a reading \( \ell : A^* \to A^* \)
    \( x_1 \ldots x_k \mapsto x_{\sigma(1)} \ldots x_{\sigma(k)} \) \( \sigma \) permutation on \( \{1, \ldots, k\} \)
  - right-to-left (resp. left-to-right) \( : \ell_r \) (resp. \( \ell_l \)).
    \( \ell_r(1342543) = 3452431 \)
    \( \ell_l(1342543) = 1342543 \)
  - a one-element insertion map \( I : D_A \times A \to D_A \),
    by iteration, insertion map \( I^* : D_A \times A^* \to D_A \) :
    \( I^*(d, x_1 \ldots x_n) = I^*(I(d, y_1), y_2 \ldots y_n) \)
    \( y_1 \ldots y_n = \ell(x_1 \ldots x_n) \)
  - a reading map \( R : D_A \to A^* \):
    \( I^*(\emptyset, \ell(-))R = \text{Id}_{D_A} \),
    \( R(\emptyset) \) is the empty word,
    \( A \subseteq R(D_A) \subseteq A^* \).

- Constructor \( C_{\mathcal{S}} \) of \( \mathcal{S} : I^*(\emptyset, \ell(-)) : A^* \to D_A \).

- A right (resp. left) SDS : insertion map w. r. t \( \ell_l \) (resp. \( \ell_r \)).
String Data Structures

Example: Young SDSs $\mathcal{Y}^\text{row}_n = (Y_t, l_l, S_r, R_{col})$ and $\mathcal{Y}^\text{col}_n = (Y_t, l_r, S_l, R_{col})$

- $\mathcal{Y}^\text{row}_n$ set of (Young) tableaux:

$$t = \begin{array}{cccccc}
1 & 1 & 1 & 2 & 4 & 4 & 4 \\
2 & 2 & 3 & 3 & 5 & 7 \\
4 & 5 & 5 & 6 \\
6 & 8
\end{array}$$

$R_{col}(t) = 6421852153163254744$

- Schensted’s right insertion $S_r : S_r \left( \begin{array}{c}
1 \\
2 \\
3 \\
4 \\
5 \\
6
\end{array} \right), 2$

- Schensted’s left insertion $S_l : S_l \left( \begin{array}{c}
1 \\
2 \\
3 \\
4 \\
5 \\
6
\end{array} \right), 2$
String Data Structures

- \( I, J : D_A \times A \rightarrow D_A \text{ commute} \) if \( J(I(d, x), y) = I(J(d, y), x) \).

- A left (resp. right) SDS \((D_A, \ell, J, R)\) (resp. \((D_A, \ell, I, R)\)) \text{ commutes to} a right (resp. left) SDS \((D_A, \ell, I, R)\) (resp. \((D_A, \ell, I, R)\)) if \( I \) and \( J \) commute.

\[
y \rightarrow (d \leftarrow x) = (y \rightarrow d) \leftarrow x
\]

- \( S = (D_A, \ell, I, R) \) is \text{ associative} if

\[
\ast_S : D_A \times D_A \rightarrow D_A \\
(d, d') \mapsto d \ast_S d' = I^*(d, \ell(R(d')))
\]

is associative.

**Theorem.** Let \( S \) be a right (resp. left) SDS. If there is a left (resp. right) SDS \( T \) that commutes to \( S \), then \( S \) and \( T \) are associative.

- **Structure monoid** \( M(S) \) associated to \( S = (D_A, \ell, I, R) \) is presented by

\[
\mathcal{R}(S) = \langle D_A \mid \gamma_{d, d'} : d|d' \rightarrow d \ast_S d', \ \forall d, d' \in D_A \rangle.
\]

- \( \mathcal{R}(S) \) is terminating,

- If \( S \) is associative, then \( \mathcal{R}(S) \) is locally confluent.
String Data Structures

- **Reading** of $\mathcal{R}(S)$:
  \[
  \mathcal{R}(A, S) = \langle A \mid \gamma_{d, d'} : R(d)R(d') \rightarrow R(d \ast_S d'), \ \forall d, d' \in D_A \rangle.
  \]
  - If $S$ is associative, then $\mathcal{R}(A, S)$ is locally confluent.

- $S = (D_A, \ell, I, R)$ is *compatible* with an equivalence relation $\sim$ on $A^*$ if:
  - $w \sim w'$ implies $I^*(d, w) = I^*(d, w')$, $\forall d \in D_A$, $\forall w, w' \in A^*$,
  - $RC_S(w) \sim w$, $\forall w \in A^*$.

**Theorem.** Let $S$ be a right SDS compatible with $\sim_S$ induced by $\mathcal{R}(A, S)$. The map $C_S$ induces $A^*/\sim_S \simeq (D_A, \ast_S)$ with the inverse induced by $R$. One says that $\mathcal{R}(S)$ and $\mathcal{R}(A, S)$ are *Tietze-equivalent*.

  - Let $\sim$ be an equivalence relation on the free monoid $K^*$ over $K$. $S \subset K^*$ satisfies the **cross-section property (c.s.p)** for $K^*/\sim$ if each equivalence class w.r.t $\sim$ contains exactly one element of $S$.

  - Let $S$ be a right associative SDS compatible with $\sim_S$ induced by $\mathcal{R}(A, S)$.
    - If $\mathcal{R}(A, S)$ is terminating, then the set of normal forms w.r.t $\mathcal{R}(S)$ satisfies the c.s.p for $M(S) \iff$ the set of normal forms w.r.t $\mathcal{R}(A, S)$ satisfies the c.s.p for $M(S)$.

**Corollary.** Let $S$ be a right associative SDS such that $\mathcal{R}(A, S)$ is terminating. Then $\mathcal{R}(S)$ and $\mathcal{R}(A, S)$ are Tietze-equivalent and the set of normal forms w.r.t $\mathcal{R}(A, S)$ satisfies the c.s.p for $M(S)$. 
String Data Structures

Example: Young SDSs $\mathcal{Y}^\text{row}_n = (Yt_n, \ell_1, S_r, R_{\text{col}})$ and $\mathcal{Y}^\text{col}_n = (Yt_n, \ell_r, S_l, R_{\text{col}})$

- The **plactic monoid** of rank $n$ is presented by the **Knuth presentation** whose set of generators is $\{1, \ldots, n\}$ submitted to the relations:

$$zxy \rightarrow xzy \quad \text{for} \quad x \leq y < z \quad \text{and} \quad yzx \rightarrow yxz \quad \text{for} \quad x < y \leq z.$$

- (Schensted, 1961). $S_r$ and $S_l$ commute: $S_r(S_l(t, x), y) = S_l(S_r(t, y), x)$.
  - then the SDSs $\mathcal{Y}^\text{row}_n$ and $\mathcal{Y}^\text{col}_n$ are associative,
  - then the SRSs $R(\mathcal{Y}^\text{row}_n)$ and $R(\mathcal{Y}^\text{col}_n)$ are convergent.

- The Knuth presentation is Tietze-equivalent to $R(\{1, \ldots, n\}, \mathcal{Y}^\text{row}_n)$.

- (Knuth, 1970). $\mathcal{Y}^\text{row}_n$ is compatible with the equivalence relation induced by the Knuth presentation.
  - then $R(\mathcal{Y}^\text{row}_n)$ is a convergent presentation of the plactic monoid,
  - then the set $Yt_n$ satisfies the c.s.p for the plactic monoid.
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3. Coherent Presentations and String Data Structures
Coherent Presentations and String Data Structures

Change of generators of an SDS $\mathbb{S} = (D_A, \ell, I, R)$

- A binary relation $|$ on $D_A$ is **compatible** with $R$ if $R(d|d') = R(d)R(d')$, where $d|d'$ denotes $(d, d') \in |$.

- **Generating set** $Q \subset D_A$ w.r.t such a binary relation:
  - $A \subseteq R(Q)$,
  - $d = c_1|\ldots|c_k \in D_A$, with $c_1, \ldots, c_k \in Q$.

- **SDS** $\mathbb{S}_Q = (D_A, \ell_Q, I_Q, R_Q)$ on a generating set $Q$:
  - $\ell_Q(c_1 \ldots c_k) = c_{\sigma(1)} \ldots c_{\sigma(k)} \in Q^*$, where $\sigma$ is a permutation on $\{1, \ldots, k\}$,
  - $I_Q : D_A \times Q \longrightarrow D_A$, $I_Q(d, c) = I^*(d, R(c))$,
  - $R_Q : D_A \longrightarrow Q^*$, $R_Q(d) = c_1|\ldots|c_k$ is the decomposition of $d$ w.r.t $|$.

- **A reduced presentation**: $Q$ generating set w.r.t $|$ compatible with $R$.

  $$\mathcal{R}(Q, \mathbb{S}) = \langle Q \mid \gamma_{c,c'} : c|c' \longrightarrow R_Q(c \ast_{\mathbb{S}} c'), \ c, c' \in Q, \ c|c' \notin D_A \rangle.$$  

**Lemma.** Let $\mathbb{S}$ be an associative SDS and $Q$ be a generating set w.r.t $|$ compatible with $R$. If $\mathcal{R}(Q, \mathbb{S})$ is normalizing, then $\mathcal{R}(\mathbb{S})$ and $\mathcal{R}(Q, \mathbb{S})$ are Tietze-equivalent.
Example: Young SDS $\mathcal{Y}_n^{col} = (\mathcal{Y}_n, l_r, S_l, R_{col})$

- $\text{Col}_n$: set of tableaux with only one column.
- $|$ : concatenation of columns in $\mathcal{Y}_n$.
  - $d = c_1 | \ldots | c_k \in \mathcal{Y}_n$, where $c_1, \ldots, c_k$ are the columns of $d$ from left to right,
  - $R_{col}(d) = R_{col}(c_1) \ldots R_{col}(c_k)$,
  - $|$ is compatible with $R_{col}$,
  - $\text{Col}_n$ is a generating set w.r.t $|$. 

- $R_{\text{Col}_n}: \mathcal{Y}_n \rightarrow \text{Col}_n^*$ writes a tableau as the concatenation of its columns from left to right.

- $\mathcal{Y}_n^{row}$ is associative:
  - $\mathcal{R}(\text{Col}_n, \mathcal{Y}_n^{col})$ is normalizing,
  - then $\mathcal{R}(\mathcal{Y}_n^{col})$ and $\mathcal{R}(\text{Col}_n, \mathcal{Y}_n^{col})$ are Tietze-equivalent.

- $\mathcal{R}(\text{Col}_n, \mathcal{Y}_n^{col})$ is a finite convergent presentation of the plactic monoid.
  - (Bokut, Chen, Chen, Li, 2015), (Cain, Gray, Malheiro, 2015).
Coherent Presentations and String Data Structures

Coherence by insertions.

- **Normalisation strategy (n.s)** for an SRS: mapping $\sigma$ of every generator $u$ to a rewriting step from $u$ to a chosen normal form $\hat{u}$.
  - leftmost one $\sigma^\top$, rightmost one $\sigma^\bot$.
- A normalization strategy $\sigma$ of $\mathcal{R}(Q, S)$ computes $C_S$ if it reduces any $c_1|\ldots|c_n \in Q^*$ to $R_Q(c_1 \ast_S \ldots \ast_S c_n)$.

**Theorem.** Let $S$ be an associative SDS such that $\mathcal{R}(Q, S)$ terminating. If there exists a n.s that computes $C_S$, then the set of normal forms of $\mathcal{R}(Q, S)$ satisfies the c.s.p for $M(S)$.

If $\sigma^\top$ computes $C_S$, then $\mathcal{R}(Q, S)$ is extended into a coherent presentation:

\[
\sigma^\top \quad \sigma^\top_{cc',c''} \quad R_Q(c \ast_S c' \ast_S c'') \quad \forall c, c', c'' \in Q.
\]
Coherent Presentations and String Data Structures

Coherence by insertions. $\sigma^\top$ (resp. $\sigma^\perp$) w.r.t $R(Q, S)$ for a right SDS $S$. Suppose $T$ commutes to $S$. If $\sigma^\top$ computes $C_S$, then $R(Q, S)$ can be extended into a coherent presentation by adjunction of

$$\sigma^\top_{cc'}c'' \quad \sigma^\top_{cc'}c''$$

$\sigma^\top$ (resp. $\sigma^\perp$) $\leadsto$ right (resp. left) insertion of $S$ (resp. $T$).

**Example:** Young SDS $\gamma_n^{col} = (Y_t, \ell_r, S_l, R_{col})$ : $\sigma^\top$ w. r. t $R(Col_n, \gamma_n^{col})$ computes $C_{\gamma_n^{col}}$. Then $R(Col_n, \gamma_n^{col})$ is extended into a coherent presentation:

$$R_{Col_n}(c \ast_{\gamma_n^{col}} c') = c_1 | c_2, \quad R_{Col_n}(c_2 \ast_{\gamma_n^{col}} c'') = c_3 | c_4, \quad R_{Col_n}(c_1 \ast_{\gamma_n^{col}} c_3) = c_3' | c_5, \quad R_{Col_n}(c_1' \ast_{\gamma_n^{row}} c_2') = c_1' | c_2', \quad R_{Col_n}(c_1' \ast_{\gamma_n^{row}} c_4') = c_3' | c_4', \quad R_{Col_n}(c_2' \ast_{\gamma_n^{row}} c_4) = c_5 | c_4.$$


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Conclusion

- We study the confluence of SRS whose rules are defined by insertion algorithm using the notion of SDS.
  - If a right SDS and a left SDS presenting a monoid commute
    - these SDSs are associatives,
    - the SRS presenting the structure monoid is confluent,
    - we obtain a minimal locally confluent presentation of the monoid,
    - we obtain a cross-section property for the monoid.

- Applications on the **Chinese monoid** generated by the set \{1, \ldots, n\} and subject to the relations \( zyx = zxy = yzx \), for \( x \leq y \leq z \).
  - construct an SDS associated to the insertion algorithm in **Chinese staircases**,
  - deduce the confluence of the reduced presentation of this monoid,
  - extend this presentation into a finite coherent presentation of this monoid.

- The **sylvester monoid** is generated by \{1, \ldots, n\} and subject to the relations \( zxvy = xzvy \), for \( x \leq y < z \) and \( v \in \{1, \ldots, n\}^* \).
  - it can be described using the notion of **binary search trees**,
  - we expect that our methods should conduce to a coherent presentation of this monoid induced by the insertion algorithm in a binary search tree.
Thank you for your attention!