Coherence of monoids by insertions

Nohra Hage

Université Saint-Joseph de Beyrouth
Ecole Supérieure d'Ingénieurs de Beyrouth

Joint works with Philippe Malbos

7th International Workshop on Confluence, IWC 2018

July 7th 2018, Oxford, United Kingdom

Plan :

1. **Motivations**
2. **String Data Structures**
3. **Coherent presentations and String data structures**
4. **Conclusion**

Motivations

- **Data structures** describe a way to organize and to store a collection of structured data.
  - appear in **combinatorial algebra**, **combinatorics** and **fundamental computer science**,
  - describe combinatorial structures : **arrays**, **tableaux**, **staircases**, **binary search trees**...
  - array structures can be used to study **plactic**, **Chinese**, **hypoplactic** and **sylvester monoids**...
- Study **string rewriting systems** (SRS) :
  - normal forms can be described using a data structure,
  - rewriting rules are induced by insertion algorithms.
- Introduce the notion of **string data structure** (SDS) :
  - the data are constructed using an insertion algorithm,
  - they are described by words through a reading map.

Motivations

Plactic monoids :

- (Young, 1900) : Young tableaux

1	1	1	2	4	4	4
2	2	3	3	5	7	
4	5	5	6			
6	8					

- (Schensted, 1961) : left and right insertions on Young tableaux
  - $u \in \{1, \dots, n\}^* \rightsquigarrow P(u)$.
- (Knuth, 1970) : $u, v \in \{1, \dots, n\}^*$, $\quad u \sim v \Leftrightarrow P(u) = P(v)$.
  - $\sim$ coincides with the congruence generated by **Knuth relations** :

$$zxy = xzy \quad \text{for } x \leq y < z \quad \text{and} \quad yzx = yxz \quad \text{for } x < y \leq z.$$

- (Lascoux, Schützenberger, 1981) : **plactic monoid** = $\{1, \dots, n\}^* / \sim$.
- Applications on **combinatorics**, **representation theory**, **rewriting theory**...

Motivations

Plactic monoids :

- (Kubat, Okninski, 2014) : $n > 4 \rightsquigarrow$ no finite completion of the Knuth presentation w.r.t the lexicographic order.

- (Cain, Gray, Malheiro, 2015), (Bokut, Chen, Chen, Li , 2015) : finite completions obtained by adding new generators :
  - column or row generators $\rightsquigarrow$ convergent presentation of plactic monoids.

- (Hage, Malbos, 2017) : coherent presentations of the plactic monoid giving all the relations among the relations of its presentations.
  - **Confluence property** is essential to obtain such coherence results.
  - **Commutation** of right and left insertions $\rightsquigarrow$ confluence of the presentation.

- Study these confluence results in a general algebraic framework using the notion of string data structure.

- Explicit **coherent presentation** of the monoid presented by an SDS :
  - a presentation of the monoid (generators + rewriting rules describing the insertion algorithms)
  - extended by **homotopy generators** of all the relations among the insertion algorithms.

Plan :

1. **Motivations**

2. **String Data Structures**

String Data Structures

- **String data structure** (**SDS**) : $\mathbb{S} = (\mathrm{D}_A, \ell, I, R)$ on an alphabet A :
  - a **set** D_A,
  - a **reading** $\ell : A^* \longrightarrow A^*$

    $x_1 \dots x_k \mapsto x_{\sigma(1)} \dots x_{\sigma(k)}$ σ permutation on $\{1, \dots, k\}$
    - **right-to-left** (resp. **left-to-right**) : ℓ_r (resp. ℓ_l).

      $\ell_r(1342543) = 3452431 \qquad \ell_l(1342543) = 1342543$
  - a **one-element insertion map** $I : \mathrm{D}_A \times A \longrightarrow \mathrm{D}_A$,
    - by iteration, **insertion map** $I^* : \mathrm{D}_A \times A^* \longrightarrow \mathrm{D}_A$:

      $I^*(d, x_1 \dots x_n) = I^*(I(d, y_1), y_2 \dots y_n), \qquad y_1 \dots y_n = \ell(x_1 \dots x_n)$
  - a **reading map** $R : \mathrm{D}_A \longrightarrow A^*$:
    - $I^*(\emptyset, \ell(-))R = \mathrm{Id}_{\mathrm{D}_A}$,
    - $R(\emptyset)$ is the empty word,
    - $A \subseteq R(\mathrm{D}_A) \subseteq A^*$.
- **Constructor** $C_{\mathbb{S}}$ of $\mathbb{S}$: $I^*(\emptyset, \ell(-)) : A^* \longrightarrow \mathrm{D}_A$.
- A ***right*** (resp. ***left***) **SDS** : insertion map w. r. t ℓ_l (resp. ℓ_r).

Example : Young SDSs $\mathcal{Y}_n^{row} = (\mathrm{Yt}_n, \ell_\mathrm{l}, S_r, R_{col})$ and $\mathcal{Y}_n^{col} = (\mathrm{Yt}_n, \ell_\mathrm{r}, S_l, R_{col})$

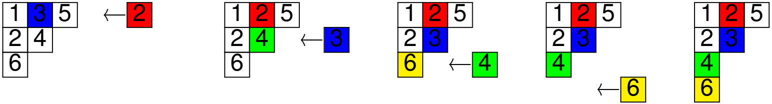
- $\mathcal{Y}_n^{row}$ set of **(Young) tableaux** :

t =

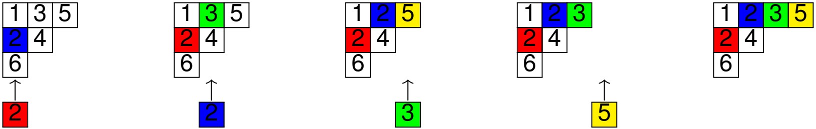
1	1	1	2	4	4	4
2	2	3	3	5	7	
4	5	5	6			
6	8					

$$R_{col}(t) = 642185215316325474 4$$

- **Schensted's right insertion** S_r : $S_r\left(\begin{smallmatrix}1&3&5\\2&4&\\6&&\end{smallmatrix}, 2\right)$



- **Schensted's left insertion** S_l : $S_l\left(\begin{smallmatrix}1&3&5\\2&4&\\6&&\end{smallmatrix}, 2\right)$



String Data Structures

- $I, J : \mathrm{D}_A \times A \longrightarrow \mathrm{D}_A$ **commute** if $J(I(d,x),y) = I(J(d,y),x)$.
- A left (resp. right) SDS $(\mathrm{D}_A, \ell_{\mathrm{r}}, J, R)$ (resp. $(\mathrm{D}_A, \ell_{\mathrm{l}}, J, R)$) **commutes to** a right (resp. left) SDS $(\mathrm{D}_A, \ell_{\mathrm{l}}, I, R)$ (resp. $(\mathrm{D}_A, \ell_{\mathrm{r}}, I, R)$) if I and J commute.

$$y \rightarrow (d \leftarrow x) = (y \rightarrow d) \leftarrow x$$

- $\mathbb{S} = (\mathrm{D}_A, \ell, I, R)$ is **associative** if

$$\begin{aligned} \star_{\mathbb{S}} : \mathrm{D}_A \times \mathrm{D}_A &\longrightarrow \mathrm{D}_A \\ (d, d') &\mapsto d \star_{\mathbb{S}} d' = I^*(d, \ell(R(d'))) \end{aligned}$$

  is associative.

Theorem. Let $\mathbb{S}$ be a right (resp. left) SDS. If there is a left (resp. right) SDS $\mathbb{T}$ that commutes to $\mathbb{S}$, then $\mathbb{S}$ and $\mathbb{T}$ are associative.

- **Structure monoid** $\mathbf{M}(\mathbb{S})$ associated to $\mathbb{S} = (\mathrm{D}_A, \ell, I, R)$ is presented by

$$\mathcal{R}(\mathbb{S}) = \langle\, \mathrm{D}_A \mid \gamma_{d,d'} : d|d' \longrightarrow d \star_{\mathbb{S}} d', \;\; \forall d, d' \in \mathrm{D}_A \,\rangle.$$

  - $\mathcal{R}(\mathbb{S})$ is terminating,
  - If $\mathbb{S}$ is associative, then $\mathcal{R}(\mathbb{S})$ is locally confluent.

- **Reading** of $\mathcal{R}(\mathbb{S})$:

$$\mathcal{R}(A,\mathbb{S}) = \langle\, A \mid \gamma_{d,d'} : R(d)R(d') \longrightarrow R(d \star_{\mathbb{S}} d'), \ \ \forall d, d' \in \mathrm{D}_A \,\rangle.$$

  - If $\mathbb{S}$ is associative, then $\mathcal{R}(A,\mathbb{S})$ is locally confluent.

- $\mathbb{S} = (\mathrm{D}_A, \ell, I, R)$ is **compatible** with an equivalence relation $\sim$ on A^* if :
  - $w \sim w'$ implies $I^*(d,w) = I^*(d,w')$, $\forall d \in \mathrm{D}_A$, $\forall w, w' \in A^*$,
  - $RC_{\mathbb{S}}(w) \sim w$, $\ \forall w \in A^*$.

Theorem. Let $\mathbb{S}$ be a right SDS compatible with $\sim_{\mathbb{S}}$ induced by $\mathcal{R}(A,\mathbb{S})$. The map $C_{\mathbb{S}}$ induces $A^*/\sim_{\mathbb{S}} \ \simeq\ (\mathrm{D}_A, \star_{\mathbb{S}})$ with the inverse induced by R. One says that $\mathcal{R}(\mathbb{S})$ and $\mathcal{R}(A,\mathbb{S})$ are **Tietze-equivalent**.

- Let $\sim$ be an equivalence relation on the free monoid K^* over K. $S \subset K^*$ satisfies the **cross-section property (c.s.p)** for $K^*/\sim$ if each equivalence class w.r.t $\sim$ contains exactly one element of S.

- Let $\mathbb{S}$ be a right associative SDS compatible with $\sim_{\mathbb{S}}$ induced by $\mathcal{R}(A,\mathbb{S})$.
  - If $\mathcal{R}(A,\mathbb{S})$ is terminating, then the set of normal forms w.r.t $\mathcal{R}(\mathbb{S})$ satisfies the c.s.p for $\mathbf{M}(\mathbb{S})$ $\Leftrightarrow$ the set of normal forms w.r.t $\mathcal{R}(A,\mathbb{S})$ satisfies the c.s.p for $\mathbf{M}(\mathbb{S})$.

Corollary. Let $\mathbb{S}$ be a right associative SDS such that $\mathcal{R}(A,\mathbb{S})$ is terminating. Then $\mathcal{R}(\mathbb{S})$ and $\mathcal{R}(A,\mathbb{S})$ are Tietze-equivalent and the set of normal forms w.r.t $\mathcal{R}(A,\mathbb{S})$ satisfies the c.s.p for $\mathbf{M}(\mathbb{S})$.

String Data Structures

Example : Young SDSs $\mathcal{Y}_n^{row} = (\mathrm{Yt}_n, \ell_\mathrm{l}, S_r, R_{col})$ and $\mathcal{Y}_n^{col} = (\mathrm{Yt}_n, \ell_\mathrm{r}, S_l, R_{col})$

- The **plactic monoid** of rank n is presented by the **Knuth presentation** whose set of generators is $\{1, \ldots, n\}$ submitted to the relations :

$$zxy \longrightarrow xzy \;\text{ for }\; x \leq y < z \quad \text{and} \quad yzx \longrightarrow yxz \;\text{ for }\; x < y \leq z.$$

- (Schensted,1961). S_r and S_l commute : $S_r(S_l(t,x),y) = S_l(S_r(t,y),x)$.
  - then the SDSs $\mathcal{Y}_n^{row}$ and $\mathcal{Y}_n^{col}$ are associative,
  - then the SRSs $\mathcal{R}(\mathcal{Y}_n^{row})$ and $\mathcal{R}(\mathcal{Y}_n^{col})$ are convergent.
- The Knuth presentation is Tietze-equivalent to $\mathcal{R}(\{1, \ldots, n\}, \mathcal{Y}_n^{row})$.
- (Knuth, 1970). $\mathcal{Y}_n^{row}$ is compatible with the equivalence relation induced by the Knuth presentation.
  - then $\mathcal{R}(\mathcal{Y}_n^{row})$ is a convergent presentation of the plactic monoid,
  - then the set Yt_n satisfies the c.s.p for the plactic monoid.

Plan :

1. **Motivations**
2. **String Data Structures**
3. **Coherent Presentations and String Data Structures**

Coherent Presentations and String Data Structures

Change of generators of an SDS $\mathbb{S} = (\mathrm{D}_A, \ell, I, R)$

- A binary relation $|$ on D_A is **compatible** with R if $R(d|d') = R(d)R(d')$, where $d|d'$ denotes $(d, d') \in |$.

  - **Generating set** $Q \subset \mathrm{D}_A$ w.r.t such a binary relation :
    - $A \subseteq R(Q)$,
    - $d = c_1| \ldots |c_k \in \mathrm{D}_A$, with $c_1, \ldots, c_k \in Q$.

  - SDS $\mathbb{S}_Q = (\mathrm{D}_A, \ell_Q, I_Q, R_Q)$ on a generating set Q :
    - $\ell_Q(c_1 \ldots c_k) = c_{\sigma(1)} \ldots c_{\sigma(k)} \in Q^*$, where σ is a permutation on $\{1, \ldots, k\}$,
    - $I_Q : \mathrm{D}_A \times Q \longrightarrow \mathrm{D}_A, \qquad I_Q(d, c) = I^*(d, R(c))$,
    - $R_Q : \mathrm{D}_A \longrightarrow Q^*, \qquad R_Q(d) = c_1| \ldots |c_k$ is the decomposition of d w.r.t $|$.

- **A reduced presentation :** Q generating set w.r.t $|$ compatible with R.

$$\mathcal{R}(Q, \mathbb{S}) = \langle\, Q \mid \gamma_{c,c'} : c|c' \longrightarrow R_Q(c \star_{\mathbb{S}} c'), \;\; c, c' \in Q, \;\; c|c' \notin \mathrm{D}_A \,\rangle.$$

Lemma. Let $\mathbb{S}$ be an associative SDS and Q be a generating set w.r.t $|$ compatible with R. If $\mathcal{R}(Q, \mathbb{S})$ is normalizing, then $\mathcal{R}(\mathbb{S})$ and $\mathcal{R}(Q, \mathbb{S})$ are Tietze-equivalent.

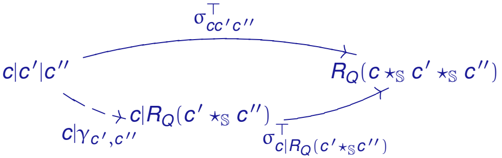
Example : Young SDS $\mathcal{Y}_n^{col} = (\mathrm{Yt}_n, \ell_r, S_l, R_{col})$

- Col_n : set of tableaux with only one column.
- $|$: concatenation of columns in Yt_n.
  - $d = c_1|\ldots|c_k \in \mathrm{Yt}_n$, where $c_1, \ldots, c_k$ are the columns of d from left to right,
  - $R_{col}(d) = R_{col}(c_1)\ldots R_{col}(c_k)$,
  - $|$ is compatible with R_{col},
  - Col_n is a generating set w.r.t $|$.
- $R_{\mathrm{Col}_n} : \mathrm{Yt}_n \longrightarrow \mathrm{Col}_n^*$ writes a tableau as the concatenation of its columns from left to right.
- $\mathcal{Y}_n^{row}$ is associative :
  - $\mathcal{R}(\mathrm{Col}_n, \mathcal{Y}_n^{col})$ is normalizing,
  - then $\mathcal{R}(\mathcal{Y}_n^{col})$ and $\mathcal{R}(\mathrm{Col}_n, \mathcal{Y}_n^{col})$ are Tietze-equivalent.
- $\mathcal{R}(\mathrm{Col}_n, \mathcal{Y}_n^{col})$ is a finite convergent presentation of the plactic monoid.
  - (Bokut, Chen, Chen, Li, 2015), (Cain, Gray, Malheiro, 2015).

Coherence by insertions.

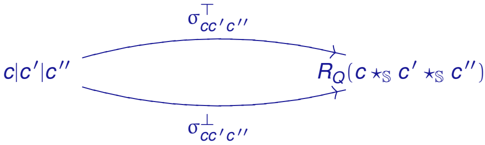
- **Normalisation strategy (n.s)** for an SRS : mapping σ of every generator u to a rewriting step from u to a chosen normal form $\widehat{u}$.
  - **leftmost** one $\sigma^{\top}$, **rightmost** one $\sigma^{\perp}$.
- A normalization strategy σ of $\mathcal{R}(Q,\mathbb{S})$ **computes** $C_{\mathbb{S}}$ if it reduces any $c_1|\ldots|c_n \in Q^*$ to $R_Q(c_1 \star_{\mathbb{S}} \ldots \star_{\mathbb{S}} c_n)$.

Theorem. Let $\mathbb{S}$ be an associative SDS such that $\mathcal{R}(Q,\mathbb{S})$ terminating. If there exists a n.s that computes $C_{\mathbb{S}}$, then the set of normal forms of $\mathcal{R}(Q,\mathbb{S})$ satisfies the c.s.p for $\mathbf{M}(\mathbb{S})$.
If $\sigma^{\top}$ computes $C_{\mathbb{S}}$, then $\mathcal{R}(Q,\mathbb{S})$ is extended into a coherent presentation :



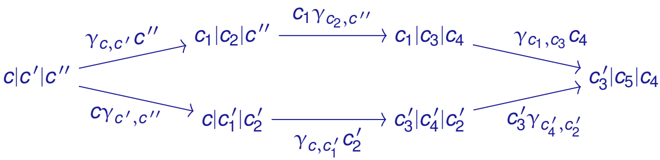
$\forall c, c', c'' \in Q.$

Coherence by insertions. $\sigma^{\top}$ (resp. $\sigma^{\perp}$) w.r.t $\mathcal{R}(Q, \mathbb{S})$ for a right SDS $\mathbb{S}$. Suppose $\mathbb{T}$ commutes to $\mathbb{S}$. If $\sigma^{\top}$ computes $C_{\mathbb{S}}$, then $\mathcal{R}(Q, \mathbb{S})$ can be extended into a coherent presentation by adjunction of



$\sigma^{\top}$ (resp. $\sigma^{\perp}$) $\rightsquigarrow$ right (resp. left) insertion of $\mathbb{S}$ (resp. $\mathbb{T}$).

Example : Young SDS $\mathcal{Y}_n^{col} = (\mathrm{Yt}_n, \ell_r, S_l, R_{col})$: $\sigma^{\top}$ w. r. t $\mathcal{R}(\mathrm{Col}_n, \mathcal{Y}_n^{col})$ computes $C_{\mathcal{Y}_n^{col}}$.Then $\mathcal{R}(\mathrm{Col}_n, \mathcal{Y}_n^{col})$ is extended into a coherent presentation :



$R_{\mathrm{Col}_n}(c \star_{\mathcal{Y}_n^{col}} c') = c_1|c_2$, $R_{\mathrm{Col}_n}(c_2 \star_{\mathcal{Y}_n^{col}} c'') = c_3|c_4$, $R_{\mathrm{Col}_n}(c_1 \star_{\mathcal{Y}_n^{col}} c_3) = c_3'|c_5$,
$R_{\mathrm{Col}_n}(c'' \star_{\mathcal{Y}_n^{row}} c') = c_1'|c_2'$, $R_{\mathrm{Col}_n}(c_1' \star_{\mathcal{Y}_n^{row}} c) = c_3'|c_4'$, $R_{\mathrm{Col}_n}(c_2' \star_{\mathcal{Y}_n^{row}} c_4') = c_5|c_4$.

Plan :

1. **Motivations**
2. **String Data Structures**
3. **Coherent Presentations and String Data Structures**
4. **Conclusion**

Conclusion

- We study the confluence of SRS whose rules are defined by insertion algorithm using the notion of SDS.
  - If a right SDS and a left SDS presenting a monoid commute
    - these SDSs are associatives,
    - the SRS presenting the structure monoid is confluent,
    - we obtain a minimal locally confluent presentation of the monoid,
    - we obtain a cross-section property for the monoid.
- Applications on the **Chinese monoid** generated by the set $\{1, \ldots, n\}$ and subject to the relations $zyx = zxy = yzx$, for $x \leq y \leq z$.
  - construct an SDS associated to the insertion algorithm in **Chinese staircases**,
  - deduce the confluence of the reduced presentation of this monoid,
  - extend this presentation into a finite coherent presentation of this monoid.
- The **sylvester monoid** is generated by $\{1, \ldots, n\}$ and subject to the relations $zxvy = xzvy$, for $x \leq y < z$ and $v \in \{1, \ldots, n\}^*$.
  - it can be described using the notion of **binary search trees**,
  - we expect that our methods should conduce to a coherent presentation of this monoid induced by the insertion algorithm in a binary search tree.

Thank you for your attention !