Convergence of Simultaneously and Sequentially Unraveled TRSs for Normal Conditional TRSs

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Oriented Conditional Term Rewriting System (CTRS)

- A set $\mathcal{R}$ of oriented conditional rules $\ell \to r \Leftarrow s_1 \twoheadrightarrow t_1, \ldots, s_k \twoheadrightarrow t_k$
- Reduction $\to_{\mathcal{R}}$ is recursively defined as follows:

$$C[\ell\sigma] \to_{\mathcal{R}} C[r\sigma] \text{ iff } \left(\begin{array}{c} \exists \ell \to r \Leftarrow s_1 \twoheadrightarrow t_1, \ldots, s_k \twoheadrightarrow t_k \in \mathcal{R}. \\ s_1\sigma \to_{\mathcal{R}}^* t_1\sigma \wedge \cdots \wedge s_k\sigma \to_{\mathcal{R}}^* t_k\sigma \end{array} \right)$$

Example

$$\mathcal{R}_0 = \left\{ \begin{array}{lrcl} (1) & \mathsf{even}(0) \to \mathsf{true} & & \\ (2) & \mathsf{even}(\mathsf{s}(x)) \to \mathsf{false} & \Leftarrow & \mathsf{even}(x) \twoheadrightarrow \mathsf{true} \\ (3) & \mathsf{even}(\mathsf{s}(x)) \to \mathsf{true} & \Leftarrow & \mathsf{even}(x) \twoheadrightarrow \mathsf{false} \end{array} \right\}$$

- even(s(s(0)))

- Analysis of properties for CTRSs is much more complicated than that of TRSs

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- $\mathsf{even}(\mathsf{s}(\mathsf{s}(0))) \to_{(3)}$
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    - $\mathsf{even}(0) \to_{(1)} \mathsf{true}$

- $\mathsf{even}(\mathsf{s}(\mathsf{s}(0))) \to_{(2)}$
  - $\mathsf{even}(\mathsf{s}(0)) \to_{(3)}^{?} \mathsf{true}$

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Example

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- $\mathsf{even}(\mathsf{s}(\mathsf{s}(0))) \to_{(3)} \mathsf{true}$
  - $\mathsf{even}(\mathsf{s}(0)) \to_{(2)} \mathsf{false}$
    - $\mathsf{even}(0) \to_{(1)} \mathsf{true}$
- $\mathsf{even}(\mathsf{s}(\mathsf{s}(0))) \to_{(2)}$
  - $\mathsf{even}(\mathsf{s}(0)) \to_{(3)}^{?} \mathsf{true}$
    - $\mathsf{even}(0) \not\to_{\mathcal{R}_0} \mathsf{false}$

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Transformational Approach to CTRS Analysis

- Transform a CTRS into an unconditional TRS and analyze the TRS
  - Techniques for TRSs can be applied
  - Soundness w.r.t. reduction is necessary for some properties
- Existing transformations are
  - unravelings [Marchiori, 1996, Marchiori, 1997, Ohlebusch, 2001]
  - structure-preserving transformations [Viry, 1999, Şerbănuţă and Roşu, 2006]

  which are defined for normal 1-CTRSs and 3-DCTRSs

Normal 1-CTRSs and 3-DCTRSs

- $\mathcal{R}$ is a normal 1-CTRS if $\forall \ell \to r \Leftarrow s_1 \twoheadrightarrow t_1, \ldots, s_k \twoheadrightarrow t_k \in \mathcal{R}$,
  - $\mathcal{V}ar(\ell) \supseteq \mathcal{V}ar(r, s_1, t_1, \ldots, s_k, t_k)$, and
  - $t_1, \ldots, t_k$ are ground normal forms of $\mathcal{R}_u = \{\ell \to r \mid \ell \to r \Leftarrow c \in \mathcal{R}\}$
- $\mathcal{R}$ is a 3-DCTRS if $\forall \ell \to r \Leftarrow s_1 \twoheadrightarrow t_1, \ldots, s_k \twoheadrightarrow t_k \in \mathcal{R}$,
  - $\mathcal{V}ar(\ell, s_1, t_1, \ldots, s_k, t_k) \supseteq \mathcal{V}ar(r)$, and
  - $\forall i.\ \mathcal{V}ar(s_i) \subseteq \mathcal{V}ar(\ell, t_1, \ldots, t_{i-1})$
    - Evaluate conditions from left to right
- Normal 1-CTRSs are 3-DCTRSs

Example (contd.)

$$\mathcal{R}_0 = \left\{ \begin{array}{llcl} (1) & \mathsf{even}(0) \to \mathsf{true} & & \\ (2) & \mathsf{even}(\mathsf{s}(x)) \to \mathsf{false} & \Leftarrow & \mathsf{even}(x) \twoheadrightarrow \mathsf{true} \\ (3) & \mathsf{even}(\mathsf{s}(x)) \to \mathsf{true} & \Leftarrow & \mathsf{even}(x) \twoheadrightarrow \mathsf{false} \end{array} \right\} \text{ is a normal 1-CTRS}$$

Unravelings [Marchiori, 1996, Marchiori, 1997, Ohlebusch, 2001]

- For $\rho : \ell \to r \Leftarrow s_1 \twoheadrightarrow t_1, \ldots, s_k \twoheadrightarrow t_k \in \mathcal{R}$,

$$\mathbb{U}_{sim}(\rho) = \left\{ \begin{array}{rcl} \ell & \to & \mathsf{U}^\rho(s_1, \ldots, s_k, \overrightarrow{\mathcal{V}ar(\ell)}) \\ \mathsf{U}^\rho(t_1, \ldots, t_n, \overrightarrow{\mathcal{V}ar(\ell)}) & \to & r \end{array} \right\}$$

$$\mathbb{U}_{seq}(\rho) = \left\{ \begin{array}{rcl} \ell & \to & \mathsf{U}_1^\rho(s_1, \overrightarrow{\mathcal{V}ar(\ell)}) \\ \mathsf{U}_1^\rho(t_1, \overrightarrow{\mathcal{V}ar(\ell)}) & \to & \mathsf{U}_2^\rho(s_2, \overrightarrow{\mathcal{V}ar(\ell, t_1)}) \\ \mathsf{U}_2^\rho(t_2, \overrightarrow{\mathcal{V}ar(\ell, t_1)}) & \to & \mathsf{U}_3^\rho(s_3, \overrightarrow{\mathcal{V}ar(\ell, t_1, t_2)}) \\ & \vdots & \\ \mathsf{U}_k^\rho(t_k, \overrightarrow{\mathcal{V}ar(\ell, t_1, \ldots, t_{k-1})}) & \to & r \end{array} \right\}$$

  where $\mathsf{U}^\rho, \mathsf{U}_1^\rho, \ldots, \mathsf{U}_k^\rho$ (U symbols) are newly introduced function symbols

- $\mathbb{U}(\ell \to r) = \{\ell \to r\}$ and $\mathbb{U}(\mathcal{R}) = \bigcup_{\rho \in \mathcal{R}} \mathbb{U}(\rho)$ where $\mathbb{U}$ is $\mathbb{U}_{sim}$ or $\mathbb{U}_{seq}$

- $\to_{\mathcal{R}} \subseteq \to^+_{\mathbb{U}(\mathcal{R})}$

Example ([Giesl and Arts, 2001, pp. 42–43], `278.trs` in Cops)

$$\mathcal{R}_1 = \left\{ \begin{array}{l} \alpha : \mathsf{p}(y,x) \to \mathsf{p}(\mathsf{a}(\mathsf{mp}(\mathsf{slf},\mathsf{nil}),\mathsf{s2}(x,y)),x) \\ \qquad\qquad \Leftarrow \quad \mathsf{leq}(x,\mathsf{len}(y)) \twoheadrightarrow \mathsf{true},\ \mathsf{even}(\mathsf{s1}(x,y)) \twoheadrightarrow \mathsf{false} \\ \qquad\qquad \vdots \end{array} \right\}$$

$$\mathbb{U}_{sim}(\alpha) = \left\{ \begin{array}{r} \mathsf{p}(y,x) \to \mathsf{U}^{\alpha}(\ \mathsf{leq}(x,\mathsf{len}(y)),\ \mathsf{even}(\mathsf{s1}(x,y)),\ y,x) \\ \mathsf{U}^{\alpha}(\ \mathsf{true},\ \mathsf{false},\ y,x) \to \mathsf{p}(\mathsf{a}(\mathsf{mp}(\mathsf{slf},\mathsf{nil}),\mathsf{s2}(x,y)),x) \end{array} \right\}$$

$$\mathbb{U}_{seq}(\alpha) = \left\{ \begin{array}{c} \mathsf{p}(y,x) \to \mathsf{U}^{\alpha}_1(\mathsf{leq}(x,\mathsf{len}(y)),y,x) \\ \mathsf{U}^{\alpha}_1(\mathsf{true},y,x) \to \mathsf{U}^{\alpha}_2(\mathsf{even}(\mathsf{s1}(x,y)),y,x) \\ \mathsf{U}^{\alpha}_2(\mathsf{false},y,x) \to \mathsf{p}(\mathsf{a}(\mathsf{mp}(\mathsf{slf},\mathsf{nil}),\mathsf{s2}(x,y)),x) \end{array} \right\}$$

Operational-Termination and Confluence via Unravelings

Theorem ([Durán et al., 2008])

A 3-DCTRS $\mathcal{R}$ is operationally terminating if $\mathbb{U}_{seq}(\mathcal{R})$ is terminating

Theorem ([Gmeiner et al., 2010, Gmeiner et al., 2012, Gmeiner et al., 2013])

- *A weakly-left-linear normal 1-CTRS $\mathcal{R}$ is confluent if $\mathbb{U}_{sim}(\mathcal{R})$ is confluent*
- *A weakly-left-linear 3-DCTRS $\mathcal{R}$ is confluent if $\mathbb{U}_{seq}(\mathcal{R})$ is confluent*

- Both $\mathbb{U}_{sim}$ and $\mathbb{U}_{seq}$ are applicable to normal 1-CTRSs
  - CO3 [Nishida et al., 2015] uses $\mathbb{U}_{sim}$ and $\mathbb{U}_{seq}$ in order
- Which is better to prove OP-termination, confluence, etc, $\mathbb{U}_{sim}$ or $\mathbb{U}_{seq}$?
  - or equivalent?

Purpose and Results

Purpose

Comparison of $\mathbb{U}_{sim}$ and $\mathbb{U}_{seq}$ w.r.t. termination and confluence

Results

We show that for a normal 1-CTRS $\mathcal{R}$,

1. $\mathbb{U}_{sim}(\mathcal{R})$ is orthogonal iff $\mathbb{U}_{seq}(\mathcal{R})$ is so
2. if $\mathbb{U}_{sim}(\mathcal{R})$ is terminating, then
   - (i) $\mathbb{U}_{seq}(\mathcal{R})$ is terminating
   - (ii) $\mathbb{U}_{sim}(\mathcal{R})$ is locally confluent iff $\mathbb{U}_{seq}(\mathcal{R})$ is so

- 2. implies that if $\mathbb{U}_{sim}(\mathcal{R})$ is terminating, then
  - $\mathbb{U}_{sim}(\mathcal{R})$ is convergent iff $\mathbb{U}_{seq}(\mathcal{R})$ is so
- CO3 uses the following criteria for confluence
  - orthogonality
  - termination and local-confluence (joinability of critical pairs)

Assumption for Readability

- Let $\mathcal{R}$ be a normal 1-CTRS
  - Then, $\forall i.\ \mathcal{V}ar(\ell, t_1, \ldots, t_i) = \mathcal{V}ar(\ell)$ because $t_1, \ldots, t_k$ are ground
- $k = 2$ for $\rho : \ell \to r \Leftarrow s_1 \twoheadrightarrow t_1, \ldots, s_k \twoheadrightarrow t_k \in \mathcal{R}$
- Then,

$$\mathbb{U}_{sim}(\rho) = \left\{ \begin{array}{rcl} \ell & \to & \mathsf{U}^{\rho}(s_1, s_2, \overrightarrow{\mathcal{V}ar(\ell)}) \\ \mathsf{U}^{\rho}(t_1, t_2, \overrightarrow{\mathcal{V}ar(\ell)}) & \to & r \end{array} \right\}$$

$$\mathbb{U}_{seq}(\rho) = \left\{ \begin{array}{rcl} \ell & \to & \mathsf{U}^{\rho}_1(s_1, \overrightarrow{\mathcal{V}ar(\ell)}) \\ \mathsf{U}^{\rho}_1(t_1, \overrightarrow{\mathcal{V}ar(\ell)}) & \to & \mathsf{U}^{\rho}_2(s_2, \overrightarrow{\mathcal{V}ar(\ell)}) \\ \mathsf{U}^{\rho}_2(t_2, \overrightarrow{\mathcal{V}ar(\ell)}) & \to & r \end{array} \right\}$$

Contents of This Talk

$\mathbb{U}_{sim}(\mathcal{R})$-Orthogonality = $\mathbb{U}_{seq}(\mathcal{R})$-Orthogonality

- orthogonal = left-linear + non-overlapping

Theorem

$\mathbb{U}_{sim}(\mathcal{R})$ *is orthogonal iff* $\mathbb{U}_{seq}(\mathcal{R})$ *is so*

$\mathbb{U}_{sim}(\mathcal{R})$-Orthogonality $=$ $\mathbb{U}_{seq}(\mathcal{R})$-Orthogonality

- orthogonal $=$ left-linear $+$ non-overlapping

Theorem

$\mathbb{U}_{sim}(\mathcal{R})$ is orthogonal iff $\mathbb{U}_{seq}(\mathcal{R})$ is so

Proof (Sketch).

$$\mathbb{U}_{sim}(\rho : \ell \to r \Leftarrow s_1 \twoheadrightarrow t_1, s_2 \twoheadrightarrow t_2) = \left\{ \begin{array}{r} \ell \to \mathsf{U}^{\rho}(s_1, s_2, \overrightarrow{\mathcal{V}ar(\ell)}) \\ \mathsf{U}^{\rho}(t_1, t_2, \overrightarrow{\mathcal{V}ar(\ell)}) \to r \end{array} \right\}$$

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- Recall that t_1, t_2 are ground normal form of $\mathcal{R}_u = \{\ell \to r \mid \ell \to r \Leftarrow c \in \mathcal{R}\}$
- $\mathbb{U}_{sim}(\rho)$ is left-linear iff $\mathbb{U}_{seq}(\rho)$ is so
- $\mathbb{U}_{sim}(\rho)$ has no overlap with $\mathbb{U}_{sim}(\mathcal{R})$ iff $\mathbb{U}_{seq}(\rho)$ has no overlap with $\mathbb{U}_{seq}(\mathcal{R})$ □

Contents of This Talk

1. Background
2. Orthogonality
3. Termination
4. Local Confluence
5. Experiments
6. Conclusion

$\mathbb{U}_{sim}(\mathcal{R})$-Termination Implies $\mathbb{U}_{seq}(\mathcal{R})$-Termination

Theorem

If $\mathbb{U}_{sim}(\mathcal{R})$ is terminating, then $\mathbb{U}_{seq}(\mathcal{R})$ is so

$\mathbb{U}_{sim}(\mathcal{R})$-Termination Implies $\mathbb{U}_{seq}(\mathcal{R})$-Termination

Theorem

If $\mathbb{U}_{sim}(\mathcal{R})$ is terminating, then $\mathbb{U}_{seq}(\mathcal{R})$ is so

Proof (Sketch).

An infinite dependency chain of $\mathbb{U}_{seq}(\mathcal{R})$ can be transformed into an infinite dependency chain of $\mathbb{U}_{sim}(\mathcal{R})$

- There exists a tree homomorphism ϕ s.t. [Nishida et al., 2012]
  - if $s \to^*_{\mathbb{U}_{seq}(\mathcal{R})} t$, then $\phi(s) \to^*_{\mathbb{U}_{sim}(\mathcal{R})} \phi(t)$
  - if $s\ (\to_{\varepsilon, DP(\mathbb{U}_{seq}(\mathcal{R}))} \circ \to^*_{>\varepsilon, \mathbb{U}_{seq}(\mathcal{R})})^+\ t$ and neither s nor t is U-rooted,
    then $\phi(s)\ (\to_{\varepsilon, DP(\mathbb{U}_{sim}(\mathcal{R}))} \circ \to^*_{>\varepsilon, \mathbb{U}_{sim}(\mathcal{R})})^+\ \phi(t)$

$$\mathbb{U}_{sim}(\rho : \ell \to r \Leftarrow s_1 \twoheadrightarrow t_1, s_2 \twoheadrightarrow t_2) = \left\{ \begin{array}{r} \ell \to \mathsf{U}^\rho(s_1, s_2, \overrightarrow{\mathcal{V}ar(\ell)}) \\ \mathsf{U}^\rho(t_1, t_2, \overrightarrow{\mathcal{V}ar(\ell)}) \to r \end{array} \right\}$$

$$\mathbb{U}_{seq}(\rho) = \left\{ \begin{array}{r} \ell \to \mathsf{U}^\rho_1(s_1, \overrightarrow{\mathcal{V}ar(\ell)}) \\ \mathsf{U}^\rho_1(t_1, \overrightarrow{\mathcal{V}ar(\ell)}) \to \mathsf{U}^\rho_2(s_2, \overrightarrow{\mathcal{V}ar(\ell)}) \\ \mathsf{U}^\rho_2(t_2, \overrightarrow{\mathcal{V}ar(\ell)}) \to r \end{array} \right\}$$

□

Converse does NOT Hold

Theorem

If $\mathbb{U}_{sim}(\mathcal{R})$ *is terminating, then* $\mathbb{U}_{seq}(\mathcal{R})$ *is so*

Example

$$\mathcal{R}_2 = \{\ \mathsf{a} \to \mathsf{b} \Leftarrow \mathsf{c} \twoheadrightarrow \mathsf{d}, \mathsf{a} \twoheadrightarrow \mathsf{e}\ \}$$

- $\mathbb{U}_{sim}(\mathcal{R}_2) = \left\{ \begin{array}{r} \mathsf{a} \to \mathsf{U}_1(\mathsf{c}, \mathsf{a}) \\ \mathsf{U}_1(\mathsf{d}, \mathsf{e}) \to \mathsf{b} \end{array} \right\}$ is NOT terminating
- $\mathbb{U}_{seq}(\mathcal{R}_2) = \left\{ \begin{array}{r} \mathsf{a} \to \mathsf{U}_2(\mathsf{c}) \\ \mathsf{U}_2(\mathsf{d}) \to \mathsf{U}_2(\mathsf{a}) \\ \mathsf{U}_3(\mathsf{e}) \to \mathsf{b} \end{array} \right\}$ is terminating

Contents of This Talk

1. Background
2. Orthogonality
3. Termination
4. Local Confluence
5. Experiments
6. Conclusion

$\mathbb{U}_{sim}(\mathcal{R})$-LCR Implies $\mathbb{U}_{seq}(\mathcal{R})$-LCR under Termination

Theorem

If $\mathbb{U}_{sim}(\mathcal{R})$ is terminating, then

- *$\mathbb{U}_{sim}(\mathcal{R})$ is locally confluent iff $\mathbb{U}_{seq}(\mathcal{R})$ is so*

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Proof (Sketch).

Let $\rightarrow$ be $\rightarrow_{\mathbb{U}_{seq}(\mathcal{R})}$, and $\rightarrow$ be $\rightarrow_{\mathbb{U}_{sim}(\mathcal{R})}$

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Proof (Sketch).

Let $\rightarrow$ be $\rightarrow_{\mathbb{U}_{seq}(\mathcal{R})}$, and $\rightarrow$ be $\rightarrow_{\mathbb{U}_{sim}(\mathcal{R})}$

($\Rightarrow$)

- Assume

$$\begin{array}{ccccc} & & s & & \\ & \swarrow & & \searrow & \\ t & & & & t' \\ \downarrow * & & & & \downarrow * \\ u & & & & u' \end{array}$$

with $\mathbb{U}_{seq}(\mathcal{R})$-nfs u, u',

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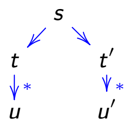
Let $\rightarrow$ be $\rightarrow_{\mathbb{U}_{seq}(\mathcal{R})}$, and $\rightarrow$ be $\rightarrow_{\mathbb{U}_{sim}(\mathcal{R})}$

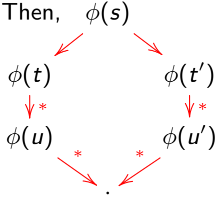
($\Rightarrow$)

- Assume



with $\mathbb{U}_{seq}(\mathcal{R})$-nfs u, u',

- Then,



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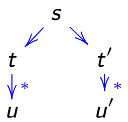
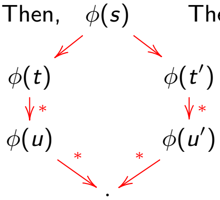


with $\mathbb{U}_{seq}(\mathcal{R})$-nfs u, u',

- Then,



Then, $u = u'$

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Proof (Sketch).

Let $\textcolor{blue}{\to}$ be $\to_{\textcolor{blue}{\mathbb{U}_{seq}(\mathcal{R})}}$, and $\textcolor{red}{\to}$ be $\to_{\textcolor{red}{\mathbb{U}_{sim}(\mathcal{R})}}$

($\Rightarrow$)

- Assume

$$\begin{array}{ccc} & s & \\ \swarrow & & \searrow \\ t & & t' \\ \downarrow * & & \downarrow * \\ u & & u' \end{array}$$

with $\mathbb{U}_{seq}(\mathcal{R})$-nfs u, u',

- Then,

$$\begin{array}{ccc} & \phi(s) & \\ \swarrow & & \searrow \\ \phi(t) & & \phi(t') \\ \downarrow * & & \downarrow * \\ \phi(u) & & \phi(u') \\ & {}^{*}\searrow \quad \swarrow^{*} & \\ & \cdot & \end{array}$$

Then, $u = u'$

($\Leftarrow$)

- Assume

$$\begin{array}{ccc} & s & \\ \swarrow & & \searrow \\ t & & t' \end{array}$$

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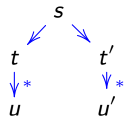
Let $\rightarrow$ be $\rightarrow_{\mathbb{U}_{seq}(\mathcal{R})}$, and $\rightarrow$ be $\rightarrow_{\mathbb{U}_{sim}(\mathcal{R})}$

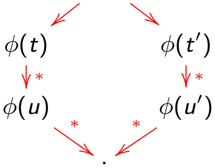
($\Rightarrow$)

- Assume



with $\mathbb{U}_{seq}(\mathcal{R})$-nfs u, u',

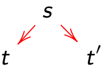
- Then, $\phi(s)$



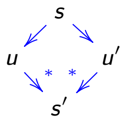
Then, $u = u'$

($\Leftarrow$)

- Assume



- Then,



with $\phi(u) = t$ and $\phi(u') = t'$

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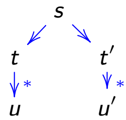
Proof (Sketch).

Let $\rightarrow$ be $\rightarrow_{\mathbb{U}_{seq}(\mathcal{R})}$, and $\rightarrow$ be $\rightarrow_{\mathbb{U}_{sim}(\mathcal{R})}$

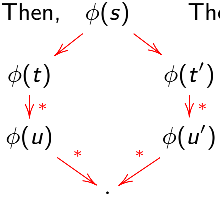
($\Rightarrow$)

- Assume



with $\mathbb{U}_{seq}(\mathcal{R})$-nfs u, u',

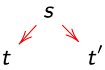
- Then,



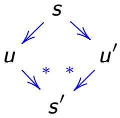
Then, $u = u'$

($\Leftarrow$)

- Assume

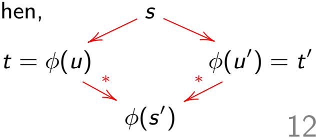


- Then,



with $\phi(u) = t$ and $\phi(u') = t'$

- Then,



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Decidability

- Both orthogonality and local confluence under termination are decidable
  - A proof for $\mathbb{U}_{sim}(\mathcal{R})$ exists iff a proof for $\mathbb{U}_{seq}(\mathcal{R})$ exists
- Termination is undecidable in general
  - Not easy to compare $\mathbb{U}_{sim}$ and $\mathbb{U}_{seq}$ w.r.t. all existing criteria
  - Empirical comparison for 4 benchmarks from Cops with termination provers
    - ⋆ 51 normal 1-CTRSs
    - ⋆ 10 of 51 are transformed into TRSs by removing infeasible rules
    - ⋆ 37 of 41 have at most one condition, i.e., $\mathbb{U}_{sim} = \mathbb{U}_{seq}$ for them

result	AProVE		NaTT		CO3	
	$\mathbb{U}_{sim}$	$\mathbb{U}_{seq}$	$\mathbb{U}_{sim}$	$\mathbb{U}_{seq}$	$\mathbb{U}_{sim}$	$\mathbb{U}_{seq}$
YES	4	4	1	1	1	1
NO	0	0	0	0	—	—
MAYBE	0	0	3	3	3	3

Proving Confluence by CO3

- 4 normal 1-CTRSs from Cops

result	CO3	
	$\mathbb{U}_{sim}$	$\mathbb{U}_{seq}$
YES	1	1
NO	1	1
MAYBE	2	2

- Their powers for proving confluence are equivalent, except for termination

Contents of This Talk

Conclusion

We showed that for a normal 1-CTRS $\mathcal{R}$,

- $\mathbb{U}_{sim}(\mathcal{R})$ is orthogonal iff $\mathbb{U}_{seq}(\mathcal{R})$ is so
- if $\mathbb{U}_{sim}(\mathcal{R})$ is terminating, then
  - $\mathbb{U}_{seq}(\mathcal{R})$ is terminating
  - $\mathbb{U}_{sim}(\mathcal{R})$ is locally confluent iff $\mathbb{U}_{seq}(\mathcal{R})$ is so

Future Work

- Compare $\mathbb{U}_{seq}$ and $\mathbb{U}_{conf}$ [Gmeiner et al., 2013]
- Compare $\mathbb{U}_{sim}$, $\mathbb{U}_{seq}$, and $\mathbb{U}_{conf}$ w.r.t. other confluence criteria

result	CO3 1.4	CO3 1.5			
	$\mathbb{U}^{+}_{sim} + \mathbb{U}_{conf}$	$\mathbb{U}_{sim}$	$\mathbb{U}^{+}_{sim}$	$\mathbb{U}_{seq}$	$\mathbb{U}_{conf}$
YES	22	17	21	17	22
NO	12	12	12	12	12
MAYBE	17	22	18	22	17

- CO3 1.5 doesn't use $\mathbb{U}_{sim}$

References

Durán, F., Lucas, S., Marché, C., Meseguer, J., and Urbain, X. (2008).
Proving operational termination of membership equational programs.
Higher-Order and Symbolic Computation, 21(1-2):59–88.

Giesl, J. and Arts, T. (2001).
Verification of Erlang processes by dependency pairs.
Appl. Algebra Eng. Commun. Comput., 12(1/2):39–72.

Gmeiner, K., Gramlich, B., and Schernhammer, F. (2010).
On (un)soundness of unravelings.
In *Proc. RTA 2010*, volume 6 of *LIPIcs*, pages 119–134. Schloss Dagstuhl – Leibniz-Zentrum für Informatik.

Gmeiner, K., Gramlich, B., and Schernhammer, F. (2012).
On soundness conditions for unraveling deterministic conditional rewrite systems.
In *Proc. RTA 2012*, volume 15 of *LIPIcs*, pages 193–208. Schloss Dagstuhl – Leibniz-Zentrum für Informatik.

Gmeiner, K., Nishida, N., and Gramlich, B. (2013).
Proving confluence of conditional term rewriting systems via unravelings.
In *Proc. IWC 2013*, pages 35–39.

References (cont.)

Marchiori, M. (1996).
Unravelings and ultra-properties.
In *Proc. ALP 1996*, volume 1139 of *Lecture Notes in Computer Science*, pages 107–121. Springer.

Marchiori, M. (1997).
On deterministic conditional rewriting.
Computation Structures Group, Memo 405, MIT Laboratory for Computer Science.

Nishida, N., Kuroda, T., Yanagisawa, M., and Gmeiner, K. (2015).
CO3: a COnverter for proving COnfluence of COnditional TRSs.
In *Proc. IWC 2015*, page 42.

Nishida, N., Sakai, M., and Sakabe, T. (2012).
Soundness of unravelings for conditional term rewriting systems via ultra-properties related to linearity.
Logical Methods in Computer Science, 8(3):1–49.

Ohlebusch, E. (2001).
Termination of logic programs: Transformational methods revisited.
Appl. Algebra Eng. Commun. Comput., 12(1/2):73–116.

References (cont.)

Şerbănuţă, T.-F. and Roşu, G. (2006).
Computationally equivalent elimination of conditions.
In *Proc. RTA 2006*, volume 4098 of *Lecture Notes in Computer Science*, pages 19–34. Springer.

Viry, P. (1999).
Elimination of conditions.
J. Symb. Comput., 28(3):381–401.