Critical pairs for Gray categories

Simon Forest

July 7th 2018

Context

Rewriting for algebraic theories

Rewriting can be used to study algebraic theories

- **Algebraic theory**: given by a **signature** and **equations**
  signature for monoids

  m :  e : 

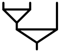
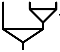
  equations for monoids

- **Term rewriting system**: obtained by orienting the equations

Gray-categories

- In which world lives  $\rightarrow$ ?

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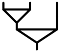
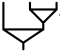
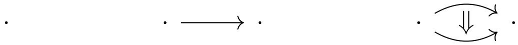
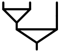
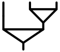
- In which world lives  $\to$ ?
- General answer: in a *higher category*
  - category: a structure with objects and morphisms (or 1-cells) between objects
  - higher category: a category with n+1-cells between n-cells



  -  would be a 2-cell and  $\to$  a 3-cell

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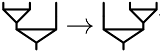
- In which world lives ?
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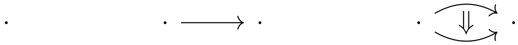
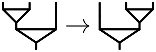


  -  would be a 2-cell and  a 3-cell
- In this work: Gray-categories
  - 3-dimensional categories
  - have nice properties for doing rewriting

Coherence

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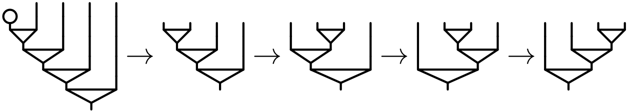

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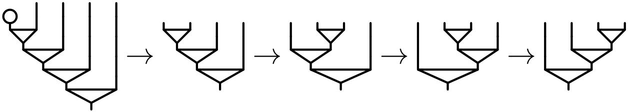

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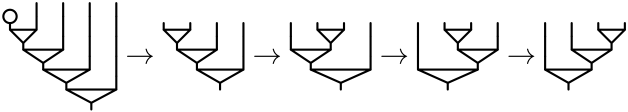

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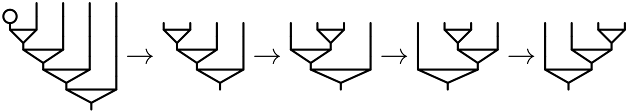

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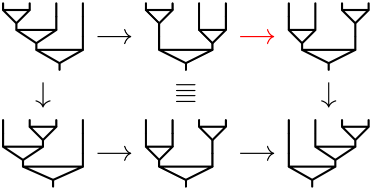
- **Coherence**: between two 2-cells, at most one zigzag up to $\equiv$



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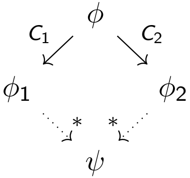

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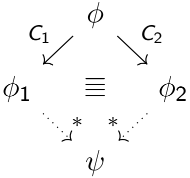

Coherence from rewriting

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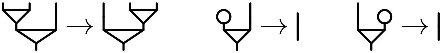
- **Good rewriting system and congruence** Start from a "good" orientation for the isos of the considered structure



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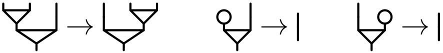

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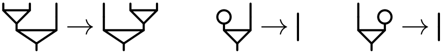

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Compute critical pairs and take $\equiv$ such that

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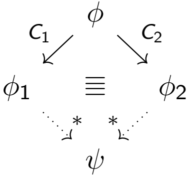

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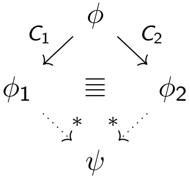

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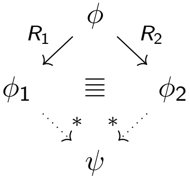

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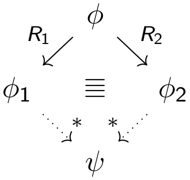

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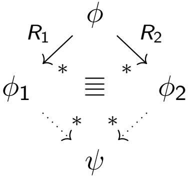

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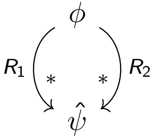

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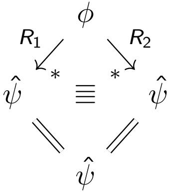

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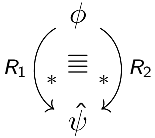

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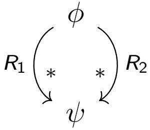

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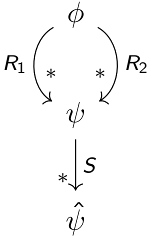

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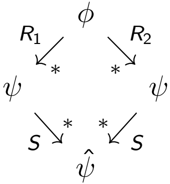

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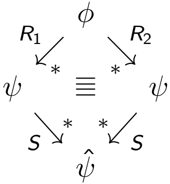

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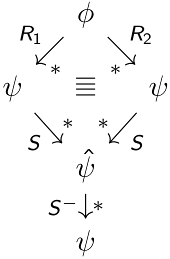

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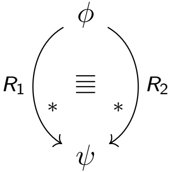

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  Third case: zigzags (paths with inverses)

  $\rightarrow$ Analogous to the proof of the Church-Rosser lemma

Rewriting in Gray-categories

Signatures

A signature S is given by:

- a set of elementary 2-dimensional diagrams called 2-generators

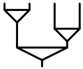
{  }

- some typing information about the source and target of these diagrams

Terms

- **slice**: a 2-generators with identities on the left and the right

- **terms** (or 2-**cells**): a sequence of composable slices

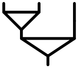
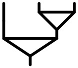


in particular, in this formalism, the following cell does not exist

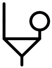
because there is only one 2-generator per slice

Rewriting system

- A *rewriting system* is given by:
  - a signature S
  - a set P of rewrite rules (called 3-generators) on the terms of the signature

A :  $\rightarrow$ 

L :  $\rightarrow$

R :  $\rightarrow$

Gray category structure

- Remember: the underlying structures are Gray-categories
- The precise definition of Gray-cat? Out of the scope of IWC
- But the axioms of Gray-cat induce properties on the rewriting system
  - Interchangers
  - Parallel paths identification
  - Naturally equivalent paths identification

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- Let S a **signature** and $\alpha, \beta \in$ S and *u* a sequence of identities

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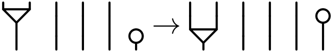
- Gray-cats induce **interchanger** $X_{\alpha,u,\beta}$: rewrite rule that exchanges α and β when separated by u

$X_{m,\bar{3},e}$: 

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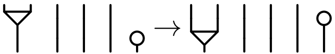
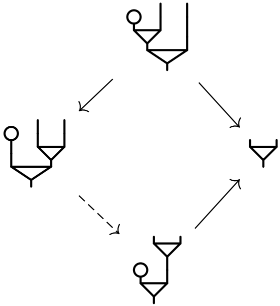


- Nice, because we had branchings that could not be closed



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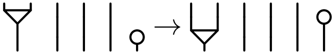
$X_{m,\bar{3},e}$: 

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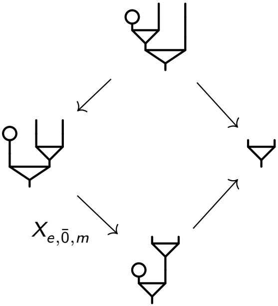
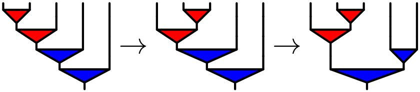


- From now on, interchangers are allowed rewriting steps

Parallel paths

Let $\equiv$ a congruence

- Consider the following two paths:



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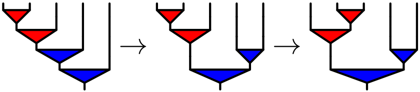

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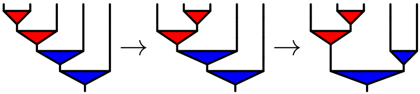

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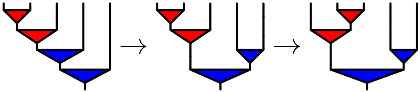

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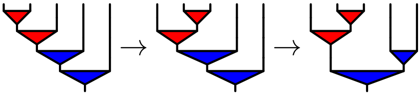

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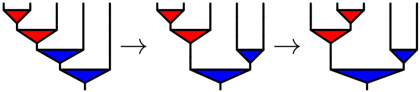

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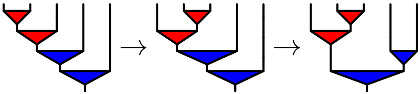

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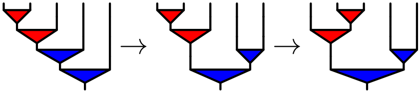

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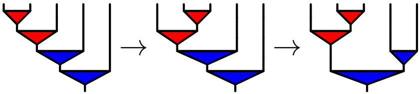
- Happily: in a Gray-cat, two parallel paths are equal

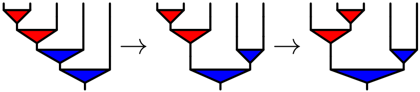
Parallel paths

Let $\equiv$ a congruence

- Consider the following two paths:



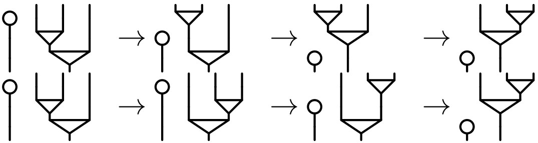
and



- **Parallel paths**: the two paths obtained by applying two rules at independent positions
- Note: for coherence, these two paths need to be $\equiv$-equivalent
- Happily: in a Gray-cat, two parallel paths are equal
- So, we only consider $\equiv$ that relates parallel paths

Naturally-equivalent paths

- Consider the two following rewriting paths:



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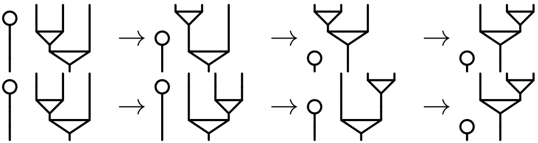

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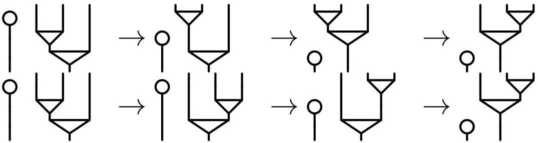

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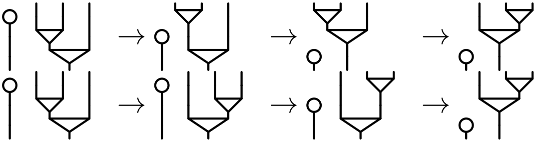

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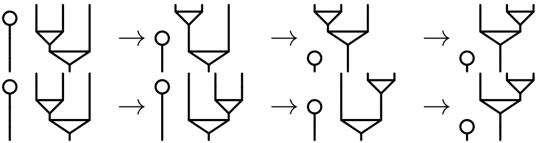

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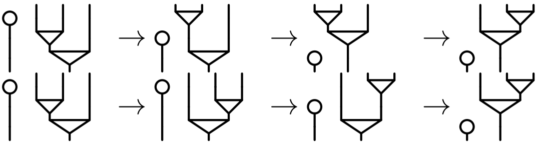

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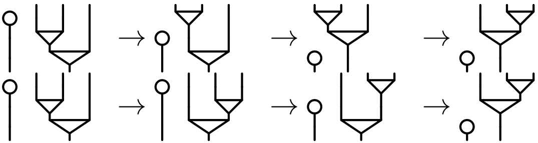

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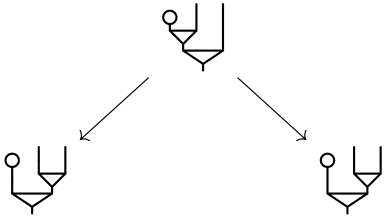
- Solution: squares given by "critical branchings"

Computing critical branchings

Critical branchings

Let $P_1 : \phi \to \psi_1$, $P_2 : \phi \to \psi_2$ a local branching:

- it is *trivial* when $P_1 = P_2$



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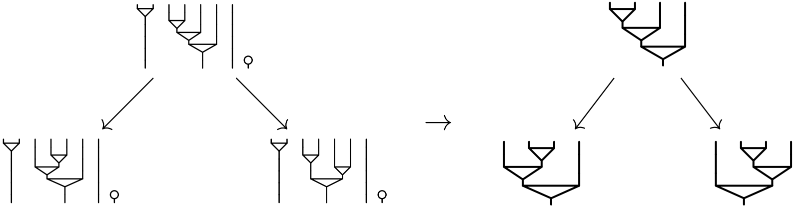

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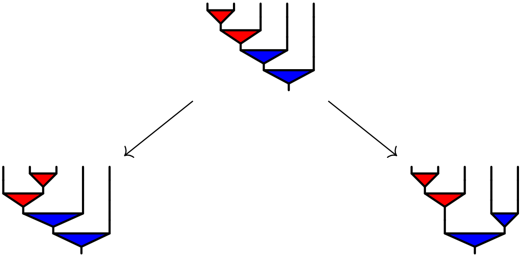

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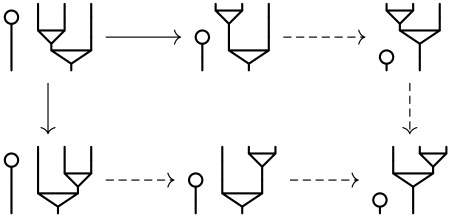

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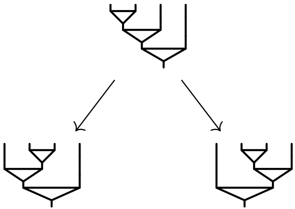

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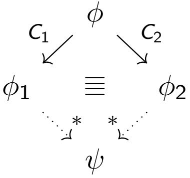
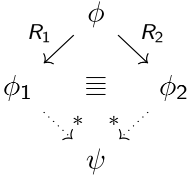
Theorem(Critical pair lemma): if critical branchings are confluent then all local branchings are confluent

$\forall(C_1, C_2)$

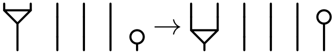


then $\forall(R_1, R_2)$

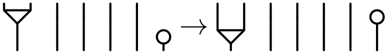


Finite number of critical pairs

- There is an infinite number of interchangers



$X_{m,\bar{3},e}$



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$X_{m,\bar{n},e}$ for all n

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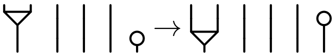

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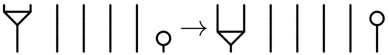

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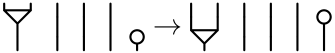

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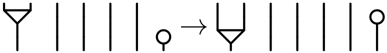

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Theorem: A finite number of operational rules (and ...) gives **a finite number of critical branchings**.
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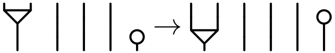

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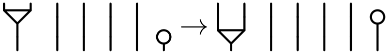

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- Concerning computability

  An algorithm exists to compute the critical branchings

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Three kinds of branchings:

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  - they are handled as "natural branchings"

Representing terms

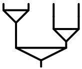
- **slice**: a 2-generators α with identities on the left and the right l, r written $l *_0 \alpha *_0 r$

$$\bar{3} *_0 m *_0 \bar{2} \quad :$$ 

- **terms** (or 2-**cells**): a sequence

$$(l_1 *_0 \alpha_1 *_0 r_1) *_1 \dots *_1 (l_n *_0 \alpha_n *_0 r_n)$$

  of composable slices

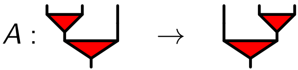
$$(m *_0 \bar{2}) *_1 (\bar{1} *_0 m) *_1 m :$$ 

Representing rewriting steps

Rewriting step: a rewrite rule in a context

- identities on the left and the right l, r
- 2-cells top and bottom τ, β

Start from a rewrite rule, say:



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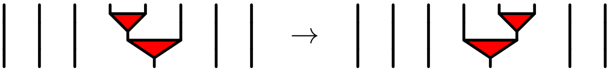
Add 1-cells (wires) to the left and the right:

$(l *_0 A *_0 r)$:



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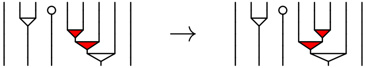
$\tau *_1 (l *_0 A *_0 r) *_1 \beta :$ 

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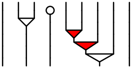
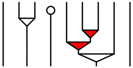
$\tau *_1 (l *_0 A *_0 r) *_1 \beta :$  $\rightarrow$ 

Write $E[\ \_\ ]$ for $\tau *_1 (l *_0 \_ *_0 r) *_1 \beta$

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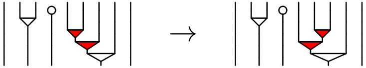

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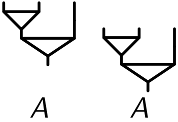
Branching: a pair $(E[R_1], E[R_2])$ of rewriting steps in a context

Computing critical branchings between two rules

Critical branchings between A and A ?

- foreach possible superposition



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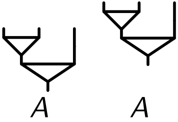

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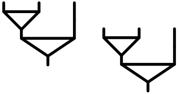

A A

- try to unify the slices that are overlapping

m $(m * \bar{1})$

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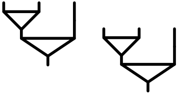

A A

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$m *_0 \bar{1}$
$l_1 = \bar{0}$
$r_1 = \bar{1}$

$m *_0 \bar{1}$
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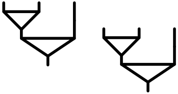

A $\quad$ A

- try to unify the slices that are overlapping
- find the top and bottom 2-cells

$(A *_0 \bar{1})$ $\quad$ A

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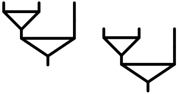

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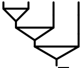
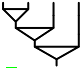


$$1 *_1 (A *_0 \bar{1}) *_1 m$$
$$\tau_1 = 1$$
$$\beta_1 = m$$



$$(m *_0 \bar{2}) *_1 A *_1 1$$
$$\tau_2 = (m *_0 \bar{2})$$
$$\beta_2 = 1$$

Computing critical branchings between two rules

Critical branchings between A and A ?

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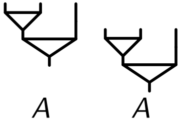


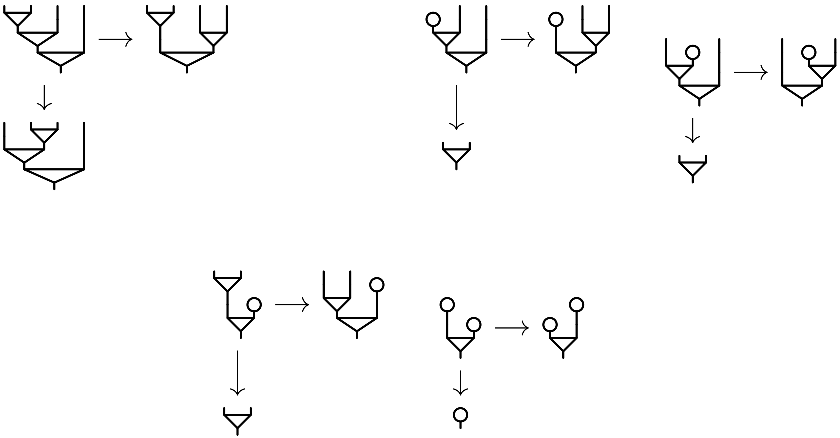
- try to unify the slices that are overlapping
- find the top and bottom 2-cells
- dedude the contexts of the critical branching

$$E_1[\ \_\ ] = (1 *_1 (\bar{0} *_0 \ \_\ *_0 \bar{1}) *_1 m)$$

$$E_2[\ \_\ ] = ((m *_0 \bar{2}) *_1 \ \_\ *_1 1)$$

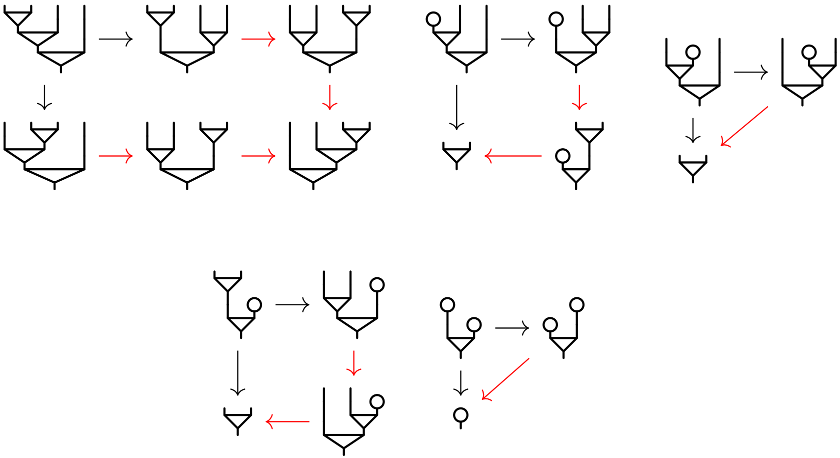
Example of monoids

With monoids, we find five critical pairs



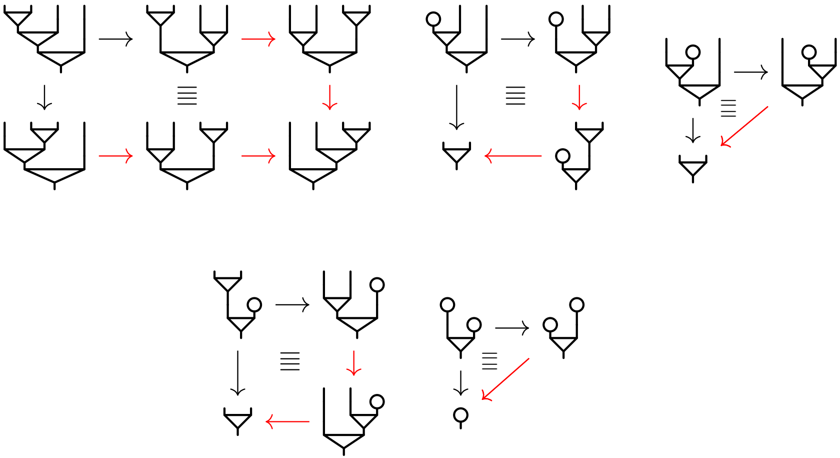
Example of monoids

With monoids, we find five critical pairs and they are confluent



Example of monoids

With monoids, we find five critical pairs and they are confluent



We deduce constraints on ≡ for coherence

Other examples

- Adjunctions

$$\mathsf{S} = \{\cup, \cap\}$$

$$\mathsf{P} = \{zig : \cup\cap \to |, zag : \cap\cup \to |\}$$

- Self-dualities

$$\mathsf{S} = \{\ \cup, \cap\ \}$$

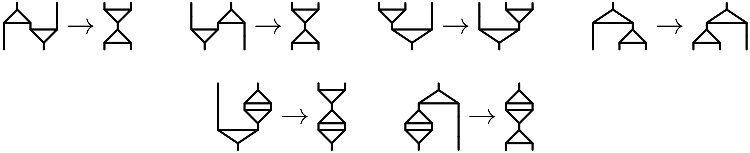
$$\mathsf{P} = \{zig : \cup\cap \to |, zag : \cap\cup \to |\}$$

- Frobenius monoid

Frobenius monoid (without units)

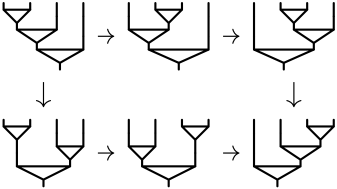
Signature

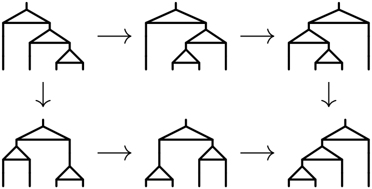
Rules

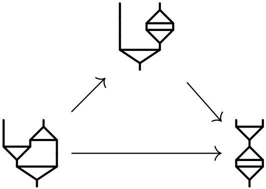


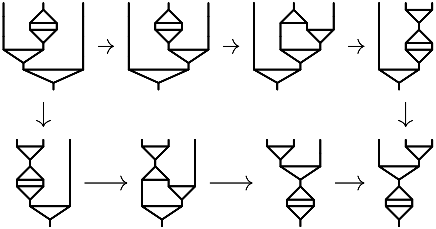
Coherence relations

19 relations found by the algorithm

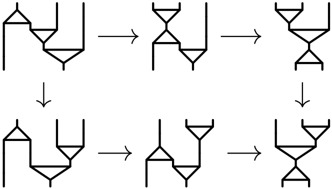


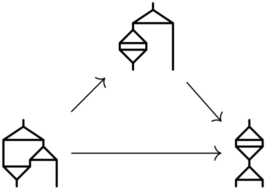


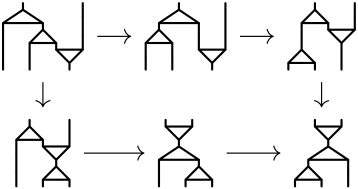


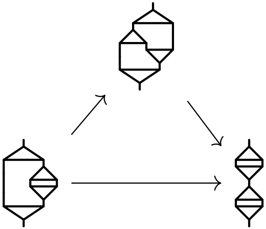


Coherence relations

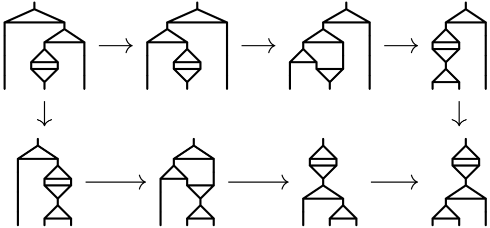


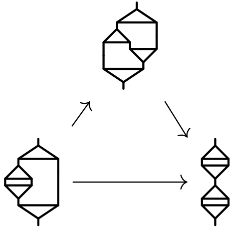


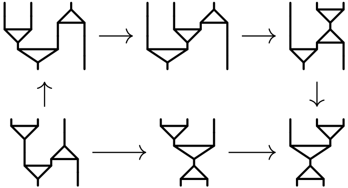


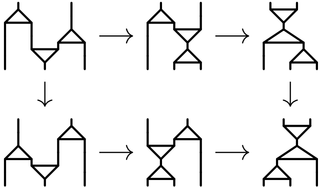


Coherence relations









Conclusion

Conclusion

- rewriting in Gray-category setting
- coherence conditions from critical pairs
- finite number of critical branchings from a finite rewriting system
- and an algorithm to compute them