ISR 2008
Term Rewriting Systems
Exercise Session Monday July 21

1 Term Algebra

Exercise 1.1. Let Σ be the signature consisting of the nullary symbol $\mathsf{0}$, the unary symbol S, and the binary symbols A and M. Consider the term

$$s = \mathsf{A}(\mathsf{A}(\mathsf{0}, \mathsf{S}(\mathsf{S}(\mathsf{x}))), \mathsf{M}(\mathsf{A}(\mathsf{S}(\mathsf{y}), \mathsf{S}(\mathsf{0})), \mathsf{S}(\mathsf{S}(\mathsf{x}))))$$

(a) Picture the term s as a tree.

(b) Give all subterms of s.

(c) What are the positions of function symbols?

(d) Give two different contexts C such that $s = C[\mathsf{0}]$.

Exercise 1.2. [A2.5] Consider the term

$$s = \mathsf{A}(\mathsf{A}(\mathsf{0}, \mathsf{S}(\mathsf{0})), \mathsf{A}(\mathsf{S}(\mathsf{S}(\mathsf{0})), \mathsf{A}(\mathsf{0}, \mathsf{0})))$$

(a) Compute $s|_{21}$.

(b) Compute $s[\mathsf{A}(\mathsf{0}, \mathsf{S}(\mathsf{0}))]_{121}$.

Exercise 1.3. [T2.1.3]

(a) Give a direct inductive definition of one-hole contexts.

(b) Give a definition of $C[t]$ by recursion on the (direct inductive) definition of one-hole contexts.

Exercise 1.4. [W1.2] Consider the signature consisting of the nullary symbol $\mathsf{0}$, the unary symbol S, and the binary symbols A and M. Let $\sigma = \{x \mapsto \mathsf{0}, \mathsf{y} \mapsto \mathsf{S}(\mathsf{0}), \mathsf{z} \mapsto \mathsf{A}(\mathsf{x}, \mathsf{y})\}$. Compute $s\sigma$ for the following terms s:

(a) z

(b) $\mathsf{A}(0, \mathsf{S}(0))$

(c) $\mathsf{M}(\mathsf{A}(\mathsf{x}.\mathsf{M}(\mathsf{y}, \mathsf{z})), \mathsf{S}(\mathsf{x}))$

Exercise 1.5. [BN3.1] Let Σ be the signature consisting of the binary function symbol f. Let s be a term over Σ and Var.

What are the possible shapes of positions of s?

Exercise 1.6. [BN3.2]

(a) Let $pq \in \mathsf{Pos(s)}$. Prove the following:

$$s|_{pq} = (s|_p)|_q$$
$$s[t]_{pq} = (s|_p)[t]_q$$

(b) Let $p, q \in \mathsf{Pos(s)}$ with $p \parallel q$. Prove that for all t we have:

$$(s[t]_p)|_q = s|_q$$

Exercise 1.7. [BN3.3] Prove that composition of substitutions is associative.

2 Term Rewriting Systems

Exercise 2.1. [W15 Opgave 1.1] Consider the signature Σ consisting of a binary function symbol f, a unary function symbol g, and a constant symbol a. Which of the following are proper rewrite rules:

(a) $x \to \mathsf{a}$

(b) $\mathsf{a} \to x$

(c) $\mathsf{a} \to \mathsf{a}$

(d) $\mathsf{f}(x) \to \mathsf{g}(x)$

(e) $\mathsf{g}(x) \to \mathsf{f}(x, y)$

(f) $\mathsf{f}(x, x) \to \mathsf{f}(x, y)$

(g) $\mathsf{g}(x) \to \mathsf{f}(x, x)$

Exercise 2.2. Consider the TRS for addition and multiplication:

$$\begin{array}{rcl} \mathsf{A}(0, \mathsf{y}) & \to & y \\ \mathsf{A}(\mathsf{S}(\mathsf{x}), \mathsf{y}) & \to & \mathsf{S}(\mathsf{A}(\mathsf{x}, \mathsf{y})) \\ \mathsf{M}(0, \mathsf{y}) & \to & 0 \\ \mathsf{M}(\mathsf{S}(\mathsf{x}), \mathsf{y}) & \to & \mathsf{A}(\mathsf{M}(\mathsf{x}, \mathsf{y}), \mathsf{y}) \end{array}$$

Give all possible ways of computing the normal form of

$$\mathsf{M}(\mathsf{S}(\mathsf{S}(0)), \mathsf{S}(\mathsf{S}(0)))$$

Exercise 2.3. Consider combinatory logic (CL):

$$\begin{array}{rcl} \mathsf{S\,x\,y\,z} & \to & (x\,z)\,(y\,z) \\ \mathsf{K\,x\,y} & \to & x \\ \mathsf{I\,x} & \to & x \end{array}$$

Give the three possible reductions of the term

$$\mathsf{S\,I\,(K\,x)\,I\,y}$$

Exercise 2.4. This exercise is concerned with fixed points in CL. We defined the following combinators (in the course):

$$\begin{array}{rcl} \mathsf{B} & := & \mathsf{S\,(K\,S)\,K} \\ \mathsf{D} & := & \mathsf{S\,I\,I} \end{array}$$

First show the following (for any x, y, z):

$$\begin{array}{rcl} \mathsf{B}\,x\,y\,z & \to^* & x\,(y\,z) \\ \mathsf{D}\,x & \to^* & x\,x \end{array}$$

Then, use B and D to construct, given a CL-term F, a fixed point for F, that is, a term P_F with the following property:

$$P_F \to^* F\,P_F$$

Exercise 2.5. Define a TRS for insertion sort.

Exercise 2.6. Give an example of a TRS consisting of two rewrite rules that is not confluent.

Exercise 2.7. Give an example of a TRS that satisfies UN but is not confluent.

3 Equational Reasoning

Exercise 3.1. [BN3.6] Consider the equational specification $\mathcal{E}$ for groups:

$$\begin{array}{rcl} (x * y) * z & = & x * (y * z) \\ x * e & = & x \\ x * \bar{x} & = & e \end{array}$$

Recall from the slides that in the corresponding TRS we have $x \leftrightarrow^* e * x$. Show that

$$\mathcal{E} \vdash x = e * x$$

Show also the following (this was mentioned during the course):

$$e = \bar{x} * x$$

Exercise 3.2. [BN3.7] Consider the equational specification $\mathcal{E}$ defined by the following equations:

$$\begin{array}{rcl} f(x, f(y,z)) & = & f(f(x,y),z) \\ f(f(x,y),x) & = & x \end{array}$$

(a) Show that in the corresponding TRS we have

$$\begin{array}{rcl} f(x,x) & \leftrightarrow^* & x \\ f(f(x,y),z) & \leftrightarrow^* & f(x,z) \end{array}$$

(b) Give also the corresponding derivations:

$$\mathcal{E} \vdash f(x,x) = x$$
$$\mathcal{E} \vdash f(f(x,y),z) = f(x,z)$$

Exercise 3.3. [A] Consider the following equational system:

$$\begin{array}{rcl} f(x) & = & x \\ f(f(a)) & = & g(x,x) \\ g(x, f(x)) & = & b \end{array}$$

Which of the following equations can be derived:

(a) $a = b$

(b) $g(x,y) = g(y,x)$

(c) $g(f(a), a) = f(b)$

Exercise 3.4. [T7.1.3] Recall the following definitions.

If (Σ, R) is a pseudo-TRS, then the corresponding equational specification is $(\Sigma, R^=)$ with

$$R^= = \{l = r \mid l \to r \in R\}$$

If (Σ, E) is an equational specification, then the corresponding pseudo-TRS is $(\Sigma, E^\leftrightarrow)$ with

$$E^\leftrightarrow = \{l \to r \mid l = r \in E \mathrm{or} r = l \in E\}$$

(a) Let (Σ, R) be a TRS. Prove the following: $(\Sigma, R^=) \vdash s = t$ if and only if $s \leftrightarrow_R^* t$.

(b) Let (Σ, E) be an equational specification. Prove the following: $(\Sigma, E) \vdash s = t$ if and only if $s \leftrightarrow_{E^\leftrightarrow}^* t$.

Exercise 3.5. [W5.1] Consider again the equational specification $\mathcal{E}$ for groups:

$$\begin{array}{rcl} (x * y) * z & = & x * (y * z) \\ x * e & = & x \\ x * \bar{x} & = & e \end{array}$$

Show that in every model of $\mathcal{E}$ the following equation is valid:

$$(x \bar{*} y) = \bar{x} * \bar{y}$$

Exercise 3.6. [W5.2] Consider the equational specification $\mathcal{E}$ defined by the following equations:

$$\begin{array}{rcl} \mathsf{A}(\mathsf{0},\mathsf{y}) & = & y \\ \mathsf{A}(\mathsf{S}(\mathsf{x}),\mathsf{y}) & = & \mathsf{S}(\mathsf{A}(\mathsf{x},\mathsf{y})) \end{array}$$

Construct a model for $\mathcal{E}$ in which A is not commmutative.

Exercise 3.7. [T Proposition 2.5.5] Let (σ, R) be a TRS and let $(\sigma, R^{=})$ be the corresponding equational specification. Assume that $\mathcal{A}$ is a model for $(\sigma, R^{=})$ such that: for all (ground) normal forms s, s' of (σ, R), if $\mathcal{A} \models s = s'$ then $s = s'$ (so s and s' are syntactically equal).

Show that the TRS (σ, R) has the property (ground-)UN.

Exercise 3.8. [T2.5.6] Consider the TRS for addition and multiplication:

$$\begin{array}{rcl} \mathsf{A}(\mathsf{0},\mathsf{y}) & \to & y \\ \mathsf{A}(\mathsf{S}(\mathsf{x}),\mathsf{y}) & \to & \mathsf{S}(\mathsf{A}(\mathsf{x},\mathsf{y})) \\ \mathsf{M}(\mathsf{0},\mathsf{y}) & \to & \mathsf{0} \\ \mathsf{M}(\mathsf{S}(\mathsf{x}),\mathsf{y}) & \to & \mathsf{A}(\mathsf{M}(\mathsf{x},\mathsf{y}),\mathsf{y}) \end{array}$$

(a) What are the closed normal forms of the TRS?

(b) Show that the TRS has the property ground-UN.

Exercise 3.9. [BN3.13] Consider the equational specification defined by the following equations (it is not the same as the specification for groups!):

$$\begin{array}{rcl} (x * y) * z & = & x * (y * z) \\ e * x & = & x \\ x * \bar{x} & = & e \end{array}$$

Show that the equation $x * e = x$ is derivable. Hint: construct a model in which is it not satisfied.

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Exercise Session Tuesday July 22

1 Completion

Exercise 1.1. [T2.7.27] Consider the following TRS:

$$
\begin{array}{rcl}
x \cdot \emptyset & \to & x \\
\emptyset \cdot x & \to & x \\
x/\emptyset & \to & x \\
\emptyset/x & \to & \emptyset \\
x/x & \to & \emptyset \\
(x \cdot y)/z & \to & (x/z) \cdot (y/(z/x)) \\
z/(x \cdot y) & \to & (z/x)/y
\end{array}
$$

Prove local confluence by considering all critical pairs.

Exercise 1.2. Consider the TRS defined by the following rewrite rules:

$$
\begin{array}{rcl}
\neg(\mathsf{true}) & \to & \mathsf{false} \\
\neg(\neg(x)) & \to & x \\
\mathsf{and}(\mathsf{true}, x) & \to & x \\
\mathsf{and}(\mathsf{false}, x) & \to & \mathsf{false} \\
\mathsf{or}(x, y) & \to & \neg(\mathsf{and}(\neg(x), \neg(y)))
\end{array}
$$

Complete the TRS by adding one rule.

Exercise 1.3. Consider the following equational specification $\mathcal{E}$:

$$
\begin{array}{rcl}
f(f(f(x))) & = & g(x) \\
g(f(g(x))) & = & f(x)
\end{array}
$$

(a) Compute a complete TRS for $\mathcal{E}$.

Hint: Think of a good orientation of the equations.

(b) Is the equation $g(g(x)) = x$ valid?

Exercise 1.4. [R] Complete the following equational specification:

$$\begin{array}{rcl} f(f(f(x))) & = & x \\ f(f(f(f(f(x))))) & = & x \end{array}$$

Exercise 1.5. [R] Complete the following equational specification:

$$f(g(f(x))) = x$$

Exercise 1.6. [R] Complete the following equational specification:

$$(x \cdot y) \cdot (y \cdot z) = y$$

Exercise 1.7. [T7.4.3] Consider the following equational specification:

$$f(f(x)) \quad = \quad g(x)$$

Give three different completions of this specification.

Exercise 1.8. [T2.7.19] Prove, using the critical pair lemma, the following: if a TRS is terminating (or: SN), and has finitely many rules, then both local confluence and confluence are decidable.

Exercise 1.9. [T7.4.3] Consider the following equational specification:

$$f(g(f(x)) \quad = \quad g(f(x))$$

Argue why the completion algorithm applied to this specification does not terminate.

2 Modularity

Exercise 2.1. [W1.22] Recall the definition of the property $\mathrm{UN}^{\rightarrow}$:

> if $s \rightarrow^* n$ and $s \rightarrow^* n'$ with n and n' normal forms, then $n = n'$.

Give an example showing that $\mathrm{UN}^{\rightarrow}$ is not modular.

Exercise 2.2. [T5.10.3] Recall the definition of the property NF:

> if $s \leftrightarrow^* n$ with n a normal form, then $s \rightarrow^* n$.

Give an example showing that NF is not modular.

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Exercise Session Wednesday July 23

1 Orthogonality

Exercise 1.1. [W2.1] Show that the following TRSs are not confluent:

(a)

$$\mathsf{f}(\mathsf{f}(x)) \quad \to \quad a$$

(b)

$$\begin{aligned} \mathsf{f}(\mathsf{g}(x), y) \quad &\to \quad x \\ \mathsf{g}(\mathsf{a}) \quad &\to \quad \mathsf{b} \end{aligned}$$

(c)

$$\begin{aligned} \mathsf{or}(x, y) \quad &\to \quad x \\ \mathsf{or}(x, y) \quad &\to \quad y \end{aligned}$$

Exercise 1.2. [W2.8] Define a TRS and illustrate the two cases of the proof that orthogonality implies local confluence by concrete examples.

Exercise 1.3. [T4.7.5] Show that the inclusions $\to \subseteq \mathrel{-\!\!\!\!\Vert\!\!\!\!\to} \subseteq \to^*$ are proper.

Exercise 1.4. [T1.3.1] Assume $\to \subseteq \mathrel{-\!\!\!\!\Vert\!\!\!\!\to} \subseteq \to^*$ and suppose that $\mathrel{-\!\!\!\!\Vert\!\!\!\!\to}$ has the diamond property. Prove the following:

(a) $\mathrel{-\!\!\!\!\Vert\!\!\!\!\to}$ is confluent.

(b) $\mathrel{-\!\!\!\!\Vert\!\!\!\!\to}^* = \to^*$.

(c) $\to$ is confluent.

Solution: in the notes by Bas Luttik

Exercise 1.5. Assume that $x^\sigma \twoheadrightarrow\!\!\!\!\!\!\shortmid\;\; x^\tau$ for substitutions σ and τ. Prove that $s^\sigma \twoheadrightarrow\!\!\!\!\!\!\shortmid\;\; s^\tau$ for every term s.

Exercise 1.6. [TRS] Consider the TRS defined by the following rules:

$$\begin{array}{rcl} f(g(x),y) & \to & f(f(g(x),y),y) \\ a & \to & g(a) \end{array}$$

Then:

(a) Is the TRS confluent?

(b) Perform a complete development of the term $f(g(g(x)),a)$.

(c) Join the following two diverging reductions:

$$f(g(g(x)),a) \to f(f(g(g(x)),a),a) \to f(f(g(g(x)),a),g(a))$$

and

$$f(g(g(x)),a) \to f(g(g(x)),g(a))$$

Exercise 1.7. [TRS] Consider the TRS defined by the following rules:

$$\begin{array}{rcl} f(x) & \to & g(x,x,a) \\ a & \to & b \end{array}$$

Compute the descendants of the redex a in $f(a)$ after the following reduction:

$$f(a) \to g(a,a,a) \to g(a,b,a)$$

Exercise 1.8. [TRS] Consider the TRS defined by the following rules:

$$\begin{array}{rcl} f(g(x),y) & \to & f(x,f(y,x)) \\ a & \to & b \\ b & \to & c \end{array}$$

(a) Is the TRS confluent?

(b) Give a complete development of the term $f(g(b),g(y))$.

Exercise 1.9. [T4.1.9] Consider the following rewrite rules:

$$\begin{array}{rcl} a & \to & a \\ f\,a\,a & \to & f\,a\,a \\ f\,x\,a & \to & f\,a\,a \\ f\,a\,x & \to & f\,a\,a \\ f\,a\,x & \to & f\,x\,a \\ f\,x\,y & \to & f\,y\,x \end{array}$$

Determine a maximal subset of rewrite rules that is weakly orthogonal.

Exercise 1.10. [T4.1.6(vi)] Consider the following rewrite rules:

$$\begin{array}{rcl} \mathsf{S}(\mathsf{P}(\mathsf{x})) & \to & x \\ \mathsf{P}(\mathsf{S}(\mathsf{x})) & \to & x \end{array}$$

Compute the two critical pairs and conclude that the TRS is weakly orthogonal.

Exercise 1.11. [BNp200] Consider the TRS defined by the following rules:

$$\begin{array}{rcl} f(x,x) & \to & x \\ a & \to & g(a) \end{array}$$

Note that this TRS is not left-linear and is not terminating. Show that it is confluent.

Exercise 1.12. [BN9.2] Consider the TRS defined by the following rules:

$$\begin{array}{rcl} g(x)+y & \to & f(x+h(y)) \\ h(x)+g(a) & \to & f(h(x)+g(x)) \\ h(a)+g(y) & \to & f(h(y)+g(y)) \\ x+h(y) & \to & f(g(x)+y) \end{array}$$

Show that it is confluent.

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Exercise Session Friday July 25

1 Termination

Exercise 1.1. [T2.3.9] Prove or disprove termination of the following TRSs:

(a)

$$\begin{array}{rcl} g(x) & \rightarrow & f(x) \\ f(x) & \rightarrow & f(f(g(x))) \end{array}$$

(b)

$$xg(f(x)) \rightarrow f(f(g(x)))$$

(c)

$$g(f(x)) \rightarrow f(f(g(g(x))))$$

(d)

$$f(f(x)) \rightarrow f(g(f(x)))$$

Solution: in the notes by Bas Luttik

2 Modularity

Exercise 2.1. [BNp200] Consider the TRS defined by the following rules:

$$\begin{array}{rcl} f(x,x) & \rightarrow & x \\ a & \rightarrow & g(a) \end{array}$$

Note that this TRS is not left-linear and is not terminating. Show that it is confluent.

Exercise 2.2. [BN9.2] Consider the TRS defined by the following rules:

$$\begin{array}{rcl} g(x) + y & \to & f(x + h(y)) \\ h(x) + g(a) & \to & f(h(x) + g(x)) \\ h(a) + g(y) & \to & f(h(y) + g(y)) \\ x + h(y) & \to & f(g(x) + y) \end{array}$$

Show that it is confluent.

Exercise 2.3. [BN9.3] Show that confluence is modular for terminating systems.

Exercise 2.4. [W1.22] Recall the definition of the property $UN^{\to}$: if $s \to^* n$ and $s \to^* n'$ with n and n' normal forms, then $n = n'$.

Give an example showing that $UN^{\to}$ is not a modular.

Exercise 2.5. [T5.10.3] Recall the definition of the property NF: if $s = n$ (in another notation: $s \leftrightarrow^* n$) with n a normal form, then $s \to^* n$.

Give an example showing that NF is not modular.

3 Strategies

Exercise 3.1. [TRS] Consider the TRS defined by the following rules:

$$\begin{array}{rcl} g(x, y) & \to & h(f(x, y)) \\ f(x, b) & \to & f(h(x), a) \\ f(x, h(y)) & \to & f(a, h(x)) \\ a & \to & b \end{array}$$

Consider the following term:

$$t = g(f(f(a, b), h(g(a, a))), g(a, b))$$

Rewrite t according to the following strategies:

(a) leftmost-innermost,

(b) parallel-innermost,

(c) leftmost-outermost,

(d) parallel-outermost,

(e) full-substitution.

Exercise 3.2. Consider the TRS for addition and multiplication:

$$\begin{array}{rcl} \mathsf{plus}(0, \mathsf{y}) & \to & y \\ \mathsf{plus}(\mathsf{S}(\mathsf{x}), \mathsf{y}) & \to & \mathsf{S}(\mathsf{plus}(\mathsf{x}, \mathsf{y})) \\ \mathsf{times}(0, \mathsf{y}) & \to & 0 \\ \mathsf{times}(\mathsf{S}(\mathsf{x}), \mathsf{y}) & \to & \mathsf{plus}(\mathsf{times}(\mathsf{x}, \mathsf{y}), \mathsf{y}) \end{array}$$

Reduce the following term

$$\mathsf{times}(\mathsf{times}(\mathsf{S}(0), \mathsf{S}(0)), \mathsf{plus}(0, \mathsf{S}(0)))$$

with respect to the following strategies:

(a) leftmost-innermost,

(b) parallel-innermost,

(c) leftmost-outermost,

(d) parallel-outermost,

(e) full-substitution.

Exercise 3.3. Consider the CL term

$$\mathsf{S}\,\mathsf{clI}\,\mathsf{clI}\,(\mathsf{I}\,\mathsf{clK})\,(\mathsf{K}\,\mathsf{clK}\,(\mathsf{I}\,\mathsf{clI}))$$

to normal form using the following strategies: with respect to the following strategies:

(a) leftmost-innermost,

(b) parallel-innermost,

(c) leftmost-outermost,

(d) parallel-outermost,

(e) full-substitution.

Exercise 3.4. Consider the CL term

$$clK\,(\mathsf{S}\,\mathsf{K}\,\mathsf{K}\,\mathsf{I})\,(\mathsf{I}\,\mathsf{I})$$

and reduce it according to the strategies leftmost-outermost, full-substitution, parallel-innermost.