ISR 2008
Term Rewriting Systems
Exercise Session Friday July 25

1 Termination

Exercise 1.1. Prove termination of the following TRS:

$$\begin{array}{rcl} \neg(\mathsf{true}) & \to & \mathsf{false} \\ \neg(\neg(x)) & \to & x \\ \mathsf{and}(\mathsf{true}, x) & \to & x \\ \mathsf{and}(\mathsf{false}, x) & \to & \mathsf{false} \\ \mathsf{or}(x, y) & \to & \neg(\mathsf{and}(\neg(x), \neg(y))) \\ \neg(\mathsf{false}) & \to & \mathsf{true} \end{array}$$

(This is the locally confluent TRS from Exercise 1.2 on completion on Tuesday July 22.)

Solution:

Exercise 1.2. Show termination of the following TRS:

$$\begin{array}{rcl} \mathsf{A}(\mathsf{0}, \mathsf{y}) & \to & y \\ \mathsf{A}(\mathsf{S}(\mathsf{x}), \mathsf{y}) & \to & \mathsf{S}(\mathsf{A}(\mathsf{x}, \mathsf{y})) \\ \mathsf{M}(\mathsf{0}, \mathsf{y}) & \to & \mathsf{0} \\ \mathsf{M}(\mathsf{S}(\mathsf{x}), \mathsf{y}) & \to & \mathsf{A}(\mathsf{M}(\mathsf{x}, \mathsf{y}), \mathsf{y}) \end{array}$$

Exercise 1.3. Prove termination of the following TRS:

$$\begin{array}{rcl} f(f(x)) & \to & g(x) \\ g(f(x)) & \to & f(g(x)) \end{array}$$

(This is a completion of the TRS from Exercise 1.7 on Tuesday July 22.)
Solution:

Exercise 1.4. Prove termination of the following TRS:

$$g(x) \to f(f(x))$$

(This is a completion of the TRS from Exercise 1.7 on Tuesday July 22.)
Solution:

Exercise 1.5. [T2.3.9] Prove or disprove termination of the following TRSs:

(a)

$$\begin{array}{rcl} g(x) & \to & f(x) \\ f(x) & \to & f(f(g(x))) \end{array}$$

Solution:

(b)

$$xg(f(x)) \quad \to \quad f(f(g(x)))$$

Solution:

(c)

$$g(f(x)) \quad \to \quad f(f(g(g(x))))$$

Solution:

(d)

$$f(f(x)) \quad \to \quad f(g(f(x)))$$

Solution:

Exercise 1.6. [Wp73] Show termination using a polynomial interpretation of the following TRS:

$$\begin{array}{rcl} x + x & \to & x \\ x + \delta & \to & x \\ x \times (y + xz) & \to & (x \times y) + (x \times z) \end{array}$$

Exercise 1.7. [Wp74] Show termination using a polynomial interpretation of the following TRS:

$$\begin{array}{rcl} f(c, x) & \to & x \\ f(g(x), x) & \to & c \\ f(f(x, y), z) & \to & f(x, f(y, z)) \\ h(x, y) & \to & f(x, g(y)) \end{array}$$

Exercise 1.8. [T6.2.15] Show termination using a polynomial interpretation of the following TRS:

$$\begin{array}{rcl} f(g(x)) & \to & h(x) \\ h(x) & \to & g(f(x)) \end{array}$$

Solution:

Exercise 1.9. [T3.2.15] Show termination using a polynomial interpretation of the following TRS:

$$\begin{array}{rcl} 0 + x & \to & x \\ x + 0 & \to & x \\ x * 0 & \to & 0 \\ x * (y + z) & \to & (x * z) + (y * x) \end{array}$$

Solution:

Exercise 1.10. [W4.2] Show the following: a TRS $\mathcal{R}$ is terminating if and only if there is a reduction order $>$ such that $l > r$ for every rewrite rule $l \to r$ of $\mathcal{R}$.

Exercise 1.11. [W4.1] Show the following: a TRS $\mathcal{R}$ is terminating (SN) if and only if all redexes in $\mathcal{R}$ are terminating (SN).

Exercise 1.12. [W4.3] Is the TRS consisting of the rewrite rule

$$-(x + y) \to (-(-x) + y) + y$$

terminating?

Exercise 1.13. Show termination of the following TRS:

$$\begin{array}{rcl} g(x, y) & \to & h(x, y) \\ h(f(x), y) & \to & f(g(x, y)) \end{array}$$

2 Strategies

Exercise 2.1. [TRS] Consider the TRS defined by the following rules:

$$\begin{array}{rcl} g(x, y) & \to & h(f(x, y)) \\ f(x, b) & \to & f(h(x), a) \\ f(x, h(y)) & \to & f(a, h(x)) \\ a & \to & b \end{array}$$

Consider the following term:

$$t = g(f(f(a, b), h(g(a, a))), g(a, b))$$

Rewrite t according to the following strategies:

(a) leftmost-innermost,

(b) parallel-innermost,

(c) leftmost-outermost,

(d) parallel-outermost,

(e) full-substitution.

Exercise 2.2. Consider the TRS for addition and multiplication:

$$\begin{array}{rcl} \mathsf{A}(0,\mathsf{y}) & \to & y \\ \mathsf{A}(\mathsf{S}(\mathsf{x}),\mathsf{y}) & \to & \mathsf{S}(\mathsf{A}(\mathsf{x},\mathsf{y})) \\ \mathsf{M}(0,\mathsf{y}) & \to & 0 \\ \mathsf{M}(\mathsf{S}(\mathsf{x}),\mathsf{y}) & \to & \mathsf{A}(\mathsf{M}(\mathsf{x},\mathsf{y}),\mathsf{y}) \end{array}$$

Reduce the following term

$$\mathsf{M}(\mathsf{M}(\mathsf{S}(0),\mathsf{S}(0)),\mathsf{A}(0,\mathsf{S}(0)))$$

with respect to the following strategies:

(a) leftmost-innermost,

(b) parallel-innermost,

(c) leftmost-outermost,

(d) parallel-outermost,

(e) full-substitution.

Exercise 2.3. Consider the CL term

$$\mathsf{S}\,\mathsf{cII}\,\mathsf{cII}\,(\mathsf{I}\,\mathsf{clK})\,(\mathsf{K}\,\mathsf{clK}\,(\mathsf{I}\,\mathsf{cII}))$$

to normal form using the following strategies: with respect to the following strategies:

(a) leftmost-innermost,

(b) parallel-innermost,

(c) leftmost-outermost,

(d) parallel-outermost,

(e) full-substitution.

Exercise 2.4. Consider the CL term

$$clK\,(\mathsf{S}\,\mathsf{K}\,\mathsf{K}\,\mathsf{I})\,(\mathsf{I}\,\mathsf{I})$$

and reduce it according to the strategies leftmost-outermost, full-substitution, parallel-innermost.