# ISR 2008
# Term Rewriting Systems
# Exercise Session Monday July 21

## 1 Term Algebra

**Exercise 1.1.** Let $\Sigma$ be the signature consisting of the nullary symbol $\mathsf{0}$, the unary symbol $\mathsf{S}$, and the binary symbols $\mathsf{A}$ and $\mathsf{M}$. Consider the term

$$s = \mathsf{A}(\mathsf{A}(\mathsf{0}, \mathsf{S}(\mathsf{S}(\mathsf{x}))), \mathsf{M}(\mathsf{A}(\mathsf{S}(\mathsf{y}), \mathsf{S}(\mathsf{0})), \mathsf{S}(\mathsf{S}(\mathsf{x}))))$$

(a) Picture the term $s$ as a tree.

(b) What is $\mathsf{Fun}(s)$?

(c) What is $\mathsf{Var}(s)$?

(d) Give all subterms of $s$.

(e) What are the positions of function symbols?

(f) Give two different contexts $C$ such that $s = C[\mathsf{0}]$.

**Exercise 1.2.** [A2.5] Consider the term

$$s = \mathsf{A}(\mathsf{A}(\mathsf{0}, \mathsf{S}(\mathsf{0})), \mathsf{A}(\mathsf{S}(\mathsf{S}(\mathsf{0})), \mathsf{A}(\mathsf{0}, \mathsf{0})))$$

(a) Compute $s|_{21}$.

(b) Compute $s[\mathsf{A}(\mathsf{0}, \mathsf{S}(\mathsf{0}))]_{121}$.

**Exercise 1.3.** [T2.1.3]

(a) Give a direct inductive definition of one-hole contexts.

(b) Give a definition of $C[t]$ by recursion on the (direct inductive) definition of one-hole contexts.

**Exercise 1.4.** [W1.2] Consider the signature consisting of the nullary symbol $\mathsf{0}$, the unary symbol $\mathsf{S}$, and the binary symbols $\mathsf{A}$ and $\mathsf{M}$. Let $\sigma = \{x \mapsto \mathsf{0}, \mathsf{y} \mapsto \mathsf{S(0)}, \mathsf{z} \mapsto \mathsf{A(x,y)}\}$. Compute $s\sigma$ for the following terms $s$:

(a) $z$

(b) $\mathsf{A(0, S(0))}$

(c) $\mathsf{M(A(x.M(y,z)), S(x))}$

**Exercise 1.5.** [BN3.1] Let $\Sigma$ be the signature consisting of the binary function symbol $\mathsf{f}$. Let $s$ be a term over $\Sigma$ and $\mathsf{Var}$.

What are the possible shapes of positions of $s$?

**Exercise 1.6.** [BN3.2]

(a) Let $pq \in \mathsf{Pos(s)}$. Prove the following:

$$s|_{pq} = (s|_p)|_q$$
$$s[t]_{pq} = (s|_p)[t]_q$$

(b) Let $p, q \in \mathsf{Pos(s)}$ with $p \parallel q$. Prove that for all $t$ we have:

$$(s[t]_p)|_q = s|_q$$

**Exercise 1.7.** [BN3.3] Prove that composition of substitutions is associative.

# 2 Term Rewriting Systems

**Exercise 2.1.** [W15 Opgave 1.1] Consider the signature $\Sigma$ consisting of a binary function symbol $\mathsf{f}$, a unary function symbol $\mathsf{g}$, and a constant symbol $\mathsf{a}$. Which of the following are proper rewrite rules:

(a) $x \to \mathsf{a}$

(b) $\mathsf{a} \to x$

(c) $\mathsf{a} \to \mathsf{a}$

(d) $\mathsf{f}(x) \to \mathsf{g}(x)$

(e) $\mathsf{g}(x) \to \mathsf{f}(x, y)$

(f) $\mathsf{f}(x, x) \to \mathsf{f}(x, y)$

(g) $\mathsf{g}(x) \to \mathsf{f}(x, x)$

**Exercise 2.2.** Consider the TRS for addition and multiplication:

$$\begin{array}{rcl} \mathsf{A}(0,\mathsf{y}) & \to & y \\ \mathsf{A}(\mathsf{S}(\mathsf{x}),\mathsf{y}) & \to & \mathsf{S}(\mathsf{A}(\mathsf{x},\mathsf{y})) \\ \mathsf{M}(0,\mathsf{y}) & \to & 0 \\ \mathsf{M}(\mathsf{S}(\mathsf{x}),\mathsf{y}) & \to & \mathsf{A}(\mathsf{M}(\mathsf{x},\mathsf{y}),\mathsf{y}) \end{array}$$

Give all possible ways of computing the normal form of

$$\mathsf{M}(\mathsf{S}(\mathsf{S}(0)),\mathsf{S}(\mathsf{S}(0)))$$

**Exercise 2.3.** Consider combinatory logic (CL):

$$\begin{array}{rcl} \mathsf{S\,x\,y\,z} & \to & (x\,z)\,(y\,z) \\ \mathsf{K\,x\,y} & \to & x \\ \mathsf{I\,x} & \to & x \end{array}$$

Give the three possible reductions of the term

$$\mathsf{S\,I\,(K\,x)\,I\,y}$$

**Exercise 2.4.** Define a TRS for insertion sort.

**Exercise 2.5.** Give an example of a TRS consisting of two rewrite rules that is not confluent.

**Exercise 2.6.** Give an example of a TRS that satisfies UN but is not confluent.

# 3 Equational Reasoning

**Exercise 3.1.** [BN3.6] Consider the equational specification $\mathcal{E}$ for groups:

$$\begin{array}{rcl} (x * y) * z & = & x * (y * z) \\ x * e & = & x \\ x * \bar{x} & = & e \end{array}$$

Recall from the slides that in the corresponding TRS we have $x \leftrightarrow^* e * x$. Show that

$$\mathcal{E} \vdash x = e * x$$

**Exercise 3.2.** [BN3.7] Consider the equational specification $\mathcal{E}$ defined by the following equations:

$$\begin{array}{rcl} f(x, f(y,z)) & = & f(f(x,y),z) \\ f(f(x,y),x) & = & x \end{array}$$

(a) Show that in the corresponding TRS we have

$$\begin{array}{rcl} f(x,x) & \leftrightarrow^* & x \\ f(f(x,y),z) & \leftrightarrow & f(x,z) \end{array}$$

(b) Give also the corresponding derivations:

$$\mathcal{E} \vdash f(x, x) = x$$
$$\mathcal{E} \vdash f(f(x, y), z) = f(x, z)$$

**Exercise 3.3.** [A] Consider the following equational system:

$$\begin{array}{rcl} f(x) & = & x \\ f(f(a)) & = & g(x, x) \\ g(x, f(x)) & = & b \end{array}$$

Which of the following equations can be derived:

(a) $a = b$

(b) $g(x, y) = g(y, x)$

(c) $g(f(a), a) = f(b)$

**Exercise 3.4.** [T7.1.3] Recall the following definitions.

If $(\Sigma, R)$ is a pseudo-TRS, then the corresponding equational specification is $(\Sigma, R^{=})$ with

$$R^{=} = \{l = r \mid l \to r \in R\}$$

If $(\Sigma, E)$ is an equational specification, then the corresponding pseudo-TRS is $(\Sigma, E^{\leftrightarrow})$ with

$$E^{\leftrightarrow} = \{l \to r \mid l = r \in E \text{or} r = l \in E\}$$

(a) Let $(\Sigma, R)$ be a TRS. Prove the following: $(\Sigma, R^{=}) \vdash s = t$ if and only if $s \leftrightarrow_R^* t$.

(b) Let $(\Sigma, E)$ be an equational specification. Prove the following: $(\Sigma, E) \vdash s = t$ if and only if $s \leftrightarrow_{E^{\leftrightarrow}}^* t$.

**Exercise 3.5.** [W5.1] Consider again the equational specification $\mathcal{E}$ for groups:

$$\begin{array}{rcl} (x * y) * z & = & x * (y * z) \\ x * e & = & x \\ x * \bar{x} & = & e \end{array}$$

Show that in every model of $\mathcal{E}$ the following equation is valid:

$$\overline{(x * y)} = \bar{x} * \bar{y}$$

**Exercise 3.6.** [W5.2] Consider the equational specification $\mathcal{E}$ defined by the following equations:

$$\begin{array}{rcl} \mathsf{A}(0, \mathsf{y}) & = & y \\ \mathsf{A}(\mathsf{S}(\mathsf{x}), \mathsf{y}) & = & \mathsf{S}(\mathsf{A}(\mathsf{x}, \mathsf{y})) \end{array}$$

Construct a model for $\mathcal{E}$ in which $\mathsf{A}$ is not commmutative.

**Exercise 3.7.** [T Proposition 2.5.5] Let $(\sigma, R)$ be a TRS and let $(\sigma, R^=)$ be the corresponding equational specification. Assume that $\mathcal{A}$ is a model for $(\sigma, R^=)$ such that: for all (ground) normal forms $s, s'$ of $(\sigma, R)$, if $\mathcal{A} \models s = s'$ then $s = s'$ (so $s$ and $s'$ are syntactically equal).

Show that the TRS $(\sigma, R)$ has the property (ground-)UN.

**Exercise 3.8.** [T2.5.6] Consider the TRS for addition and multiplication:

$$\begin{array}{rcl} \mathsf{A}(\mathsf{0},\mathsf{y}) & \to & y \\ \mathsf{A}(\mathsf{S}(\mathsf{x}),\mathsf{y}) & \to & \mathsf{S}(\mathsf{A}(\mathsf{x},\mathsf{y})) \\ \mathsf{M}(\mathsf{0},\mathsf{y}) & \to & \mathsf{0} \\ \mathsf{M}(\mathsf{S}(\mathsf{x}),\mathsf{y}) & \to & \mathsf{A}(\mathsf{M}(\mathsf{x},\mathsf{y}),\mathsf{y}) \end{array}$$

(a) What are the closed normal forms of the TRS?

(b) Show that the TRS has the property ground-UN.

**Exercise 3.9.** [BN3.13] Consider the equational specification defined by the following equations (it is not the same as the specification for groups!):

$$\begin{array}{rcl} (x * y) * z & = & x * (y * z) \\ e * x & = & x \\ x * \bar{x} & = & e \end{array}$$

Show that the equation $x * e = x$ is derivable. Hint: construct a model in which is it not satisfied.