Reducing Confluence of LCTRSs to Confluence of TRSs*

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Abstract

We present a transformation from logically constrained term rewrite systems (LCTRSs) to plain term rewrite systems (TRSs) such that critical pairs of the latter correspond to constrained critical pairs of the former. This allows us to transfer confluence results for TRSs based on critical pair conditions to LCTRSs.

1 Introduction

Numerous techniques exist to (dis)prove confluence of plain TRSs. For LCTRSs much less is known. Kop and Nishida [1] established (weak) orthogonality as sufficient confluence criteria for LCTRSs. Joinability of critical pairs for terminating systems is implicit in [4]. Very recently, strong closedness for linear LCTRSs and (almost) parallel closedness for left-linear LCTRSs were established [2]. The proofs of these results were obtained by *replaying* existing proofs for TRSs in a constrained setting, involving a non-trivial effort. For more advanced confluence criteria, this is not feasible.

In this paper we present a simple transformation from LCTRSs to TRSs which allows us to relate results for the latter to the former. This transformation is presented in the next section and used in Section 3 to prove that (almost) development closed left-linear LCTRSs are confluent by *reusing* the corresponding result for TRSs obtained by van Oostrom [3].

We assume familiarity with the basic notions of term rewriting. In the remainder of this introductory section we recall a few key notions for LCTRSs. For more background information we refer to [1, 2, 4]. We assume a many-sorted signature $\mathcal{F} = \mathcal{F}_{\mathsf{te}} \cup \mathcal{F}_{\mathsf{th}}$. For every sort ι in $\mathcal{F}_{\mathsf{th}}$ we have a non-empty set $\mathcal{V}\mathsf{al}_\iota \subseteq \mathcal{F}_{\mathsf{th}}$ of value symbols, such that all $c \in \mathcal{V}\mathsf{al}_\iota$ are constants of sort ι. We demand $\mathcal{F}_{\mathsf{te}} \cap \mathcal{F}_{\mathsf{th}} \subseteq \mathcal{V}\mathsf{al}$ where $\mathcal{V}\mathsf{al} = \bigcup_\iota \mathcal{V}\mathsf{al}_\iota$. In the case of integers this results in an infinite signature with $\mathbb{Z} \subseteq \mathcal{V}\mathsf{al} \subseteq \mathcal{F}_{\mathsf{th}}$. A term in $\mathcal{T}(\mathcal{F}_{\mathsf{th}}, \mathcal{V})$ is called a *logical* term. Ground logical terms are mapped to values by an interpretation $\mathcal{J}$: $[\![f(t_1,\dots,t_n)]\!] = f_{\mathcal{J}}([\![t_1]\!],\dots,[\![t_n]\!])$. Logical terms of sort bool are called *constraints*. A constraint φ is *valid* if $[\![\varphi\gamma]\!] = \top$ for all substitutions γ such that $\gamma(x) \in \mathcal{V}\mathsf{al}$ for all $x \in \mathcal{V}\mathsf{ar}(\varphi)$. A *constrained rewrite rule* is a triple $\ell \to r\ [\varphi]$ where $\ell, r \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ are terms of the same sort such that $\mathsf{root}(\ell) \in \mathcal{F}_{\mathsf{te}} \setminus \mathcal{F}_{\mathsf{th}}$ and φ is a constraint. We denote the set $\mathcal{V}\mathsf{ar}(\varphi) \cup (\mathcal{V}\mathsf{ar}(r) \setminus \mathcal{V}\mathsf{ar}(\ell))$ of *logical* variables in $\ell \to r\ [\varphi]$ by $\mathcal{LV}\mathsf{ar}(\ell \to r\ [\varphi])$. We write $\mathcal{EV}\mathsf{ar}(\ell \to r\ [\varphi])$ for the set $\mathcal{V}\mathsf{ar}(r) \setminus (\mathcal{V}\mathsf{ar}(\ell) \cup \mathcal{V}\mathsf{ar}(\varphi))$. A set of constrained rewrite rules is called an LCTRS. A substitution σ is said to *respect* a rule $\ell \to r\ [\varphi]$, denoted by $\sigma \vDash \ell \to r\ [\varphi]$, if $\mathcal{D}\mathsf{om}(\sigma) = \mathcal{V}\mathsf{ar}(\ell) \cup \mathcal{V}\mathsf{ar}(r) \cup \mathcal{V}\mathsf{ar}(\varphi)$, $\sigma(x) \in \mathcal{V}\mathsf{al}$ for all $x \in \mathcal{LV}\mathsf{ar}(\ell \to r\ [\varphi])$, and $\varphi\sigma$ is valid. Moreover, a constraint φ is respected by σ, denoted by $\sigma \vDash \varphi$, if $\sigma(x) \in \mathcal{V}\mathsf{al}$ for all $x \in \mathcal{V}\mathsf{ar}(\varphi)$ and $\varphi\sigma$ is valid. We call $f(x_1,\dots,x_n) \to y\ [y = f(x_1,\dots,x_n)]$ with a fresh variable y and $f \in \mathcal{F}_{\mathsf{th}} \setminus \mathcal{V}\mathsf{al}$ a *calculation rule*. The set of all calculation rules induced by the signature $\mathcal{F}_{\mathsf{th}}$ of an LCTRS $\mathcal{R}$ is denoted by $\mathcal{R}_{\mathsf{ca}}$ and we abbreviate $\mathcal{R} \cup \mathcal{R}_{\mathsf{ca}}$ to $\mathcal{R}_{\mathsf{rc}}$. A rewrite step $s \to_{\mathcal{R}} t$ satisfies $s|_p = \ell\sigma$ and $t = s[r\sigma]_p$ for some position p, constrained rewrite rule $\ell \to r\ [\varphi]$ in $\mathcal{R}_{\mathsf{rc}}$, and substitution σ such that $\sigma \vDash \ell \to r\ [\varphi]$.

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A *constrained term* is a pair $s\ [\varphi]$ consisting of a term s and a constraint φ. Two constrained terms $s\ [\varphi]$ and $t\ [\psi]$ are *equivalent*, denoted by $s\ [\varphi] \sim t\ [\psi]$, if for every substitution γ respecting φ there is some substitution δ respecting ψ such that $s\gamma = t\delta$, and vice versa. Let $s\ [\varphi]$ be a constrained term. If $s|_p = \ell\sigma$ for some constrained rewrite rule $\rho\colon \ell \to r\ [\psi] \in \mathcal{R}_{\mathsf{rc}}$, position p, and substitution σ such that $\sigma(x) \in \mathcal{V}\mathsf{al} \cup \mathcal{V}\mathsf{ar}(\varphi)$ for all $x \in \mathcal{LV}\mathsf{ar}(\rho)$, φ is satisfiable and $\varphi \Rightarrow \psi\sigma$ is valid then $s\ [\varphi] \to_{\mathcal{R}} s[r\sigma]_p\ [\varphi]$. The rewrite relation $\xrightarrow{\sim}_{\mathcal{R}}$ on constrained terms is defined as $\sim \cdot \to_{\mathcal{R}} \cdot \sim$ and $s\ [\varphi] \xrightarrow{\sim}_p t\ [\psi]$ indicates that the rewrite step in $\xrightarrow{\sim}_{\mathcal{R}}$ takes place at position p. Similarly, we write $s\ [\varphi] \xrightarrow{\sim}_{\geqslant p} t\ [\psi]$ if the position in the rewrite step is below position p. Note that in our definition of $\to_{\mathcal{R}}$ the constraint is not modified. This is different from [1, 2] where calculation steps $s[f(v_1,\dots,v_n)]_p\ [\varphi] \to s[v]_p\ [\varphi \wedge v = f(v_1,\dots,v_n)]$ modify the constraint. Our relation $\xrightarrow{\sim}$ is equivalent to the relation $\sim \cdot (\to_{\mathsf{ru}} \cup \to_{\mathsf{ca}}) \cdot \sim$ in [1, 2] since the constraint can be expanded as exemplified below.

Example 1. Consider the constrained term $x + \mathsf{1}\ [x > \mathsf{3}]$. Calculation steps as defined in [1, 2] permit $x + \mathsf{1}\ [x > \mathsf{3}] \to z\ [z = x + \mathsf{1} \wedge x > \mathsf{3}]$. In our setting, an initial equivalence step is required to introduce the fresh variable z and the corresponding assignment needed to perform a calculation: $x + \mathsf{1}\ [x > \mathsf{3}] \sim x + \mathsf{1}\ [z = x + \mathsf{1} \wedge x > \mathsf{3}] \to z\ [z = x + \mathsf{1} \wedge x > \mathsf{3}]$.

Our treatment allows for a much simpler definition of parallel and multi-step rewriting since we do not have to merge different constraints.

2 Transformation

Our transformation is defined below.

Definition 2. Given an LCTRS $\mathcal{R}$, the TRS $\overline{\mathcal{R}}$ consists of the following rules: (1) $\ell\tau \to r\tau$ for all $\rho\colon \ell \to r\ [\varphi] \in \mathcal{R}$ with $\tau \vDash \rho$ and $\mathcal{D}\mathsf{om}(\tau) = \mathcal{LV}\mathsf{ar}(\rho)$, and (2) $f(v_1,\dots,v_n) \to [\![f(v_1,\dots,v_n)]\!]$ for all $f \in \mathcal{F}_{\mathsf{th}} \setminus \mathcal{V}\mathsf{al}$ and $v_1,\dots,v_n \in \mathcal{V}\mathsf{al}$.

Note that $\overline{\mathcal{R}}$ typically consists of infinitely many rules.

Lemma 3. *The rewrite relations of $\mathcal{R}$ and $\overline{\mathcal{R}}$ are the same. Moreover $\xrightarrow{p}_{\mathcal{R}} = \xrightarrow{p}_{\overline{\mathcal{R}}}$ for all positions p.*

Proof. We first show $\xrightarrow{p}_{\mathcal{R}} \subseteq \xrightarrow{p}_{\overline{\mathcal{R}}}$. Assume $s \xrightarrow{p}_{\mathcal{R}} t$. So either $s = s[f(v_1,\dots,v_n)]_p \to s[v]_p = t$ for some $f \in \mathcal{F}_{\mathsf{th}} \setminus \mathcal{V}\mathsf{al}$ and $v = [\![f(v_1,\dots,v_n)]\!]$ or $s = s[\ell\sigma]_p \to s[r\sigma]_p = t$ for some $\rho\colon \ell \to r\ [\varphi] \in \mathcal{R}$ and $\sigma \vDash \rho$. In the first case $s \xrightarrow{p}_{\overline{\mathcal{R}}} t$ by the definition of $\overline{\mathcal{R}}$. In the second case we split σ into two substitutions $\tau = \{x \mapsto \sigma(x) \mid x \in \mathcal{LV}\mathsf{ar}(\rho)\}$ and $\delta = \{x \mapsto \sigma(x) \mid x \in \mathcal{V}\mathsf{ar}(\ell) \setminus \mathcal{LV}\mathsf{ar}(\rho)\}$. From $\sigma \vDash \rho$ we infer $\tau \vDash \rho$ and thus $\tau(x) \in \mathcal{V}\mathsf{al}$ for all $x \in \mathcal{LV}\mathsf{ar}(\rho)$. Hence $\sigma = \tau \cup \delta = \tau\delta$. We have $\ell\tau \to r\tau \in \overline{\mathcal{R}}$. Hence $s = s[\ell\tau\delta]_p \xrightarrow{p}_{\overline{\mathcal{R}}} s[r\tau\delta]_p = t$ as desired. To show the reverse inclusion $\xrightarrow{p}_{\overline{\mathcal{R}}} \subseteq \xrightarrow{p}_{\mathcal{R}}$ we assume $s \xrightarrow{p}_{\overline{\mathcal{R}}} t$. When the applied rule stems from a calculation rule in $\mathcal{R}$, we trivially have $s \xrightarrow{p}_{\mathcal{R}} t$. Otherwise $s = s[\ell\tau\delta]_p \xrightarrow{p}_{\overline{\mathcal{R}}} s[r\tau\delta]_p$ for some rule $\rho\colon \ell \to r\ [\varphi] \in \mathcal{R}$ with $\tau \vDash \rho$. Let $\sigma = \tau\delta$. Since $\tau(x) \in \mathcal{V}\mathsf{al}$ for all $x \in \mathcal{LV}\mathsf{ar}(\rho)$, we have $x\sigma = x\tau$ for all $x \in \mathcal{LV}\mathsf{ar}(\rho)$. Hence $\sigma \vDash \rho$ and thus $s = s[\ell\sigma]_p \xrightarrow{p}_{\mathcal{R}} s[r\sigma]_p = t$. $\square$

Since $\to_{\mathcal{R}}$ and $\to_{\overline{\mathcal{R}}}$ coincide, we drop the subscript in the sequel. Rules of type (2) in Definition 2 can be viewed as type (1) for the rule $f(x_1,\dots,x_n) \to y\ [y = f(x_1,\dots,x_n)]$ in $\mathcal{R}_{\mathsf{ca}}$ by taking $\tau = \{x_1 \mapsto v_1,\dots,x_n \mapsto v_n, y \mapsto [\![f(v_1,\dots,v_n)]\!]\}$. Hence we do not distinguish between the two cases and consider only rules of type (1) with $\rho \in \mathcal{R}_{\mathsf{rc}}$.

Definition 4. An *overlap* of an LCTRS $\mathcal{R}$ is a triple $\langle \rho_1, p, \rho_2 \rangle$ with rules $\rho_1 \colon \ell_1 \to r_1\ [\varphi_1]$ and $\rho_2 \colon \ell_2 \to r_2\ [\varphi_2]$, satisfying the following conditions: (1) ρ_1 and ρ_2 are variable-disjoint variants of rewrite rules in $\mathcal{R}_{\mathsf{rc}}$, (2) $p \in \mathcal{P}\mathsf{os}_{\mathcal{F}}(\ell_2)$, (3) ℓ_1 and $\ell_2|_p$ unify with mgu σ such that $\sigma(x) \in \mathcal{V}\mathsf{al} \cup \mathcal{V}$ for all $x \in \mathcal{LV}\mathsf{ar}(\rho_1) \cup \mathcal{LV}\mathsf{ar}(\rho_2)$, (4) $\varphi_1\sigma \wedge \varphi_2\sigma$ is satisfiable, and (5) if $p = \epsilon$ then ρ_1 and ρ_2 are not variants, or $\mathcal{V}\mathsf{ar}(r_1) \nsubseteq \mathcal{V}\mathsf{ar}(\ell_1)$. In this case we call $\ell_2\sigma[r_1\sigma]_p \approx r_2\sigma\ [\varphi_1\sigma \wedge \varphi_2\sigma \wedge \psi\sigma]$ a *constrained critical pair* obtained from the overlap $\langle \rho_1, p, \rho_2 \rangle$. Here $\psi = \bigwedge\{x = x \mid x \in \mathcal{EV}\mathsf{ar}(\rho_1) \cup \mathcal{EV}\mathsf{ar}(\rho_2)\}$. The set of all constrained critical pairs of $\mathcal{R}$ is denoted by $\mathsf{CCP}(\mathcal{R})$.

A key ingredient of our approach is to relate critical pairs of the transformed TRS to constrained critical pairs of the originating LCTRS.

Theorem 5. *For every critical pair $s \approx t$ of $\overline{\mathcal{R}}$ there exists a constrained critical pair $s' \approx t'\ [\varphi']$ of $\mathcal{R}$ and a substitution γ such that $s = s'\gamma$, $t = t'\gamma$ and $\gamma \vDash \varphi'$.*

Proof. Let $s \approx t$ be a critical pair of $\overline{\mathcal{R}}$, originating from the critical peak $\ell_2\mu\sigma[r_1\nu\sigma]_p \leftarrow \ell_2\mu\sigma = \ell_2\mu\sigma[\ell_1\nu\sigma]_p \to r_2\mu\sigma$ with variants $\rho_1 \colon \ell_1 \to r_1\ [\varphi_1]$ and $\rho_2 \colon \ell_2 \to r_2\ [\varphi_2]$ of rules in $\mathcal{R}_{\mathsf{rc}}$ without shared variables, $\mathcal{D}\mathsf{om}(\nu) = \mathcal{LV}\mathsf{ar}(\rho_1)$, $\mathcal{D}\mathsf{om}(\mu) = \mathcal{LV}\mathsf{ar}(\rho_2)$, $\nu \vDash \rho_1$, $\mu \vDash \rho_2$, $p \in \mathcal{P}\mathsf{os}_{\mathcal{F}}(\ell_2\mu)$, and σ is an mgu of $\ell_2\mu|_p$ and $\ell_1\nu$. Moreover, if $p = \epsilon$ then $\ell_1\nu \to r_1\nu$ and $\ell_2\mu \to r_2\mu$ are not variants. Define $\tau = \nu \uplus \mu$. Clearly, $\ell_1\tau = \ell_1\nu$, $r_1\tau = r_1\nu$, $\ell_2\tau = \ell_2\mu$, $r_2\tau = r_2\mu$, $\tau \vDash \rho_1$ and $\tau \vDash \rho_2$. Hence the given peak can be written as $\ell_2\tau\sigma[r_1\tau\sigma]_p \leftarrow \ell_2\tau\sigma = \ell_2\tau\sigma[\ell_1\tau\sigma]_p \to r_2\tau\sigma$ and $\tau \vDash \varphi$ where $\varphi = \varphi_1 \wedge \varphi_2 \wedge \bigwedge\{x = x \mid x \in \mathcal{EV}\mathsf{ar}(\rho_1) \cup \mathcal{EV}\mathsf{ar}(\rho_2)\}$. Since $\ell_2|_p\tau\sigma = \ell_1\tau\sigma$ there exists an mgu δ of $\ell_2|_p$ and ℓ_1, and a substitution γ such that $\delta\gamma = \tau\sigma$. Let $s' = \ell_2\delta[r_1\delta]_p$ and $t' = r_2\delta$. We claim that $\langle \rho_1, p, \rho_2 \rangle$ is an overlap of $\mathcal{R}$, resulting in the constrained critical pair $s' \approx t'\ [\varphi\delta]$. Condition (1) of Definition 4 is trivially satisfied. For condition (2) we need to show $p \in \mathcal{P}\mathsf{os}_{\mathcal{F}}(\ell_2)$. This follows from $p \in \mathcal{P}\mathsf{os}_{\mathcal{F}}(\ell_2\mu)$, $\mu(x) \in \mathcal{V}\mathsf{al}$ for every $x \in \mathcal{D}\mathsf{om}(\mu)$, and $\mathsf{root}(\ell_2\mu|_p) = \mathsf{root}(\ell_1\nu) \in \mathcal{F} \setminus \mathcal{V}\mathsf{al}$. For condition (3) it remains to show that $\delta(x) \in \mathcal{V}\mathsf{al} \cup \mathcal{V}$ for all $x \in \mathcal{LV}\mathsf{ar}(\rho_1) \cup \mathcal{LV}\mathsf{ar}(\rho_2)$. Suppose to the contrary that $\mathsf{root}(\delta(x)) \in \mathcal{F} \setminus \mathcal{V}\mathsf{al}$ for some $x \in \mathcal{LV}\mathsf{ar}(\rho_1) \cup \mathcal{LV}\mathsf{ar}(\rho_2)$. Then $\mathsf{root}(\delta(x)) = \mathsf{root}(\gamma(\delta(x))) = \mathsf{root}(\sigma(\tau(x))) \in \mathcal{F} \setminus \mathcal{V}\mathsf{al}$, which contradicts $\tau \vDash \varphi$. Condition 4 follows from the identity $\delta\gamma = \tau\sigma$ together with $\tau \vDash \varphi$ which imply $\delta\gamma \vDash \varphi$ and thus $\varphi\delta$ is satisfiable. Hence also $\varphi_1\delta \wedge \varphi_2\delta$ is satisfiable. It remains to show condition 5, so let $p = \epsilon$ and further assume that ρ_1 and ρ_2 are variants. So there exists a variable renaming π such that $\rho_1\pi = \rho_2$. In particular, $\ell_1\pi = \ell_2$ and $r_1\pi = r_2$. Let $x \in \mathcal{V}\mathsf{ar}(\ell_1)$. If $x \in \mathcal{LV}\mathsf{ar}(\rho_1) = \mathcal{D}\mathsf{om}(\nu)$ then $\tau(x) = \nu(x) \in \mathcal{V}\mathsf{al}$. Moreover, $\pi(x) \in \mathcal{LV}\mathsf{ar}(\rho_2) = \mathcal{D}\mathsf{om}(\mu)$ and thus $\tau(\pi(x)) = \mu(\pi(x)) \in \mathcal{V}\mathsf{al}$. Since $\ell_1\tau$ and $\ell_2\tau$ are unifiable, $\pi(\tau(x)) = \tau(x) = \tau(\pi(x))$. If $x \notin \mathcal{LV}\mathsf{ar}(\rho_1)$ then $\tau(x) = x$, $\pi(x) \notin \mathcal{LV}\mathsf{ar}(\rho_2)$ and similarly $\tau(\pi(x)) = \pi(x) = \pi(\tau(x))$. All in all, $\ell_1\tau\pi = \ell_1\pi\tau = \ell_2\tau$. Now, if $\mathcal{V}\mathsf{ar}(r_1) \subseteq \mathcal{V}\mathsf{ar}(\ell_1)$ then we obtain $r_1\tau\pi = r_1\pi\tau = r_2\tau$, contradicting the fact that $\ell_1\nu \to r_1\nu$ and $\ell_2\mu \to r_2\mu$ are not variants. We conclude that $s' \approx t'\ [\varphi\delta]$ is a constrained critical pair of $\mathcal{R}$. So we can take $\varphi' = \varphi\delta$. Clearly, $s = s'\gamma$ and $t = t'\gamma$. Moreover, $\gamma \vDash \varphi'$ since $\varphi'\gamma = \varphi\tau\sigma = \varphi\tau$ and $\tau \vDash \varphi$. □

The converse does not hold in general.

Example 6. Consider the LCTRS $\mathcal{R}$ consisting of the single rule $\mathsf{a} \to x\ [x = 0]$ where the variable x ranges over the integers. Since x appears on the right-hand side but not the left, we obtain a constrained critical pair $x \approx x'\ [x = 0 \wedge x' = 0]$. Since the constraint uniquely determines the values of x and x', the TRS $\overline{\mathcal{R}}$ consists of the single rule $\mathsf{a} \to 0$. Obviously $\overline{\mathcal{R}}$ has no critical pairs.

The above example also shows that orthogonality of $\overline{\mathcal{R}}$ does not imply orthogonality of $\mathcal{R}$. However, the counterexample relies somewhat on a technicality in condition (5) of Definition 4.

It only occurs when the two rules $\ell_1 \to r_1\ [\varphi_1]$ and $\ell_2 \to r_2\ [\varphi_2]$ involved in the critical pair overlap at the root and have instances $\ell_1\tau_1 \to r_1\tau_1$ and $\ell_2\tau_2 \to r_2\tau_2$ in $\overline{\mathcal{R}}$ which are variants of each other. By dealing with such cases separately we can prove the following theorem.

Theorem 7. *For every constrained critical pair $s \approx t\ [\varphi]$ of $\mathcal{R}$ and every substitution σ with $\sigma \vDash \varphi$, (1) $s\sigma = t\sigma$ or (2) there exist a critical pair $u \approx v$ of $\overline{\mathcal{R}}$ and a substitution δ such that $s\sigma = u\delta$ and $t\sigma = v\delta$.*

Proof. Let $s \approx t\ [\varphi]$ be a constrained critical pair of $\mathcal{R}$ originating from the critical peak $s = \ell_2\theta[r_1\theta]_p \leftarrow \ell_2\theta[\ell_1\theta]_p \to r_2\theta = t$ with variants $\rho_1 \colon \ell_1 \to r_1\ [\varphi_1]$ and $\rho_2 \colon \ell_2 \to r_2\ [\varphi_2]$ of rules in $\mathcal{R}_{\mathsf{rc}}$, and an mgu θ of $\ell_2|_p$ and ℓ_1 where $p \in \mathcal{P}\mathsf{os}_{\mathcal{F}}(\ell_2)$. Moreover $\theta(x) \in \mathcal{V}\mathsf{al} \cup \mathcal{V}$ for all $x \in \mathcal{LV}\mathsf{ar}(\rho_1) \cup \mathcal{LV}\mathsf{ar}(\rho_2)$, and $\varphi = \varphi_1\theta \wedge \varphi_2\theta \wedge \psi\theta$ with $\psi = \bigwedge\{x = x \mid x \in \mathcal{EV}\mathsf{ar}(\rho_1) \cup \mathcal{EV}\mathsf{ar}(\rho_2)\}$. Let σ be a substitution with $\sigma \vDash \varphi$. Hence $\theta\sigma \vDash \varphi_1 \wedge \varphi_2 \wedge \psi$ and further $\sigma(\theta(x)) \in \mathcal{V}\mathsf{al}$ for all $x \in \mathcal{LV}\mathsf{ar}(\rho_1) \cup \mathcal{LV}\mathsf{ar}(\rho_2)$. We split $\theta\sigma$ into substitutions τ_1, τ_2 and π as follows:

$$\tau_i(x) = \begin{cases} x\theta\sigma & \text{if } x \in \mathcal{LV}\mathsf{ar}(\rho_i) \\ x & \text{otherwise} \end{cases} \qquad \pi(x) = \begin{cases} x\theta\sigma & \text{if } x \in \mathcal{D}\mathsf{om}(\theta\sigma) \setminus (\mathcal{LV}\mathsf{ar}(\rho_1) \cup \mathcal{LV}\mathsf{ar}(\rho_2)) \\ x & \text{otherwise} \end{cases}$$

for $i \in \{1,2\}$. From $\theta\sigma \vDash \varphi_1 \wedge \varphi_2 \wedge \psi$ and $\mathcal{V}\mathsf{ar}(\varphi_i) \subseteq \mathcal{LV}\mathsf{ar}(\rho_i)$ we infer $\tau_i \vDash \varphi_i$ for $i \in \{1,2\}$. Since $\mathcal{D}\mathsf{om}(\tau_i) = \mathcal{LV}\mathsf{ar}(\rho_i)$, $\ell_i\tau_i \to r_i\tau_i \in \overline{\mathcal{R}}$ for $i \in \{1,2\}$. Furthermore, $\tau_i\pi = \tau_i \cup \pi$ for $i \in \{1,2\}$. Hence $\ell_2|_p\tau_2\pi = \ell_2|_p\theta\sigma = \ell_1\theta\sigma = \ell_1\tau_1\pi$, implying that $\ell_2|_p\tau_2$ and $\ell_1\tau_1$ are unifiable. Let γ be an mgu of these two terms. There exists a substitution δ such that $\gamma\delta = \pi$. Clearly $p \in \mathcal{P}\mathsf{os}_{\mathcal{F}}(\ell_2\tau_2)$. If $p \neq \epsilon$ or $\ell_1\tau_1 \to r_1\tau_1$ and $\ell_2\tau_2 \to r_2\tau_2$ are not variants, then $u \approx v$ with $u = \ell_2\tau_2\gamma[r_1\tau_1\gamma]_p$ and $v = r_2\tau_2\gamma$ is a critical pair of $\overline{\mathcal{R}}$. Moreover $t\sigma = r_2\theta\sigma = r_2\tau_2\pi = r_2\tau_2\gamma\delta = v\delta$, and similarly $s\sigma = u\delta$. Thus option (2) is satisfied. If $p = \epsilon$ and $\ell_1\tau_1 \to r_1\tau_1$ and $\ell_2\tau_2 \to r_2\tau_2$ are variants then $s\sigma = r_1\tau_1\gamma\delta = r_2\tau_2\gamma\delta = t\sigma$, fulfilling (1). $\square$

A direct consequence is that weak orthogonality of $\overline{\mathcal{R}}$ implies weak orthogonality of $\mathcal{R}$.

3 Confluence

Using Theorem 5 we can easily transfer confluence criteria for TRSs to LCTRSs. Rather than reproving the confluence results reported in [1, 4, 2], we illustrate this by extending the result of van Oostrom [3] concerning (almost) development closed critical pairs from TRSs to LCTRSs. The following result from [4] plays an important role.

Lemma 8. *Suppose $s \approx t\ [\varphi] \overset{\sim}{\to}_p u \approx v\ [\psi]$ with $\gamma \vDash \varphi$ and position p. If $p \geqslant 1$ then $s\gamma \to u\delta$ and $t\gamma = v\delta$ for some substitution δ with $\delta \vDash \psi$. If $p \geqslant 2$ then $s\gamma = u\delta$ and $t\gamma \to v\delta$ for some substitution δ with $\delta \vDash \psi$.* $\square$

Definition 9. Let $\mathcal{R}$ be an LCTRS. The multi-step relation $\twoheadrightarrow$ on terms is defined inductively as follows: (1) $x \twoheadrightarrow x$ for all variables x, (2) $f(s_1,\dots,s_n) \twoheadrightarrow f(t_1,\dots,t_n)$ if $s_i \twoheadrightarrow t_i$ with $1 \leqslant i \leqslant n$, (3) $\ell\sigma \twoheadrightarrow r\tau$ if $\ell \to r\ [\varphi] \in \mathcal{R}_{\mathsf{rc}}$, $\sigma \vDash \ell \to r\ [\varphi]$ and $\sigma \twoheadrightarrow \tau$, where $\sigma \twoheadrightarrow \tau$ denotes $\sigma(x) \twoheadrightarrow \tau(x)$ for all variables $x \in \mathcal{D}\mathsf{om}(\sigma)$.

The next definition inductively defines multi-step rewriting on constrained terms.

Definition 10. Let $\mathcal{R}$ be an LCTRS. The multi-step relation $\twoheadrightarrow$ on constrained terms is defined inductively as follows:

1. $x\ [\varphi] \twoheadrightarrow x\ [\varphi]$ for all variables x,

2. $f(s_1, \ldots, s_n)\ [\varphi] \mathrel{\multimap\!\!\!\to} f(t_1, \ldots, t_n)\ [\varphi]$ if $s_i\ [\varphi] \mathrel{\multimap\!\!\!\to} t_i\ [\varphi]$ for $1 \leqslant i \leqslant n$,

3. $\ell\sigma\ [\varphi] \mathrel{\multimap\!\!\!\to} r\tau\ [\varphi]$ if $\rho\colon \ell \to r\ [\psi] \in \mathcal{R}_{\mathsf{rc}}$, $\sigma(x) \in \mathcal{V}\mathsf{al} \cup \mathcal{V}\mathsf{ar}(\varphi)$ for all $x \in \mathcal{L}\mathcal{V}\mathsf{ar}(\rho)$, φ is satisfiable, $\varphi \Rightarrow \psi\sigma$ is valid, and $\sigma\ [\varphi] \mathrel{\multimap\!\!\!\to} \tau\ [\varphi]$.

Here $\sigma\ [\varphi] \mathrel{\multimap\!\!\!\to} \tau\ [\varphi]$ denotes $\sigma(x)\ [\varphi] \mathrel{\multimap\!\!\!\to} \tau(x)\ [\varphi]$ for all variables $x \in \mathcal{D}\mathsf{om}(\sigma)$. The multi-step rewrite relation $\mathrel{\widetilde{\multimap\!\!\!\to}}$ on constrained terms is then defined as $\sim \cdot \mathrel{\multimap\!\!\!\to} \cdot \sim$.

Lemma 11. *If $s\ [\varphi] \mathrel{\multimap\!\!\!\to} t\ [\varphi]$ then $s\delta \mathrel{\multimap\!\!\!\to} t\delta$ for all substitutions $\delta \vDash \varphi$.*

Proof. We proceed by induction on $\mathrel{\multimap\!\!\!\to}$. In case 1 we have $x\ [\varphi] \mathrel{\multimap\!\!\!\to} x\ [\varphi]$, and $x\delta \mathrel{\multimap\!\!\!\to} x\delta$ holds trivially. In case 2 we have $s = f(s_1, \ldots, s_n)$, $t = f(t_1, \ldots, t_n)$ and $s_i\ [\varphi] \mathrel{\multimap\!\!\!\to} t_i\ [\varphi]$ for $1 \leqslant i \leqslant n$. From the induction hypothesis we obtain $s_i\delta \mathrel{\multimap\!\!\!\to} t_i\delta$ for all $1 \leqslant i \leqslant n$, which further implies $s\delta \mathrel{\multimap\!\!\!\to} t\delta$. In case 3 we have $s = \ell\sigma$ and $t = r\sigma$ for some rule $\rho\colon \ell \to r\ [\psi] \in \mathcal{R}_{\mathsf{rc}}$, $\sigma(x) \in \mathcal{V}\mathsf{al} \cup \mathcal{V}\mathsf{ar}(\varphi)$ for all $x \in \mathcal{L}\mathcal{V}\mathsf{ar}(\rho)$, φ is satisfiable, $\varphi \Rightarrow \psi\sigma$ is valid, and $\sigma(x)\ [\varphi] \mathrel{\multimap\!\!\!\to} \tau(x)\ [\varphi]$ for all $x \in \mathcal{V}\mathsf{ar}(\varphi)$. From the induction hypothesis we obtain $\sigma(x)\delta \mathrel{\multimap\!\!\!\to} \tau(x)\delta$ for all $x \in \mathcal{V}\mathsf{ar}(\varphi)$. Moreover, since $\delta \vDash \varphi$ we have $\delta \vDash \psi\sigma$ and thus also $\sigma\delta \vDash \psi$. Therefore $s\delta = \ell\sigma\delta \mathrel{\multimap\!\!\!\to} r\tau\delta = t\delta$ as desired. ☐

Lemma 12. *If $s \approx t\ [\varphi] \mathrel{\widetilde{\multimap\!\!\!\to}}_{\geqslant 1} u \approx v\ [\psi]$ then for all substitutions $\sigma \vDash \varphi$ there exists a $\delta \vDash \psi$ such that $s\sigma \mathrel{\multimap\!\!\!\to} u\delta$ and $t\sigma = v\delta$.*

Proof. By unfolding the definition of $\mathrel{\widetilde{\multimap\!\!\!\to}}$ we obtain $s \approx t\ [\varphi] \sim s' \approx t'\ [\varphi'] \mathrel{\multimap\!\!\!\to}_{\geqslant 1} u' \approx v'\ [\varphi'] \sim u \approx v\ [\psi]$. Let σ be a substitution with $\sigma \vDash \varphi$. From the definition of $\sim$ we obtain a substitution τ such that $\tau \vDash \varphi'$, $s\sigma = s'\tau$ and $t\sigma = t'\tau$. As all contracted redexes in the multi-step $s' \approx t'\ [\varphi']$ are below the position 1, this corresponds to case 2 in Definition 10 with s' and t' being the first and second argument of $\approx$. Hence $s'\ [\varphi'] \mathrel{\multimap\!\!\!\to} u'\ [\varphi']$ and $t' = v'$. We therefore obtain $t'\tau = v'\tau$ and $s'\tau \mathrel{\multimap\!\!\!\to} u'\tau$ from Lemma 11. Now considering the equivalence $u' \approx v'\ [\varphi'] \sim u \approx v\ [\psi]$ together with $\tau \vDash \varphi'$ we obtain a substitution δ such that $\delta \vDash \psi$, $u'\tau = u\delta$ and $v'\tau = v\delta$. Putting this all together we have $s\sigma = s'\tau \mathrel{\multimap\!\!\!\to} u'\tau = u\delta$ and $t\sigma = t'\tau = v'\tau = v\delta$. ☐

Definition 13. A constrained critical pair $s \approx t\ [\varphi]$ is *development closed* if $s \approx t\ [\varphi] \mathrel{\widetilde{\multimap\!\!\!\to}}_{\geqslant 1} u \approx v\ [\psi]$ for some trivial $u \approx v\ [\psi]$. A constrained critical pair is *almost development closed* if it is an inner critical pair and development closed, or it is an overlay and $s \approx t\ [\varphi] \mathrel{\widetilde{\multimap\!\!\!\to}}_{\geqslant 1} \cdot \mathrel{\tilde{\to}}^{*}_{\geqslant 2} u \approx v\ [\psi]$ for some trivial $u \approx v\ [\psi]$. An LCTRS is called (almost) development closed if all its constrained critical pairs are (almost) development closed.

Lemma 14. *If a constrained critical pair $s \approx t\ [\varphi]$ is almost development closed then for all substitutions σ with $\sigma \vDash \varphi$ we have $s\sigma \mathrel{\multimap\!\!\!\to} \cdot {}^{*}\!\!\leftarrow t\sigma$.*

Proof. Let $s \approx t\ [\varphi]$ be an almost development closed constrained critical pair, and $\sigma \vDash \varphi$ some substitution. From Definition 13 we obtain $s \approx t\ [\varphi] \mathrel{\multimap\!\!\!\to}_{\geqslant 1} u' \approx v'\ [\psi'] \to^{*}_{\geqslant 2} u \approx v\ [\psi]$ where $u\tau = v\tau$ for all $\tau \vDash \psi$ for some constrained term $u' \approx v'\ [\psi']$. Looking at the first part of the sequence, $s \approx t\ [\varphi] \mathrel{\multimap\!\!\!\to}_{\geqslant 1} u' \approx v'\ [\psi']$ and $s\sigma \mathrel{\multimap\!\!\!\to} u'\delta$ where $v'\delta = t\sigma$ for some $\delta \vDash \psi'$ by Lemma 12. For the second part of the sequence $u' \approx v'\ [\psi'] \to^{*}_{\geqslant 2} u \approx v\ [\psi]$ we obtain $v'\delta \to^{*} v\gamma$, $u'\delta = u\gamma$ for some $\gamma \vDash \psi$, by repeated application of Lemma 8. Moreover we have $u\gamma = v\gamma$. Hence $s\sigma \mathrel{\multimap\!\!\!\to} u'\delta = u\gamma = v\gamma\ {}^{*}\!\!\leftarrow v'\delta = t\sigma$. ☐

Theorem 15. *If an* LCTRS $\mathcal{R}$ *is almost development closed then so is* $\overline{\mathcal{R}}$.

Proof. Take any critical pair $s \approx t$ from $\overline{\mathcal{R}}$. From Theorem 5 we know that there exists a constrained critical pair $s' \approx t'$ $[\varphi]$ in $\mathcal{R}$ where $s'\sigma = s$ and $t'\sigma = t$ for some $\sigma \vDash \varphi$. Since the constrained critical pair must be almost development closed, Lemma 14 yields $s = s'\sigma \mathrel{\circ\!\!\!\!\!\longrightarrow} \cdot \;{}^*\!\!\leftarrow t'\sigma = t$ if it is an overlay and $s = s'\sigma \mathrel{\circ\!\!\!\!\!\longrightarrow} t'\sigma = t$ otherwise. This proves that $\overline{\mathcal{R}}$ is almost development closed. ☐

Interestingly, the converse does not hold, as seen in the following example.

Example 16. Consider the LCTRS $\mathcal{R}$ with the theory LIA and consisting of the rules:

$$\mathsf{f}(x) \to \mathsf{g}(x) \quad \mathsf{f}(x) \to \mathsf{h}(x)\ [1 \leqslant x \leqslant 2] \quad \mathsf{g}(x) \to \mathsf{h}(2)\ [x = 2z] \quad \mathsf{g}(x) \to \mathsf{h}(1)\ [x = 2z + 1]$$

The TRS $\overline{\mathcal{R}}$ consists of the rules

$$\begin{array}{llll} \mathsf{f}(x) \to \mathsf{g}(x) & \mathsf{f}(1) \to \mathsf{h}(1) & \mathsf{g}(n) \to \mathsf{h}(1) & \text{for all odd } n \in \mathbb{Z} \\ & \mathsf{f}(2) \to \mathsf{h}(2) & \mathsf{g}(n) \to \mathsf{h}(2) & \text{for all even } n \in \mathbb{Z} \end{array}$$

and has two (modulo symmetry) critical pairs $\mathsf{g}(1) \approx \mathsf{h}(1)$ and $\mathsf{g}(2) \approx \mathsf{h}(2)$. Since $\mathsf{g}(1) \mathrel{\circ\!\!\!\!\!\longrightarrow} \mathsf{h}(1)$ and $\mathsf{g}(2) \mathrel{\circ\!\!\!\!\!\longrightarrow} \mathsf{h}(2)$, $\overline{\mathcal{R}}$ is almost development closed. The constrained critical pair $\mathsf{g}(x) \approx \mathsf{h}(x)$ $[1 \leqslant x \leqslant 2]$ is not almost development closed, since it is a normal form with respect to the rewrite relation on constrained terms.

This also makes intuitive sense, since a rewrite step $s \approx t$ $[\varphi] \stackrel{\sim}{\to} u \approx v$ $[\psi]$ implies that the same step can be taken on all instances $s\sigma \approx t\sigma$ where $\sigma \vDash \varphi$. However it may be the case, like in the above example, that different instances of the constrained critical pair require different steps to obtain a closing sequence, which cannot directly be modeled using rewriting on constraint terms.

Since left-linearity of $\overline{\mathcal{R}}$ is preserved and left-linear almost development closed TRSs are confluent [3] the following corollary is obtained via Theorem 15. In fact $\mathcal{R}$ only has to be linear in the variables $x \notin \mathcal{LV}\mathsf{ar}$, since that is sufficient for $\overline{\mathcal{R}}$ to be linear.

Corollary 17. *Left-linear almost development closed* LCTRS*s are confluent.* ☐

4 Conclusion

We presented a left-linearity preserving transformation from LCTRSs into TRSs such that critical pairs in the latter correspond to constrained critical pairs in the former. As a consequence, confluence results for TRSs based on restricted joinability conditions carry directly over to LCTRSs. This drastically simplifies correctness proofs (like the ones in [1, 2]) and makes the formalization of confluence proofs for LCTRSs in a proof assistant a realistic goal.

References

[1] Cynthia Kop and Naoki Nishida. Term rewriting with logical constraints. In *Proc. 9th FRoCoS*, volume 8152 of *LNAI*, pages 343–358, 2013. doi:10.1007/978-3-642-40885-4_24.

[2] Jonas Schöpf and Aart Middeldorp. Confluence criteria for logically constrained rewrite systems. In *Proc. 29th CADE*, LNAI, 2023. To appear.

[3] Vincent van Oostrom. Developing developments. *Theoretical Computer Science*, 175(1):159–181, 1997. doi:10.1016/S0304-3975(96)00173-9.

[4] Sarah Winkler and Aart Middeldorp. Completion for logically constrained rewriting. In *Proc. 3rd FSCD*, volume 108 of *LIPIcs*, pages 30:1–30:18, 2018. doi:10.4230/LIPIcs.FSCD.2018.30.