On Confluence Criteria for Non-terminating Abstract Rewriting Systems

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Abstract

We give a summary of confluence criteria based on the generalized Newman's lemma recently proposed in [6]. The mentioned criteria are applicable to wide classes of not-necessarily-terminating abstract rewriting systems, in particular those that arise from reachability relations on states of discrete-continuous (hybrid) dynamical models.

1 Introduction

Textbooks and monographs related to term rewriting and/or its applications in computer science (e.g. [7, 1, 8]) usually include a chapter on basic theory of abstract rewriting systems (ARS). Besides other material, such a chapter typically includes a definition of the confluence property for ARS and Newman's lemma [10] as a confluence criterion for terminating ARS. Although in the literature one can find many other confluence conditions applicable to (special classes of) abstract rewriting systems (e.g. Hindley-Rosen lemma [3, 12], Rosen's request lemma [12], Huet's strong confluence lemma [4], De Bruijn's lemma [2], Van Oostrom's theorem [13, Theorem 3.7] and its corollaries concerning locally decreasing and decreasing Church-Rosser ARS), arguably, Newman's lemma is the most widely known one. Among its intrinsic qualities are simple formulation, reduction of global analysis to local analysis, and the lack of assumptions about the structure of elements and/or the reduction relation. However, some new potential application areas, like confluence analysis of ARS derived from reachability relations on states of nondeterminisic discrete-continuous (hybrid) models (in particular, those that arise from modeling of cyber-physical systems that combine computational and physical processes), are beyond the scope of Newman's lemma because of the termination assumption.

In [6] we proposed a generalization of Newman's lemma that aimed to preserve its positive qualities, but extend its scope to a wide class of not-necessarily-terminating ARS. The main result of [6] was formalized and machine-checked in Isabelle 2022 proof assistant using HOL logic in [5].

In this extended abstract we give a summary of the results of [6] and additional corollaries from them.

2 Preliminaries

2.1 Standard Notions and Well-Known Facts

An abstract rewriting system (ARS) is a pair $(A, \to)$, where A is a set and $\to \subseteq A \times A$ is a binary relation called reduction[1]. We will use the following notation:

[1] In some works an ARS is allowed to have an indexed family of reduction relations. In this work we restrict attention to rewriting systems with a single reduction relation.

- $\to^+$ is the transitive closure of $\to$
- $\to^*$ is the reflexive transitive closure of $\to$
- $\neg$, $\vee$, $\wedge$, $\Rightarrow$ are logical negation, disjunction, conjunction, and implication respectively.

Also note that in this work we assume that the axiom of choice holds.
An ARS $(A, \to)$ is

- confluent, if $\forall a, b, c \in A \ (a \to^* b \wedge a \to^* c \Rightarrow \exists d \in A \ (b \to^* d \wedge c \to^* d))$
- locally confluent, if $\forall a, b, c \in A \ (a \to b \wedge a \to c \Rightarrow \exists d \in A \ (b \to^* d \wedge c \to^* d))$
- (weakly) normalizing, if $\forall a \in A \ \exists \ b \in A \ (a \to^* b \wedge \neg(\exists c \in A \ b \to c))$
- strongly normalizing, or, alternatively, terminating, or Noetherian, if there exists no infinite reduction sequence $a_1 \to a_2 \to ... \ (a_i \in A)$
- inductive, if for every (infinite) reduction sequence $a_1 \to a_2 \to ...$ (where $a_i \in A$ for all $i \geq 1$) there exists $a \in A$ such that $a_n \to^* a$ for all $n \geq 1$
- acyclic, if $\neg \exists a \in A \ (a \to^+ a)$.

Some well-known facts about these notions are [7, 1, 8, 4]:

1. Any terminating ARS is acyclic and inductive.
2. A terminating ARS is confluent if (and only if) it is locally confluent
(a variant of formulation of Newman's lemma).

2.2 Topology on ARS

For any ARS $(A, \to)$ the relation $\to^*$ can be considered as a preorder (a reflexive and transitive relation). This fact can be used to transfer some preorder-specific notions to ARS and define a topology on them as follows:

- a subset $C \subseteq A$ is a *chain* (in the preordered set $(A, \to^*)$), if
$$\forall a, b \in C \ (a \to^* b \ \vee \ b \to^* a)$$
- an element $a \in A$ is an *upper bound* of a subset $S \subseteq A$ (in the preordered set $(A, \to^*)$), if
$$\forall b \in S \ b \to^* a$$
- $a \in A$ is a *least upper bound* of a subset $S \subseteq A$ (in the preordered set $(A, \to^*)$), if
$$(\forall b \in S \ b \to^* a) \wedge \forall a' \in A \ ((\forall b \in S \ b \to^* a') \Rightarrow a \to^* a').$$
- a subset $S \subseteq A$ is *closed* [6] in the preordered set $(A, \to^*)$, if for each nonempty chain C in $(A, \to^*)$ and for each $a \in A$, if a is a least upper bound of C and $C \subseteq S$, then $a \in S$.
- a subset $S \subseteq A$ is *open* [6] in the preordered set $(A, \to^*)$, if $A \backslash S$ is closed in $(A, \to^*)$
- for a given $B \subseteq A$, a subset $S \subseteq B$ is *relatively open* [6] in B (in the preordered set $(A, \to^*)$), if S is open in the preordered set $(B, \to^* \cap (B \times B))$.

For any ARS $(A, \to)$ we will denote:

$$\mathcal{T}(A, \to) = \{S \subseteq A \mid S \text{ is open in the preordered set } (A, \to^*)\}.$$

2.3 Induction Principles for ARS and Related Facts

It is well-known (e.g. [8, paragraph 1.3.15]) that an ARS is terminating if and only if it has sound Noetherian induction principle: for every unary predicate $P : A \to \{True, False\}$, if

$$\forall a \in A\ ((\forall b \in A\ (a \to^+ b \Rightarrow P(b))) \Rightarrow P(a))$$

holds, then $\forall a \in A\ P(a)$.

We will say that an ARS $(A, \to)$

- has *sound open induction principle* [11, 6], if for every unary predicate $P : A \to \{True, False\}$ such that $\{a \in A \mid P(a)\} \in \mathcal{T}(A, \to)$ (i.e. the truth domain of P is open), if

$$\forall a \in A\ ((\forall b \in A\ (a \to^+ b \Rightarrow P(b))) \Rightarrow P(a))$$

  holds, then $\forall a \in A\ P(a)$
- is *strictly inductive* [9, 6], if $(A, \to^*)$ is a strictly inductive preordered set, i.e. every nonempty chain in $(A, \to^*)$ has a least upper bound in $(A, \to^*)$
- is *openly normalizing*, if $(A, \to)$ is (weakly) normalizing and for each $a, a' \in A$ such that a' is a normal form of a (i.e. $a \to^* a' \wedge \neg(\exists a'' \in A\ a' \to a'')$), the set

$$\{b \in A \mid a \to^* b \text{ and } a' \text{ is the only normal form of } b\ \}$$

  is relatively open in $\{b \in A \mid a \to^* b\}$ in the preordered set $(A, \to^*)$.

Proposition 1.

(1) An ARS has sound open induction principle if and only if it is acyclic and strictly inductive [6, Proposition 14] (a related fact is [11, Theorem 3.3]).

(2) Any strictly inductive ARS is inductive (but is not necessarily terminating).

(3) Any terminating ARS is acyclic, strictly inductive, and openly normalizing.

Example 1. *If $[0, 1]$ denotes the real unit interval and $\to$ denotes the standard strict order on real numbers restricted to $[0, 1]$ (i.e. $< \cap([0, 1] \times [0, 1])$), then the ARS $([0, 1], \to)$ has sound open induction principle, is strictly inductive, acyclic, openly normalizing, but is not terminating (e.g. $0.9 \to 0.99 \to 0.999 \to ...$ is an infinite reduction sequence).*

Example 2. *The ARS $(\{1\}, \to)$, where $\to = \{(1, 1)\}$, is strictly inductive, but is not acyclic. In fact, every ARS with finite set of elements is strictly inductive [6, Proposition 16].*

3 Confluence Conditions for Strictly Inductive ARS

Below we give a necessary and sufficient condition for confluence of a strictly inductive ARS (Theorem 1) and two more specialized confluence conditions (Lemma 1 and 2).

Firstly, let consider specialized conditions (they have simpler formulation).

Lemma 1. *An acyclic and strictly inductive ARS is confluent if and only if it is openly normalizing and locally confluent (a consequence of [6, Theorem 28]).*

From Proposition 1(1) and Lemma 1(1) it follows that

Lemma 2. *Let $(A, \to)$ be an ARS with sound open induction principle. Then $(A, \to)$ is confluent if and only if it is openly normalizing and locally confluent.*

Remark 3.1. *The ordinary Newman's lemma implies that if $(A, \to)$ is an ARS with sound Noetherian induction principle, then $(A, \to)$ is confluent if and only if $(A, \to)$ is locally confluent. Thus Lemma 2 can be considered as an extension of Newman's lemma that replaces Noetherian induction with open induction [11]. The open normalization condition is not needed in the case of terminating ARS, because every terminating ARS is openly normalizing.*

Note that the local confluence condition is not so useful in confluence criteria when a reduction relation $\to$ is transitive (since in this case the notions of confluence and local confluence become almost trivially equivalent). To obtain useful confluence conditions in the case of a transitive reduction relation, and also to cover cases where an ARS is not acyclic, we introduce the following notions.

Let $(A, \to)$ be an ARS and $a, a' \in A$. Then

- a is *quasi-irreducible* [6], if $\forall b \in A\ (a \to^* b \Rightarrow b \to^* a)$
- a' is a *quasi-normal form* (QNF [6]) of a, if $a \to^* a'$ and a' is quasi-irreducible
- a, a' are *QNF-equivalent* [6], if $\{b \in A \mid b \text{ is a QNF of } a\} = \{b \in A \mid b \text{ is a QNF of } a'\}$.

An ARS $(A, \to)$ is

- *quasi-normalizing* [6], if each $a \in A$ has a quasi-normal form
- *openly quasi-normalizing* [6], if $(A, \to)$ is quasi-normalizing and for all $a, a' \in A$, if a' is a quasi-normal form of a, the set

$$\{b \in A \mid a \to^* b\ \wedge\ b \text{ and } a' \text{ are QNF-equivalent }\}$$

  is relatively open in $\{b \in A \mid a \to^* b\}$ in the preordered set $(A, \to^*)$.

- *quasi-locally confluent* [6], if for each $a \in A$ there exists $S \subseteq \{a' \in A \mid a \to^+ a'\}$ such that the following two conditions hold (called in [6] the two-consistency and coinitiality condition respectively):

$$\forall b, c \in S\ \exists d \in A\ (b \to^* d \wedge c \to^* d),$$

$$\forall a' \in A\ (a \to^+ a' \Rightarrow (a' \to^* a) \vee (\exists b \in S\ b \to^* a' \wedge \neg(b \to^* a))).$$

Lemma 3. *Let $(A, \to)$ be an acyclic ARS. Then*

(1) $(A, \to)$ is quasi-normalizing if and only if $(A, \to)$ is (weakly) normalizing

(2) $(A, \to)$ is openly quasi-normalizing if and only if $(A, \to)$ is openly normalizing

(3) if $(A, \to)$ is locally confluent, then $(A, \to)$ is quasi-locally confluent [6, Proposition 27].

Theorem 1 ([6], Theorem 28)**.** *Let $(A, \to)$ be a strictly inductive ARS. Then $(A, \to)$ is confluent if and only if $(A, \to)$ is openly quasi-normalizing and quasi-locally confluent.*

Note that one cannot simply omit the strict inductivity assumption, or the open quasi-normalization, or quasi-local confluence condition from the statement of Theorem 1:

Proposition 2. *(1) There exists a strictly inductive and quasi-locally confluent ARS that is not confluent. An example of such an ARS is* $(A, \to)$*, where* $A = \{(x,t) \in \mathbb{R} \times \mathbb{R} \mid t \leq 0\}$ *and* $(x,t) \to (x',t')$ *if and only if* $t < t' \wedge x' - x \leq t' - t \wedge x \leq x'$ *(also see [6, Example 31]).*

(2) There exists a strictly inductive and openly quasi-normalizing ARS that is not confluent. An example of such an ARS is $(\{0,1,2\}, \to)$*, where* $\to = \{(0,1),(0,2)\}$*.*

(3) There exists an inductive, openly quasi-normalizing, and quasi-locally confluent ARS that is not confluent. An example of such an ARS is $(\mathbb{N}_0 \times \{0,1\}, \to)$*, where* $\mathbb{N}_0 = \{0,1,2,...\}$ *is the set of non-negative integers and* $(i,j) \to (i',j')$ *if and only if* $(1 \leq i < i' \wedge j = j') \vee (i \geq 1 \wedge i' = 0)$*.*

Note that example ARS explicitly mentioned in Proposition 2 are not confluent. An example of a strictly inductive, confluent ARS with more than one irreducible element that is in the scope of the confluence criterion given in Theorem 1 can be found in [6, Example 32]:

$([0,1] \times [0,1], \to)$, where $[0,1]$ is the real unit interval and $(a,b) \to (a',b')$ if and only if $(a < a' \vee (a = a' \wedge b' < b \wedge (a < 1 \vee b < 1))) \wedge (a - b \leq a' - b') \wedge (a = b \Rightarrow a' = b') \wedge (a \leq b \Rightarrow a' \leq b')$. Note that here $(1,0)$ and $(1,1)$ are irreducible.

Other similar examples can be constructed using reachability relations on states of nondeterministic discrete-continuous dynamical models, e.g. see a nondeterministic model of a bouncing ball in [6, Figures 4–6].

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