Formalizing Confluence and Commutation Criteria Using Proof Terms*

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Abstract

We present recent advancements concerning formalizations of state-of-the-art confluence and commutation criteria. In particular we describe formalizations of several extensions of van Oostrom's development-closedness criterion in the proof assistant Isabelle/HOL. A key component for the formalized proofs is the concept of proof terms representing multi-steps.

1 Introduction

Recently we presented the first formalized proof of van Oostrom's development-closedness criterion [4]. Since then, we were able to extend this result in several ways, which we describe here. In Section 2 we first give some basic definitions and recap proof terms representing multi-steps in term rewriting—a concept which proved to be very valuable for formalizing critical pair criteria based on multi-steps (also known as development steps). In Section 3 we present our formalization of *almost* development closed critical pairs for commutation of two term rewrite systems.[1] This is an extension of the result described in [4] in two ways: First, it weakens the joinability requirement for critical pairs which are overlays,[2] second, it uses the critical pairs between two left-linear TRSs to determine whether they commute. In Section 4 we describe our most recent extension, namely a formalized proof of the results in [3].

2 Preliminaries

We assume familiarity with the basics of term rewriting, as can be found in [1], and only recap some important definitions here. A relation $\to$ is confluent if

$$ {}^{*}\!\leftarrow \cdot \to^{*} \;\subseteq\; \to^{*} \cdot {}^{*}\!\leftarrow $$

Two relations $\to_1$ and $\to_2$ (locally) commute if

$$ {}_{1}^{*}\!\leftarrow \cdot \to_2^{*} \;\subseteq\; \to_2^{*} \cdot {}_{1}^{*}\!\leftarrow \qquad (\, {}_{1}\!\leftarrow \cdot \to_2 \;\subseteq\; \to_2^{*} \cdot {}_{1}^{*}\!\leftarrow \,) $$

We say that $\to_1$ and $\to_2$ *strongly commute* if

$$ {}_{1}\!\leftarrow \cdot \to_2 \;\subseteq\; \to_2^{=} \cdot {}_{1}^{*}\!\leftarrow $$

Strong commutation of $\to_1$ and $\to_2$ implies commutation of $\to_1$ and $\to_2$. If $\to$ commutes with itself then it is confluent. The *multi-step* relation $\twoheadrightarrow\!\!\!\!\circ_{\mathcal{R}}$ is inductively defined on terms as follows:

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[1] A more detailed description of this formalization effort will appear in [5].

[2] An idea first described by Toyama for parallel closed critical pairs in [6] and adapted by van Oostrom for development closed critical pairs [7].

- $x \mathrel{\circ\!\!\!\!\rightarrow}_{\mathcal{R}} x$ for all variables x,
- $f(s_1, \ldots, s_n) \mathrel{\circ\!\!\!\!\rightarrow}_{\mathcal{R}} f(t_1, \ldots, t_n)$ if $s_i \mathrel{\circ\!\!\!\!\rightarrow}_{\mathcal{R}} t_i$ for all $1 \leqslant i \leqslant n$, and
- $\ell\sigma \mathrel{\circ\!\!\!\!\rightarrow}_{\mathcal{R}} r\tau$ if $\ell \to r \in \mathcal{R}$ and $\sigma(x) \mathrel{\circ\!\!\!\!\rightarrow}_{\mathcal{R}} \tau(x)$ for all $x \in \mathcal{V}\mathsf{ar}(\ell)$.

A critical *overlap* $(\ell_1 \to r_1, p, \ell_2 \to r_2)_\sigma$ of two TRSs $\mathcal{R}$ and $\mathcal{S}$ consists of variants $\ell_1 \to r_1$ and $\ell_2 \to r_2$ of rewrite rules in $\mathcal{R}$ and $\mathcal{S}$ without common variables, a position $p \in \mathcal{P}\mathsf{os}_{\mathcal{F}}(\ell_2)$, and a most general unifier σ of ℓ_1 and $\ell_2|_p$. From a critical overlap $(\ell_1 \to r_1, p, \ell_2 \to r_2)_\sigma$ we obtain a critical peak $\ell_2\sigma[r_1\sigma]_p \;{}_{\mathcal{R}_1}\!\leftarrow \ell_2\sigma[\ell_1\sigma]_p = \ell_2\sigma \to_{\mathcal{R}_2} r_2\sigma$ and the corresponding *critical pair* $\ell_2\sigma[r_1\sigma]_p \;{}_{\mathcal{R}}\!\leftarrow\!\rtimes\!\!\rightarrow_{\mathcal{S}} r_2\sigma$. When $p = \epsilon$ we call $r_1\sigma \;{}_{\mathcal{R}}\!\leftarrow\!\rtimes\!\!\rightarrow_{\mathcal{S}} r_2\sigma$ an *overlay*. TRSs $\mathcal{R}$ and $\mathcal{S}$ are *development closed* if $s \mathrel{\circ\!\!\!\!\rightarrow}_{\mathcal{S}} t$ for all critical pairs $s \;{}_{\mathcal{R}}\!\leftarrow\!\rtimes\!\!\rightarrow_{\mathcal{S}} t$ and $s \mathrel{\circ\!\!\!\!\rightarrow}_{\mathcal{R}} t$ for all critical pairs $s \;{}_{\mathcal{S}}\!\leftarrow\!\rtimes\!\!\rightarrow_{\mathcal{R}} t$.

Proof terms are built from function symbols, variables, and rule symbols. We use Greek letters for rule symbols. If α is a rule symbol then $\mathsf{lhs}(\alpha)$ ($\mathsf{rhs}(\alpha)$) denotes the left-hand (right-hand) side of the rewrite rule denoted by α. Furthermore $\mathsf{var}(\alpha)$ denotes the list $(x_1, \ldots, x_n)$ of variables appearing in α in some fixed order. The length of this list is the arity of α. The list $\mathsf{vpos}(\alpha) = (p_1, \ldots, p_n)$ denotes the corresponding variable positions in $\mathsf{lhs}(\alpha)$ such that $\mathsf{lhs}(\alpha)|_{p_i} = x_i$. Given a rule symbol α with $\mathsf{var}(\alpha) = (x_1, \ldots, x_n)$ and terms $t_1, \ldots, t_n$, we write $\langle t_1, \ldots, t_n \rangle_\alpha$ for the substitution $\{\, x_i \mapsto t_i \mid 1 \leqslant i \leqslant n \,\}$. Given a proof term A, its source $\mathsf{src}(A)$ and target $\mathsf{tgt}(A)$ are computed by the following equations:

$$
\begin{aligned}
\mathsf{src}(x) &= \mathsf{tgt}(x) = x \\
\mathsf{src}(f(A_1, \ldots, A_n)) &= f(\mathsf{src}(A_1), \ldots, \mathsf{src}(A_n)) \\
\mathsf{src}(\alpha(A_1, \ldots, A_n)) &= \mathsf{lhs}(\alpha)\langle \mathsf{src}(A_1), \ldots, \mathsf{src}(A_n) \rangle_\alpha \\
\mathsf{tgt}(f(A_1, \ldots, A_n)) &= f(\mathsf{tgt}(A_1), \ldots, \mathsf{tgt}(A_n)) \\
\mathsf{tgt}(\alpha(A_1, \ldots, A_n)) &= \mathsf{rhs}(\alpha)\langle \mathsf{tgt}(A_1), \ldots, \mathsf{tgt}(A_n) \rangle_\alpha
\end{aligned}
$$

Proof terms A and B are said to be *co-initial* if they have the same source. The proof term A over TRS $\mathcal{R}$ is a witness of the multi-step $\mathsf{src}(A) \mathrel{\circ\!\!\!\!\rightarrow}_{\mathcal{R}} \mathsf{tgt}(A)$. For every multi-step there exists a proof term witnessing it. For co-initial proof terms A and B the *residual* $A \,/\, B$ is a proof term witnessing the remainder of A after contracting the redexes of B. This can be formally defined as a partial operation with several useful properties which are exploited in the proofs below. The amount of overlap between two co-initial proof terms A and B is denoted by $\blacktriangle(A, B)$ and is measured as the number of function symbols in $\mathsf{src}(A) = \mathsf{src}(B)$ that are part of a redex in both A and B. If a redex in A at position $p \in \mathsf{src}(A)$ overlaps with a redex in B at position $q \in \mathsf{src}(B)$ then we call the pair (p, q) an *overlap* between A and B. The formal definitions, as well as useful lemmata about the aforementioned operations, can be found in [4].

The formalized results of the next sections are integrated into the library IsaFoR.[3] The contributions described in this paper are located in the file `Development_Closed.thy`.

3 Almost Development Closed Critical Pairs

In [4] we described the formalized proof of van Oostrom's development-closedness criterion [7].

Theorem 1. *Left-linear development closed* TRS*s are confluent.*

[3] http://cl-informatik.uibk.ac.at/isafor

When introducing his criterion in [7] van Oostrom already gave an extension where the joining condition on overlays is weakened. This extension is modeled after Toyama's almost parallel closed critical pairs [6]. Just like Toyama's result, *almost development closed* critical pairs can be lifted to the commutation setting, resulting in the following theorem.

Theorem 2. *Let $\mathcal{R}$ and $\mathcal{S}$ be two left-linear* TRS*s. If $s \mathrel{\multimap}_{\mathcal{S}} \cdot {}_{\mathcal{R}}^{*}\!\leftarrow t$ for all critical pairs $s\ {}_{\mathcal{R}}\!\leftarrow\!\rtimes\!\rightarrow_{\mathcal{S}} t$, and $s \mathrel{\multimap}_{\mathcal{R}} t$ for all critical pairs $s\ {}_{\mathcal{S}}\!\leftarrow\!\rtimes\!\rightarrow_{\mathcal{R}} t$ which are not overlays, then $\mathcal{R}$ and $\mathcal{S}$ commute.*

In [7] it is suggested to adapt the measure (▲) used in the proof of Theorem 1 in order to obtain a proof of the extended result. This turned out to be problematic as we describe in [5]. For the formalized proof we instead add another case distinction in the step case of the proof used for Theorem 1.

Formalized proof. We show strong commutation of $\multimap_{\mathcal{R}}$ and $\multimap_{\mathcal{S}}$, which implies commutation of $\multimap_{\mathcal{R}}$ and $\multimap_{\mathcal{S}}$ and hence commutation of $\rightarrow_{\mathcal{R}}$ and $\rightarrow_{\mathcal{S}}$. Assume $t\ {}_{\mathcal{R}}\!\multimapinv\ s \multimap_{\mathcal{S}} u$ and let A be a proof term representing $s \multimap_{\mathcal{R}} t$ and let B be a proof term representing $s \multimap_{\mathcal{S}} u$. We show $t \multimap_{\mathcal{S}} v\ {}_{\mathcal{R}}^{*}\!\leftarrow u$ for some term v by well-founded induction on $▲(A, B)$.

- In the base case $▲(A, B) = 0$ which implies that $A \,/\, B$ is a proof term over $\mathcal{R}$ and $B \,/\, A$ a proof term over $\mathcal{S}$ such that $\mathsf{tgt}(A \,/\, B) = \mathsf{tgt}(B \,/\, A)$.
- In the induction step we assume $▲(A, B) > 0$. To apply the induction hypothesis we need to obtain proof terms A' over $\mathcal{R}$ and B' over $\mathcal{S}$ such that $▲(A', B') < ▲(A, B)$. We select an innermost overlap (p, q) and let α and β be the corresponding rule symbols in A and B. Moreover, let $\mathsf{vpos}(\alpha) = (p_1, \ldots, p_n), \mathsf{var}(\alpha) = (x_1, \ldots, x_n), \mathsf{vpos}(\beta) = (q_1, \ldots, q_m)$ and $\mathsf{var}(\beta) = (y_1, \ldots, y_m)$, where we assume $\{x_1, \ldots, x_n\} \cap \{y_1, \ldots, y_m\} = \varnothing$ without loss of generality. We define proof terms $\Delta_1 = s[\alpha(s|_{pp_1}, \ldots, s|_{pp_n})]_p$ and $\Delta_2 = s[\beta(s|_{qq_1}, \ldots, s|_{qq_m})]_q$. Then Δ_1 represents a single step $s \rightarrow t'$ and the residual $A \,/\, \Delta_1$ witnesses $t' \multimap t$ for some term t'. Likewise Δ_2 represents a step $s \rightarrow u'$ and $B \,/\, \Delta_2$ witnesses $u' \multimap u$ for some term u'. We distinguish three cases.

  1. For $q < p$ and $q' = p \backslash q$ we define the substitution

  $$\begin{aligned}\tau \;=\; &\{x_i \mapsto \mathsf{lhs}(\beta)|_{q'p_i} \mid 1 \leqslant i \leqslant n \text{ and } q'p_i \in \mathcal{P}\mathsf{os}(\mathsf{lhs}(\beta))\} \\ \cup\; &\{y_j \mapsto \mathsf{lhs}(\alpha)|_{q_j \backslash q} \mid 1 \leqslant j \leqslant m \text{ and } q_j \backslash q \in \mathcal{P}\mathsf{os}_{\mathcal{F}}(\mathsf{lhs}(\alpha))\}\end{aligned}$$

  which yields the critical peak [4, Lemma 7.2]

  $$\mathsf{lhs}(\beta)[\mathsf{rhs}(\alpha)\tau]_{q'}\ {}_{\mathcal{R}}\!\leftarrow\ \mathsf{lhs}(\beta)[\mathsf{lhs}(\alpha)\tau]_{q'} \;=\; \mathsf{lhs}(\beta)\tau \;\rightarrow_{\mathcal{S}}\; \mathsf{rhs}(\beta)\tau$$

  We define the position $q_\beta \in \mathcal{P}\mathsf{os}(B)$ such that $B = B[\beta(B_1, \ldots, B_m)]_{q_\beta}$ and $\mathsf{src}(B)[\ ]_q = \mathsf{src}(B[\ ]_{q_\beta})$. By the almost development closedness assumption there exists a multi-step $\mathsf{lhs}(\beta)[\mathsf{rhs}(\alpha)\tau]_{q'} \multimap_{\mathcal{S}} \mathsf{rhs}(\beta)\tau$. Let D' be a proof term representing this multi-step. We define the substitution

  $$\rho \;=\; \{y_j \mapsto B_j \mid 1 \leqslant j \leqslant m\} \,\cup\, \{x_i \mapsto \mathsf{lhs}(\beta)\langle B_1, \ldots, B_m\rangle_\beta|_{q'p_i} \mid 1 \leqslant i \leqslant n\}$$

  and show that the proof term $B' = B[D'\rho]_{q_\beta}$ witnesses a multi-step $t' \multimap_{\mathcal{S}} u$. Finally, we show $▲(A', B') < ▲(A, B)$ for $A' = A \,/\, \Delta_1$ [4, Lemma 7.8].

  2. If $p < q$ a symmetric construction yields a proof term A' witnessing $u' \multimap_{\mathcal{R}} t$ such that $▲(A', B') < ▲(A, B)$ for $B' = B \,/\, \Delta_2$.

3. If $p = q$ we can apply the same construction as in the first case, but the almost development closedness assumption yields a term v', a proof term D' witnessing $\mathsf{rhs}(\alpha)\tau \multimap_{\mathcal{S}} v'$, and a rewrite sequence $\mathsf{rhs}(\beta)\tau \to^*_{\mathcal{R}} v'$. Then $B' = B[D'\rho]_{q_\beta}$ witnesses a multi-step $t' \multimap_{\mathcal{S}} w$ for some term w. Like before, $\blacktriangle(A', B') < \blacktriangle(A, B)$ for $A' = A \mathbin{/} \Delta_1$. Moreover, $u \to^*_{\mathcal{R}} w$ since $u = \mathsf{tgt}(B[\mathsf{rhs}(\beta)\tau\rho]_{q_\beta})$ and $w = \mathsf{tgt}(B[D'\rho]_{q_\beta}) = \mathsf{tgt}(B[v'\rho]_{q_\beta})$.

The previous items allow us to apply the induction hypothesis to obtain a term v such that $t \multimap_{\mathcal{S}} v \;{}_{\mathcal{R}}^{*}\!\leftarrow u$, which completes the proof. □

4 Commutation via Relative Termination

We formalized another extension of Theorem 1 for commutation due to Hirokawa and Middeldorp [3]. It is based on local commutation together with relative termination of the *critical peak steps* between two TRSs $\mathcal{R}$ and $\mathcal{S}$. In this section, when we speak of a critical peak $t \;{}_{\mathcal{R}}\!\xleftarrow{p} s \xrightarrow{q}_{\mathcal{S}} u$ we either mean the critical peak $t \;{}_{\mathcal{R}}\!\xleftarrow{p} s \xrightarrow{\epsilon}_{\mathcal{S}} u$ or the critical peak $u \;{}_{\mathcal{S}}\!\xleftarrow{q} s \xrightarrow{\epsilon}_{\mathcal{R}} t$.

Definition 3. Let $t \;{}_{\mathcal{R}}\!\xleftarrow{p} s \xrightarrow{q}_{\mathcal{S}} u$ be a critical peak. It is $(\mathcal{R}, \mathcal{S})$-*closed* if $u \multimap_{\mathcal{R}} t$ whenever $p = \epsilon$ and $t \multimap_{\mathcal{S}} u$ whenever $q = \epsilon$. The set of all non-closed critical peak steps of $\mathcal{S}$ for $\mathcal{R}$ is defined as $\mathsf{CPS}_{\mathcal{R}}(\mathcal{S}) = \{s \to u \mid t \;{}_{\mathcal{R}}\!\xleftarrow{p} s \xrightarrow{q}_{\mathcal{S}} u$ is a critical peak which is not $(\mathcal{R}, \mathcal{S})$-closed$\}$.

Theorem 4 ([3, Theorem 4.3])**.** *Left-linear locally commuting* TRSs *$\mathcal{R}$ and $\mathcal{S}$ commute if $\mathsf{CPS}_{\mathcal{S}}(\mathcal{R}) \cup \mathsf{CPS}_{\mathcal{R}}(\mathcal{S})$ is relatively terminating with respect to $\mathcal{R} \cup \mathcal{S}$.*

Recall that a TRS $\mathcal{R}$ is relatively terminating with respect to a TRS $\mathcal{S}$ if $\mathcal{R}/\mathcal{S}$ is terminating. Here $\mathcal{R}/\mathcal{S}$ denotes the relation $\to^*_{\mathcal{S}} \cdot \to_{\mathcal{R}} \cdot \to^*_{\mathcal{S}}$. The following key lemma is needed in addition to results from the previous sections in order to prove Theorem 4. The formalized proof closely follows the paper proof in [3] and is very similar to the proof of Theorem 2.

Lemma 5. *Let $\mathcal{R}$ and $\mathcal{S}$ be left-linear* TRSs*. If $t \;{}_{\mathcal{R}}\!\leftarrow\!\!\!\circ\; s \multimap_{\mathcal{S}} u$ then*

(a) $t \multimap_{\mathcal{S}} \cdot \;{}_{\mathcal{R}}\!\leftarrow\!\!\!\circ\; u$, or

(b) $t \;{}_{\mathcal{R}}\!\leftarrow\!\!\!\circ \cdot {}_{\mathsf{CPS}_{\mathcal{S}}(\mathcal{R})}\!\leftarrow s' \to_{\mathsf{CPS}_{\mathcal{R}}(\mathcal{S})} \cdot \multimap_{\mathcal{S}} u$ and $s \to^*_{\mathcal{R}\cup\mathcal{S}} s'$ for some s'.

Formalized proof. Assume $t \;{}_{\mathcal{R}}\!\leftarrow\!\!\!\circ\; s \multimap_{\mathcal{S}} u$ and let A be a proof term representing $s \multimap_{\mathcal{R}} t$ and let B be a proof term representing $s \multimap_{\mathcal{S}} u$. Like in the proof of Theorem 2 we proceed by induction on $\blacktriangle(A, B)$.

- If $\blacktriangle(A, B) = 0$ case (a) holds (by taking the residuals $B \mathbin{/} A$ and $A \mathbin{/} B$).
- If $\blacktriangle(A, B) > 0$. We select an innermost overlap (p, q), assume without loss of generality that $q \leqslant p$ and let $q' = p \backslash q$. Then we define Δ_1, Δ_2, and τ as in the proof of Theorem 2. Hence, we obtain a critical peak $t' \;{}_{\mathcal{R}}\!\leftarrow\!\!\!\circ\; s' \multimap_{\mathcal{S}} u'$ where $t' = \mathsf{lhs}(\beta)[\mathsf{rhs}(\alpha)\tau]_{q'}$, $s' = \mathsf{lhs}(\beta)[\mathsf{lhs}(\alpha)\tau]_{q'} = \mathsf{lhs}(\beta)\tau$, and $u' = \mathsf{rhs}(\beta)\tau$. We distinguish two cases.

  1. If the peak $t' \;{}_{\mathcal{R}}\!\leftarrow\!\!\!\circ\; s' \multimap_{\mathcal{S}} u'$ is not $(\mathcal{R}, \mathcal{S})$-closed then $s' \to t' \in \mathsf{CPS}_S(R)$ and $s' \to u' \in \mathsf{CPS}_R(S)$. We define the substitution

$$\sigma \;=\; \{x_i \mapsto s|_{pp_i} \mid 1 \leqslant i \leqslant n\} \,\cup\, \{y_j \mapsto s|_{qq_j} \mid 1 \leqslant j \leqslant m\}$$

and show $\mathsf{tgt}(\Delta_1) = s[t'\sigma]_{q'}$ and $\mathsf{tgt}(\Delta_2) = s[u'\sigma]_{q'}$. Then $A \,/\, \Delta_1$ witnesses a multi-step $s[t'\sigma]_{q'} \mathrel{\circ\!\!\!\to}_{\mathcal{R}} t$ and $B \,/\, \Delta_2$ witnesses a multi-step $s[u'\sigma]_{q'} \mathrel{\circ\!\!\!\to}_{\mathcal{S}} u$. Hence

$$t \;{}_{\mathcal{R}}\!\mathrel{\leftarrow\!\!\!\circ} \cdot \;{}_{\mathsf{CPS}_{\mathcal{S}}(\mathcal{R})}\!\leftarrow\; s \;\to_{\mathsf{CPS}_{\mathcal{R}}(\mathcal{S})} \cdot \mathrel{\circ\!\!\!\to}_{\mathcal{S}} u$$

i.e., case (b) holds.

2. If the peak $t' \;{}_{\mathcal{R}}\!\mathrel{\leftarrow\!\!\!\circ}\; s' \mathrel{\circ\!\!\!\to}_{\mathcal{S}} u'$ is $(\mathcal{R},\mathcal{S})$-closed then there exists a proof term D' witnessing $t' \mathrel{\circ\!\!\!\to}_{\mathcal{S}} u'$. Hence, we can apply the same constructions as in the proof of Theorem 2 to obtain a proof term $B[D'\rho]_{q_\beta}$ such that $\mathsf{src}(B[D'\rho]_{q_\beta}) = \mathsf{src}(A \,/\, \Delta_1) = \mathsf{tgt}(\Delta_1)$, $\mathsf{tgt}(B[D'\rho]_{q_\beta}) = \mathsf{tgt}(B) = u$, and $\blacktriangle(A \,/\, \Delta_1, B[D'\rho]_{q_\beta}) < \blacktriangle(A, B)$. The induction hypothesis now yields the following two cases:
   (a) $t \mathrel{\circ\!\!\!\to}_{\mathcal{S}} \cdot \;{}_{\mathcal{R}}\!\mathrel{\leftarrow\!\!\!\circ} u$, or
   (b) $t \;{}_{\mathcal{R}}\!\mathrel{\leftarrow\!\!\!\circ} \cdot \;{}_{\mathsf{CPS}_{\mathcal{S}}(\mathcal{R})}\!\leftarrow s'' \to_{\mathsf{CPS}_{\mathcal{R}}(\mathcal{S})} \cdot \mathrel{\circ\!\!\!\to}_{\mathcal{S}} u$ and $\mathsf{tgt}(\Delta_1) \to^*_{\mathcal{R}\cup\mathcal{S}} s''$ for some s''.

   In the first case we are immediately done. In the second case it remains to show $s \to^*_{\mathcal{R}\cup\mathcal{S}} s''$. This is straightforward since Δ_1 witnesses a rewrite step $s \to_{\mathcal{R}} \mathsf{tgt}(\Delta_1)$ and we have the rewrite sequence $\mathsf{tgt}(\Delta_1) \to^*_{\mathcal{R}\cup\mathcal{S}} s''$. □

Formalized proof of Theorem 4. Assume $\mathcal{R}$ and $\mathcal{S}$ are left-linear and locally commuting. Moreover, assume $(\mathsf{CPS}_{\mathcal{S}}(\mathcal{R}) \cup \mathsf{CPS}_{\mathcal{R}}(\mathcal{S}))/(\mathcal{R}\cup\mathcal{S})$ is terminating. We show that the TRSs $\mathcal{R}$ and $\mathcal{S}$ are decreasing with respect to the conversion version of van Oostrom's decreasing diagrams technique [8, Theorem 3]. In Isabelle we use Felgenhauer's formalization [2]. We first define the labeled multi-step relations $t \mathrel{\circ\!\!\!\to}_{\mathcal{R},s} u$ if and only if $s \to^*_{\mathcal{R}\cup\mathcal{S}} t \mathrel{\circ\!\!\!\to}_{\mathcal{R}} u$ and, similarly, $t \mathrel{\circ\!\!\!\to}_{\mathcal{S},s} u$ if and only if $s \to^*_{\mathcal{R}\cup\mathcal{S}} t \mathrel{\circ\!\!\!\to}_{\mathcal{S}} u$. The relation $>$ is defined as

$$\to^+_{(\mathsf{CPS}_{\mathcal{S}}(\mathcal{R}) \cup \mathsf{CPS}_{\mathcal{R}}(\mathcal{S}))/(\mathcal{R}\cup\mathcal{S})}$$

We show that $(\{\mathrel{\circ\!\!\!\to}_{\mathcal{R},s}\}_{s\in\mathcal{T}}, \{\mathrel{\circ\!\!\!\to}_{\mathcal{S},s}\}_{s\in\mathcal{T}})$ is decreasing with respect to $>$. Note that $>$ is well-founded by assumption and transitive by definition. It remains to show that every local peak $t \;{}_{\mathcal{R},s_1}\!\mathrel{\leftarrow\!\!\!\circ}\; s \mathrel{\circ\!\!\!\to}_{\mathcal{R},s_2} u$ can be completed into a decreasing diagram with conversions as illustrated in Figure 1. To this end, we apply Lemma 5 to the peak $t \;{}_{\mathcal{R},s_1}\!\mathrel{\leftarrow\!\!\!\circ}\; s \mathrel{\circ\!\!\!\to}_{\mathcal{R},s_2} u$. We need to consider the two cases of Lemma 5:

(a) First assume there exists a term v such that $t \mathrel{\circ\!\!\!\to}_{\mathcal{S}} v \;{}_{\mathcal{R}}\!\mathrel{\leftarrow\!\!\!\circ} u$. Since $s_2 \to^*_{\mathcal{R}\cup\mathcal{S}} t \mathrel{\circ\!\!\!\to}_{\mathcal{S}} v$ we have $t \mathrel{\circ\!\!\!\to}_{\mathcal{S},s_2} v$. Similarly we have $u \mathrel{\circ\!\!\!\to}_{\mathcal{R},s_1} v$. Hence, by taking empty sequences for all conversions in Figure 1, we can complete the peak.

(b) Next assume there exist terms s', t', u' such that $t \;{}_{\mathcal{R}}\!\mathrel{\leftarrow\!\!\!\circ}\; t' \;{}_{\mathsf{CPS}_{\mathcal{S}}(\mathcal{R})}\!\leftarrow s' \to_{\mathsf{CPS}_{\mathcal{R}}(\mathcal{S})} u' \mathrel{\circ\!\!\!\to}_{\mathcal{S}} u$ and $s \to^*_{\mathcal{R}\cup\mathcal{S}} s'$. Since $\mathcal{R}$ and $\mathcal{S}$ are locally commuting, we obtain a term v such that $t' \to^*_{\mathcal{S}} v \;{}^{*}_{\mathcal{R}}\!\leftarrow u'$. We show that the peak can be completed by a conversion

$$t \xleftrightarrow[\curlyvee s_1 s_2]{*} t' \xleftrightarrow[\curlyvee s_1 s_2]{*} v \xleftrightarrow[\curlyvee s_1 s_2]{*} u' \xleftrightarrow[\curlyvee s_1 s_2]{*} u$$

with steps in $\mathcal{R}\cup\mathcal{S}$. From $t' \mathrel{\circ\!\!\!\to}_{\mathcal{R}} t$ we obtain $t' \mathrel{\circ\!\!\!\to}_{\mathcal{R},t'} t$. Moreover, $t' < s_1$ since

$$s_1 \to^*_{\mathcal{R}\cup\mathcal{S}} s \to^*_{\mathcal{R}\cup\mathcal{S}} s' \mathrel{\circ\!\!\!\to}_{\mathsf{CPS}_{S}(R)} t'$$

Similarly, $u' \mathrel{\circ\!\!\!\to}_{\mathcal{S},u'} u$ and $u' < s_2$. From $t' \to^*_{\mathcal{S}} v$ we obtain a rewrite sequence

$$t' \mathrel{\circ\!\!\!\to}_{\mathcal{S},t'} \cdots \mathrel{\circ\!\!\!\to}_{\mathcal{S},t'} v$$

So each step can be labeled with t' for which we already showed $t' < s_1$. Similarly, $u' \mathrel{\circ\!\!\!\to}_{\mathcal{R},u'} \cdots \mathrel{\circ\!\!\!\to}_{\mathcal{R},u'} v$ is obtained. □

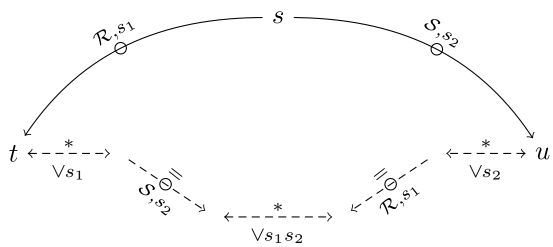

Figure 1: Decreasingness of $(\{\rightarrow\!\!\!\!\!\circ_{\mathcal{R},s}\}_{s\in\mathcal{T}}, \{\rightarrow\!\!\!\!\!\circ_{\mathcal{S},s}\}_{s\in\mathcal{T}})$.

Formalizing the results of this section turned out to be surprisingly straightforward. In order to implement $\mathsf{CPS}_{\mathcal{R}}(\mathcal{S})$, we had to add a definition of critical peaks for two different TRSs (the existing definition in IsaFoR only takes a single TRS as argument). All other relevant definitions and results were already present in IsaFoR or the Archive of Formal Proofs. In total the formalization only required a little more than 500 (new) lines of Isabelle code.

Theorem 4 subsumes Theorem 1 and its commutation version as $\mathsf{CPS}_{\mathcal{S}}(\mathcal{R}) \cup \mathsf{CPS}_{\mathcal{R}}(\mathcal{S}) = \varnothing$ and $\mathcal{R}$ and $\mathcal{S}$ are locally commuting for all left-linear TRSs $\mathcal{R}$ and $\mathcal{S}$ which are development closed. In [3] it is stated without proof that Theorem 4 can be strengthened by implementing the same weakening for overlays as in Theorem 2, hence also subsuming Theorem 2. It is however unclear how case (a) of the proof above works for this extension. The problem is that we might get a sequence $u \rightarrow^*_{\mathcal{R}} v$, instead of a multi-step, for which we would need to show that its labels are below s_1 or s_2.

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