universität innsbruck

Ground Canonical Rewrite Systems Revisited

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Outline

Two Papers

1. Wayne Snyder

   A Fast Algorithm for Generating Reduced Ground Rewriting Systems from a Set of Ground Equations

   Journal of Symbolic Computation 15(4), pp. 415 – 450, 1993

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desired: canonical rewrite system $\mathcal{R}$ for $\mathcal{E}$

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1. Introduction
2. **Canonical Rewrite Systems**
3. Snyder's Transformation
4. Ground AC Canonical Rewrite Systems

Definitions

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① (conversion) equivalent if $\leftrightarrow^*_{\mathcal{R}} = \leftrightarrow^*_{\mathcal{S}}$

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Theorem (Métivier 1983)

normalization equivalent reduced TRSs are unique up to literal similarity

Theorem (Métivier 1983)

normalization equivalent reduced ground TRSs are unique

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reduced ground TRSs are canonical

Lemma (Hirokawa et al. 2019) Ⓒ

if $\mathcal{R}$ and $\mathcal{S}$ are canonical TRSs such that $\mathsf{NF}(\mathcal{S}) \subseteq \mathsf{NF}(\mathcal{R})$ and $\to_{\mathcal{S}} \subseteq \leftrightarrow_{\mathcal{R}}^*$ then $\mathcal{R}$ and $\mathcal{S}$ are normalization equivalent

Known Results

1. congruence closure is efficient technique for solving word problems for ground ESs

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- results ② and ③ have been formalized in Isabelle/HOL

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Definition

$\mathsf{LHS}(\mathcal{R})$ / $\mathsf{RHS}(\mathcal{R})$ denotes set of left-hand / right-hand sides of rules of TRS $\mathcal{R}$

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$$\mathcal{R} \uplus \{\ell \to r\} \implies \mathcal{R}\{r \mapsto \ell\} \cup \{r \to \ell\} \qquad (\star)$$

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footnote (Snyder 1993)

The fact that this single transformation is sufficient to transform one reduced system into any other can be proved formally, but the proofs are tedious, and so we have preferred to present this intuitively.

Example (Snyder 1993)

$$f(a) \approx g(b, b) \qquad f(f(a)) \approx a \qquad f(f(f(a))) \approx a$$
$$g(b, h(a)) \approx g(b, b) \qquad h(a) \approx b \qquad i(f(a)) \approx c$$

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admits six different canonical TRSs:

$$\begin{aligned} f(g(b,b)) &\xrightarrow{1} g(b,b) \\ a &\xrightarrow{2} g(b,b) \\ h(g(b,b)) &\xrightarrow{3} b \\ i(g(b,b)) &\xrightarrow{4} c \end{aligned}$$

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$\mathcal{R} = \mathcal{R}' \uplus \{\ell \to r\}$ and $\mathcal{S} = \mathcal{R}'\{r \mapsto \ell\} \cup \{r \to \ell\}$ with $r \not\trianglelefteq \ell$

- $\mathcal{S}$ is right-reduced:

  suppose $t \in \mathrm{RHS}(\mathcal{S})$ is reducible with rule $u \to v \in \mathcal{S} \quad\implies\quad t \trianglerighteq u \quad\implies\quad u \neq r$

  $u \to v \in \mathcal{R}'\{r \mapsto \ell\} \quad\implies\quad u = u'\{r \mapsto \ell\}$ and $v = v'\{r \mapsto \ell\}$ with $u' \to v' \in \mathcal{R}'$

  $t = t'\{r \mapsto \ell\}$ with $t' \in \mathrm{RHS}(\mathcal{R}) \quad\implies\quad t' \trianglerighteq u' \quad\implies\quad \mathcal{R}$ is not right-reduced

Lemma

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if $\mathcal{R} \implies \mathcal{S}$ and $\mathcal{R}$ is canonical then $\mathcal{S}$ is canonical

Proof (cont'd)

$\mathcal{R} = \mathcal{R}' \uplus \{\ell \to r\}$ and $\mathcal{S} = \mathcal{R}'\{r \mapsto \ell\} \cup \{r \to \ell\}$ with $r \ntrianglelefteq \ell$

- $\mathcal{S}$ is right-reduced and left-reduced

Lemma

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Proof (cont'd)

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  consider $t \to u \in \mathcal{S}$ and suppose $t \notin \mathsf{NF}(\mathcal{S} \setminus \{t \to u\})$

Lemma

if $\mathcal{R} \implies \mathcal{S}$ and $\mathcal{R}$ is canonical then $\mathcal{S}$ is canonical

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$\mathcal{R} = \mathcal{R}' \uplus \{\ell \to r\}$ and $\mathcal{S} = \mathcal{R}'\{r \mapsto \ell\} \cup \{r \to \ell\}$ with $r \not\trianglelefteq \ell$

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  consider $t \to u \in \mathcal{S}$ and suppose $t \notin \mathsf{NF}(\mathcal{S} \setminus \{t \to u\})$

  ① $t = r$

Lemma

if $\mathcal{R} \implies \mathcal{S}$ and $\mathcal{R}$ is canonical then $\mathcal{S}$ is canonical

Proof (cont'd)

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  $\ell \not\trianglelefteq v \implies v \in \mathrm{LHS}(\mathcal{R}')$

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if $\mathcal{R} \implies \mathcal{S}$ and $\mathcal{R}$ is canonical then $\mathcal{S}$ is canonical

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  $\ell \ntrianglelefteq v \quad \implies \quad v \in \mathsf{LHS}(\mathcal{R}') \subseteq \mathsf{LHS}(\mathcal{R})$

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if $\mathcal{R} \implies \mathcal{S}$ and $\mathcal{R}$ is canonical then $\mathcal{S}$ is canonical

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$\mathcal{R} = \mathcal{R}' \uplus \{\ell \to r\}$ and $\mathcal{S} = \mathcal{R}'\{r \mapsto \ell\} \cup \{r \to \ell\}$ with $r \ntrianglelefteq \ell$

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if $\mathcal{R} \implies \mathcal{S}$ and $\mathcal{R}$ is canonical then $\mathcal{S}$ is canonical

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  ② $t \to u \in \mathcal{R}'\{r \mapsto \ell\}$

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  $t = t'\{r \mapsto \ell\}$ and $v = v'\{r \mapsto \ell\}$ with $t', v' \in \mathsf{LHS}(\mathcal{R}')$

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  $t = t'\{r \mapsto \ell\}$ and $v = v'\{r \mapsto \ell\}$ with $t', v' \in \mathsf{LHS}(\mathcal{R}')$

  $t' \rhd v'$

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if $\mathcal{R} \implies \mathcal{S}$ and $\mathcal{R}$ is canonical then $\mathcal{S}$ is canonical

Proof (cont'd)

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Lemma

if $\mathcal{R} \implies \mathcal{S}$ and $\mathcal{R}$ is canonical then $\mathcal{S}$ is canonical

Proof (cont'd)

$\mathcal{R} = \mathcal{R}' \uplus \{\ell \to r\}$ and $\mathcal{S} = \mathcal{R}'\{r \mapsto \ell\} \cup \{r \to \ell\}$ with $r \not\trianglelefteq \ell$

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Lemma

if $\mathcal{R} \implies \mathcal{S}$ and $\mathcal{R}$ is canonical then $\mathcal{S}$ is canonical

Proof (cont'd)

$\mathcal{R} = \mathcal{R}' \uplus \{\ell \to r\}$ and $\mathcal{S} = \mathcal{R}'\{r \mapsto \ell\} \cup \{r \to \ell\}$ with $r \ntrianglelefteq \ell$

- $\mathcal{S}$ is right-reduced and left-reduced $\implies$ $\mathcal{S}$ is canonical by theorem B

  consider $t \to u \in \mathcal{S}$ and suppose $t \notin \mathsf{NF}(\mathcal{S} \setminus \{t \to u\})$

  ① $t = r \notin \mathsf{NF}(\mathcal{R}'\{r \mapsto \ell\}) \implies r \trianglerighteq v$ with $v \in \mathsf{LHS}(\mathcal{R}'\{r \mapsto \ell\})$

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  $t = t'\{r \mapsto \ell\}$ and $v = v'\{r \mapsto \ell\}$ with $t', v' \in \mathsf{LHS}(\mathcal{R}')$

  $t' \rhd v' \implies \mathcal{R}'$ is not right-reduced ↯

Lemma

if $\mathcal{R} \implies \mathcal{S}$ and $\mathcal{R}$ is canonical then $\mathcal{S}$ is canonical and equivalent to $\mathcal{R}$

Proof (cont'd)

$\mathcal{R} = \mathcal{R}' \uplus \{\ell \to r\}$ and $\mathcal{S} = \mathcal{R}'\{r \mapsto \ell\} \cup \{r \to \ell\}$ with $r \ntrianglelefteq \ell$

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  consider $t \to u \in \mathcal{S}$ and suppose $t \notin \mathrm{NF}(\mathcal{S} \setminus \{t \to u\})$

  ① $t = r \notin \mathrm{NF}(\mathcal{R}'\{r \mapsto \ell\}) \implies r \trianglerighteq v$ with $v \in \mathrm{LHS}(\mathcal{R}'\{r \mapsto \ell\})$

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  ② $t \to u \in \mathcal{R}'\{r \mapsto \ell\}$

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  $t = t'\{r \mapsto \ell\}$ and $v = v'\{r \mapsto \ell\}$ with $t', v' \in \mathrm{LHS}(\mathcal{R}')$

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Theorem

if $\mathcal{R}$ and $\mathcal{S}$ are equivalent ground canonical TRSs then $\mathcal{R} \Longrightarrow^* \mathcal{S}$

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① $\mathsf{NF}(\mathcal{R}) \cap \mathsf{LHS}(\mathcal{S}) = \varnothing \implies \mathcal{R} = \mathcal{S}$

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$\mathsf{NF}(\mathcal{R})$ is closed under subterms

Theorem

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$\mathcal{R} = \mathcal{S}$ by theorem A

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if $\mathcal{R}$ and $\mathcal{S}$ are equivalent ground canonical TRSs then $\mathcal{R} \Longrightarrow^* \mathcal{S}$

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$\mathsf{NF}(\mathcal{R})$ is closed under subterms $\implies \mathsf{NF}(\mathcal{R}) \subseteq \mathsf{NF}(\mathcal{S})$

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$\mathcal{R} = \mathcal{S}$ by theorem A

② $\mathcal{R} \Longrightarrow^* \mathcal{S}$ by induction on $|\,\mathsf{RHS}(\mathcal{S}) \setminus \mathsf{NF}(\mathcal{R})\,|$

Theorem

if $\mathcal{R}$ and $\mathcal{S}$ are equivalent ground canonical TRSs then $\mathcal{R} \Longrightarrow^* \mathcal{S}$

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given finite ground ES $\mathcal{E}$ with k axioms

- every canonical TRS for $\mathcal{E}$ is finite
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Question

- can 2^k bound be derived from Snyder's transformation ?

Outline

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- TRS $\mathcal{R}$ is AC confluent if $(\leftarrow \cup \to \cup \sim)^* \subseteq (\to/\sim)^* \cdot \sim \cdot^* (\sim \backslash \leftarrow)$ AC Church–Rosser
- TRS $\mathcal{R}$ is AC reduced if for every rewrite rule $\ell \to r$ in $\mathcal{R}$
  1. $r \in \mathsf{NF}(\mathcal{R} \,/\, \mathsf{AC})$
  2. $\ell \in \mathsf{NF}((\mathcal{R} \setminus \{\ell \to r\}) \,/\, \mathsf{AC})$

Definitions

- $\mathcal{F}_{\mathsf{AC}} \subseteq \mathcal{F}$ is set of AC symbols
- $\sim_{\mathsf{AC}}$ is equivalence relation on $\mathcal{T}(\mathcal{F})$ induced by

$$f(x,y) \approx f(y,x) \qquad f(f(x,y),z) \approx f(x,f(y,z)) \qquad \text{for all } f \in \mathcal{F}_{\mathsf{AC}}$$

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Example

$\mathsf{f}(\mathsf{a},\mathsf{b}) \approx \mathsf{d}$ $\qquad$ $\mathsf{f}(\mathsf{b},\mathsf{c}) \approx \mathsf{e}$ $\qquad$ f AC symbol

Example

$$\mathsf{f}(\mathsf{a},\mathsf{b}) \approx \mathsf{d} \qquad \mathsf{f}(\mathsf{b},\mathsf{c}) \approx \mathsf{e} \qquad \mathsf{f} \ \text{ AC symbol}$$

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A	$\mathsf{f}(\mathsf{a},\mathsf{b}) \xrightarrow{1} \mathsf{d}$	$\mathsf{f}(\mathsf{b},\mathsf{c}) \xrightarrow{2} \mathsf{e}$	$\mathsf{f}(\mathsf{a},\mathsf{e}) \xrightarrow{3} \mathsf{f}(\mathsf{c},\mathsf{d})$
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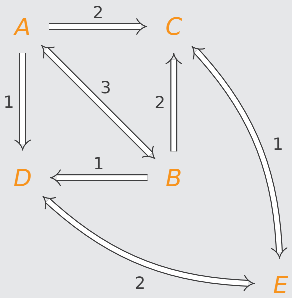
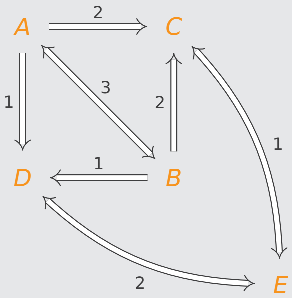
  orienting single AC critical pair results in A or B

Example (cont'd)



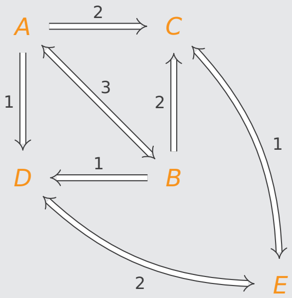
Example (cont'd)



Remarks

- reduced ground TRSs need not be AC confluent

Example (cont'd)



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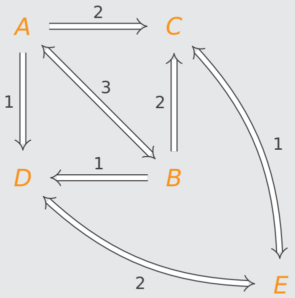

Remarks

- reduced ground TRSs need not be AC confluent
- after resolving AC critical pairs, new AC critical pairs may arise
- transformation may turn AC canonical TRS into non AC terminating TRS

Example

TRS $\mathcal{R}$

$$\mathsf{f}(\mathsf{b},\mathsf{c}) \xrightarrow{1} \mathsf{f}(\mathsf{a},\mathsf{b}) \qquad \mathsf{f}(\mathsf{c},\mathsf{d}) \xrightarrow{2} \mathsf{f}(\mathsf{d},\mathsf{a})$$

with AC symbol f is AC canonical

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with AC symbol f is AC canonical, reversing rule 1 results in TRS $\mathcal{S}$:

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TRS $\mathcal{R}$

$$\mathsf{f(b,c)} \xrightarrow{1} \mathsf{f(a,b)} \qquad \mathsf{f(c,d)} \xrightarrow{2} \mathsf{f(d,a)}$$

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$$\mathsf{f(a,b,d)} \to_{\mathcal{S}/\mathsf{AC}} \mathsf{f(b,c,d)} \to_{\mathcal{S}/\mathsf{AC}} \mathsf{f(a,b,d)}$$

Conclusion

Snyder’s transformation is not useful in AC setting

Theorem (Marché 1991)

every AC canonical presentation of finite ground ES is finite

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Summary

given finite ground ES $\mathcal{E}$ with k axioms

- every canonical TRS for $\mathcal{E}$ is finite
- different canonical TRSs for $\mathcal{E}$ have same number of rewrite rules
- total number of canonical TRSs for $\mathcal{E}$ is bounded by 2^k
- every canonical TRS for $\mathcal{E}$ can be generated by (ground) completion
- Snyder's transformation can generate all canonical TRSs for $\mathcal{E}$ from any canonical TRS

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▸ total number of canonical TRSs for $\mathcal{E}$ is bounded by 2^k	×
▸ every canonical TRS for $\mathcal{E}$ can be generated by (ground) completion	?
▸ Snyder's transformation can generate all canonical TRSs for $\mathcal{E}$ from any canonical TRS	×