On Confluence Criteria for Non-terminating Abstract Rewriting Systems

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1. Motivation & main problem

Motivation: ARS in non-discrete process modeling

Definition

An **abstract rewriting system** (**ARS**) is a pair $(X, \rightarrow)$, where

- X is a set
- $\rightarrow$ is a binary relation on X (*reduction*)

1. ARS from **reachability** relations on state spaces of nondeterministic dynamical systems
   - elements are **states**
   - $x \rightarrow^* x'$ implies that x, x' can be joined by a **trajectory**
2. ARS from **causality** relations on event structures
   - elements are **events**
   - $e \rightarrow^* e'$ implies that e **causally precedes** e'

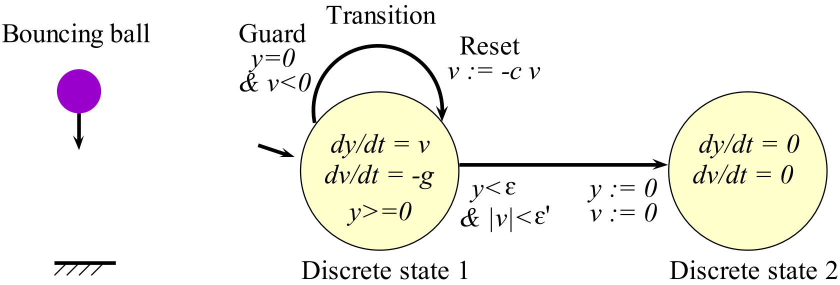
NOTE: above and on subsequent slides:

$\rightarrow^+$ denotes the transitive closure of $\rightarrow$

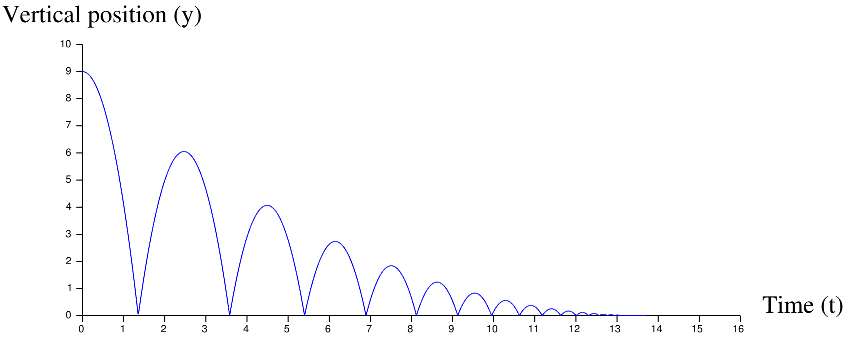
$\rightarrow^*$ denotes the reflexive transitive closure of $\rightarrow$

Example of type 1: ARS from reachability

- Consider a **nondeterministic** hybrid system:



Example of type 1: illustration of a run (partial)

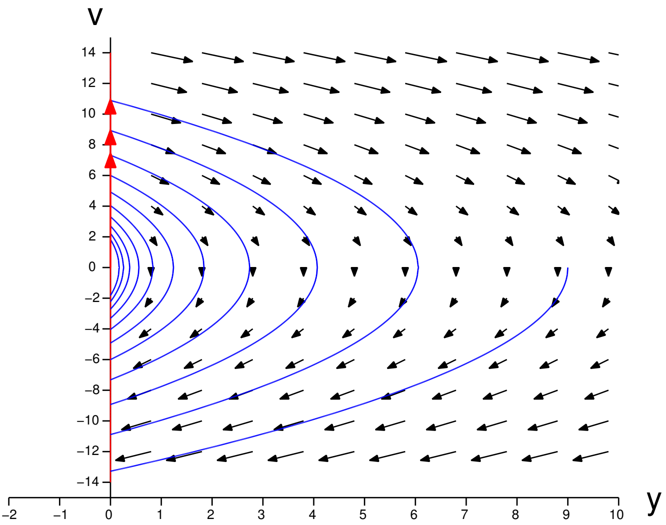


Example of type 1: ARS

Define an ARS $(S, \rightarrow)$ such that

- $S = [0, +\infty) \times \mathbb{R}$ (**continuous state space**)
- $(y_1, v_1) \rightarrow (y_2, v_2)$, if (y_2, v_2) can be **reached** from (y_1, v_1)
  - **either** via **continuous evolution** within one discrete state,
  - **or** as a result of a single **discrete transition** between discrete states (that may coincide)

Example of type 1: continuous state space



Example of type 2: "Spacetime ARS"

Consider an ARS $(E, \rightarrow)$[1], where

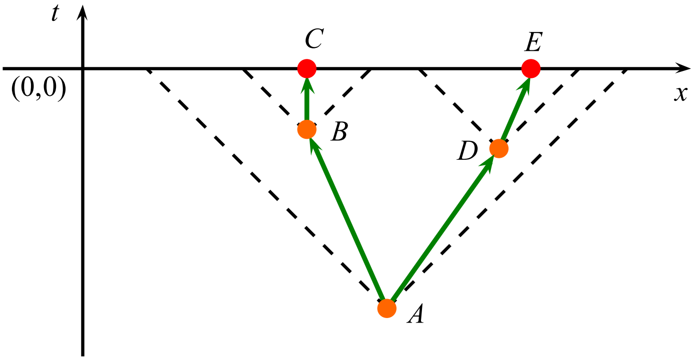
- $E = \{(x, t) \in \mathbb{R} \times \mathbb{R} \mid t \leq 0\}$
- $(x, t) \rightarrow (x', t') \Leftrightarrow (t' - t) > 0 \wedge (t' - t)^2 - (x' - x)^2 \geq 0.$

Interpretation:

- x, t are **space** and **time** coordinates
- E is a "**region of spacetime**"
- $\rightarrow$ is the **strict causal precedence** between events in (1+1)-dimensional **Minkowski spacetime**, **restricted** to E

[1] Similar examples in computer science can be constructed using e.g.: F. Mattern. *On the relativistic structure of logical time in distributed systems*

Example of type 2: illustration



examples of reducible elements

examples of irreducible elements

examples of reductions

boundaries of sets
of direct successors

Main problem

- When discussing ARS, textbooks and monographs on rewriting systems often **give priority** to ARS with properties most relevant to modeling of discrete processes, e.g. termination, countability, etc.
- Such properties **often do not hold** for ARS that arise from **continuous** and **discrete-continuous** process models.
- We suppose that this **inhibits new applications** of the theory of rewriting systems, e.g. in the domain of **cyber-physical systems** (CPS).

Topic of the talk

- **Newman's lemma** is a widely known confluence condition:

  a *terminating* ARS is confluent if it is locally confluent [2]

- However, it depends on the **termination** assumption:

  $(X, \to)$ is *terminating*, if there is *no* infinite reduction sequence $x_1 \to x_2 \to x_3 \to \ldots$ (where $x_i \in X$)

- This often **does not hold** for ARS that arise from continuous / hybrid systems.

- We propose some ways in which Newman's lemma can be **"freed" of the termination assumption**.

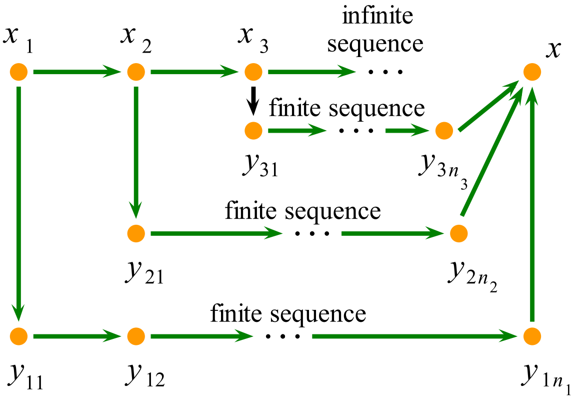
[2] E.g.: P. Malbos. *Lectures on Algebraic Rewriting*, 2019

2. Preliminaries - part 1

Recall: inductive ARS

Definition

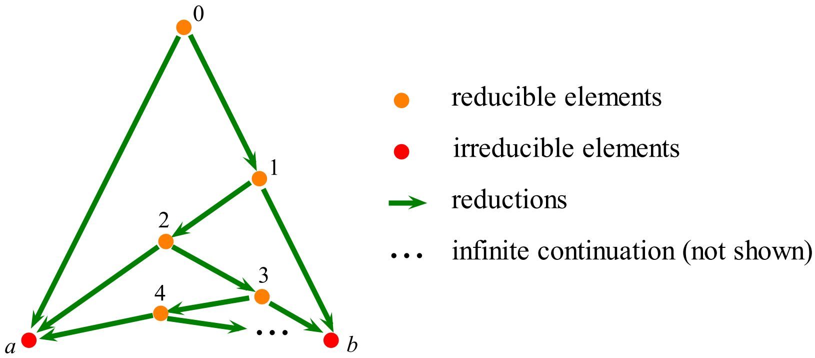
An ARS $(X, \to)$ is **inductive**, if for every reduction sequence $x_1 \to x_2 \to \ldots$ there exists $x \in X$ such that $x_n \to^* x$ for all n.



Recall: Newman's counterexample

Widely known example[3] (due to Newman) of a relation that is

- **acyclic**, **inductive**, **locally confluent**, but ***not* confluent**



Formalization: $(X, \to)$, $X = \{a, b, 0, 1, 2, ...\}$, $a = -1$, $b = -2$

- $(2n) \to a$, $(2n+1) \to b$, $n \to (n+1)$ for $n = 0, 1, 2, ...$

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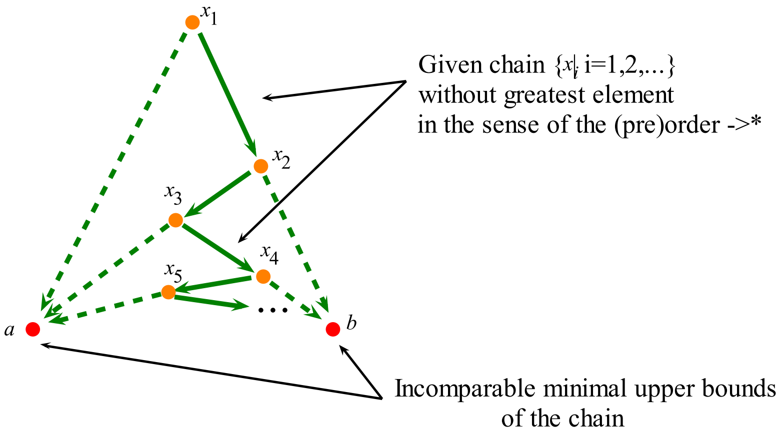
[3] G. Huet. *Confluent reductions: Abstract properties and applications to term rewriting systems*, JACM, 1980

Newman's counterexample in order-theoretic terms

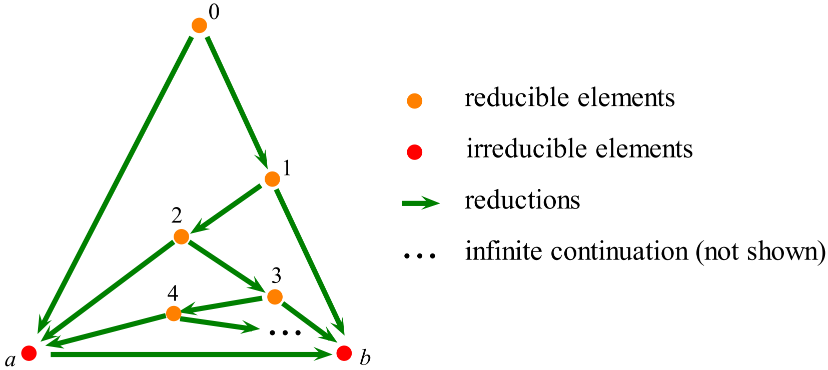
- In the (pre)ordered set $(X, \to^*)$ there is an infinite **chain** with **two (incomparable) minimal upper bounds**.
- It is easy to see that **local confluence** is not sufficient to guarantee that they have a common upper bound.



Strict inductivity

Definition

An ARS $(X, \to)$ **is strictly inductive**, if every nonempty chain in the preordered set $(X, \to^*)$ has a ***least*** upper bound



Formalization: $(X, \to)$, $X = \{a, b, 0, 1, 2, ...\}$, $a = -1$, $b = -2$,
$a \to b$, $(2n) \to a$, $(2n+1) \to b$, $n \to (n+1)$ for $n = 0, 1, 2, ...$

Strengthened Newman's counterexample

The ***strict* inductivity** assumption is **not sufficient** to guarantee the equivalence "local confluence $\Leftrightarrow$ confluence".

Example (Strengthened Newman's counterexample)

Consider an ARS $(P, \to)$, where

$$P = \mathbb{R} \times \mathbb{R} \times (\omega + 1)$$

and $(x, y, n) \to (x', y', n')$ iff one of the following holds:

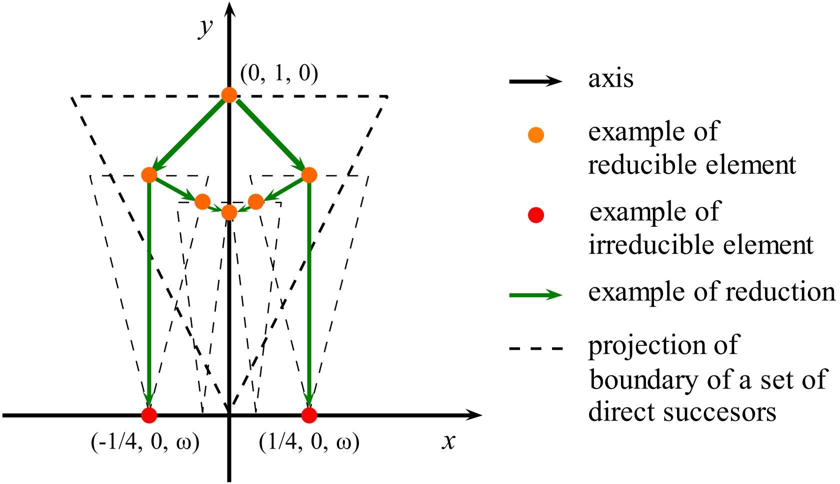
- $n < \omega \wedge n' = n + 1 \wedge |x' - x| < y'/2^{n'} \wedge 0 < y' < y$
- $n < \omega \wedge n' = \omega \wedge x' = x \wedge 0 = y' < y$.

Then $(P, \to)$ is

- **acyclic**,
- ***strictly* inductive**,
- **locally confluent**,

but is ***not* confluent**.

Strengthened Newman's counterexample illustration



Strengthened counterexample can't be countable

Theorem (Newman's lemma for possibly nonterminating ARS)

Let $(X, \to)$ *be an* ***acyclic***, ***strictly inductive*** *ARS with at most* ***countable*** *set of irreducible elements. Then* $(X, \to)$ *is* ***confluent*** *if and only if it is* ***locally confluent***.

We checked this fact in Isabelle 2022 proof assistant using a formalization of ARS notions from:

I. Ivanov. *Formalization of Generalized Newman's Lemma*, 2023, http://doi.org/10.5281/zenodo.7855691

3. Preliminaries - part 2

Recall: Noetherian induction

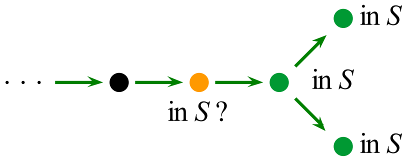
Definition

1. ARS $(X, \rightarrow)$ **has sound Noetherian induction principle (sound NIP)**, if for every $S \subseteq X$,
   if $\forall x \in X\ ((\forall y \in X\ (x \rightarrow^{+} y \Rightarrow y \in S)) \Rightarrow x \in S)$,
   then $S = X$

It is well known[4] that

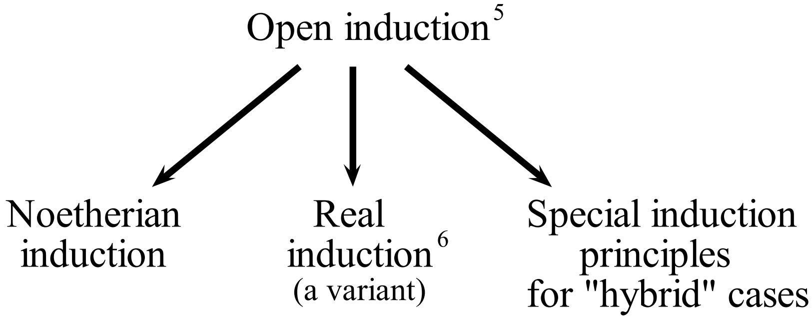
$$(X, \rightarrow) \textbf{ is terminating} \Leftrightarrow (X, \rightarrow) \textbf{ has sound NIP}.$$

Induction step: check that the orange element (•) is in S, assuming that green elements (•) are in S



[4] P. Malbos. *Lectures on Algebraic Rewriting*, 2019

More induction principles



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[5] J.C. Raoult. *Proving open properties by induction*. Information processing letters, 29(1):19–23, 1988

[6] P. Clark. *The instructor's guide to real induction*. Mathematics Magazine, 92(2):136–150, 2019

Inductive reasoning for real numbers

Example. Show that **if**

$$\frac{dy}{dx} = 1 - xy, \quad y(0) = 1 \ ,$$

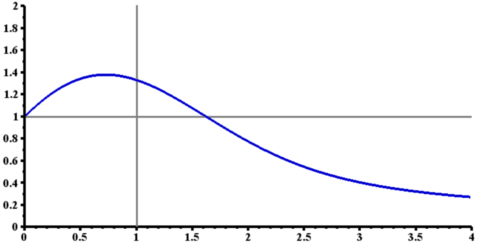
then

$$y(x) \geq 1 \text{ for all } x \in [0, 1].$$

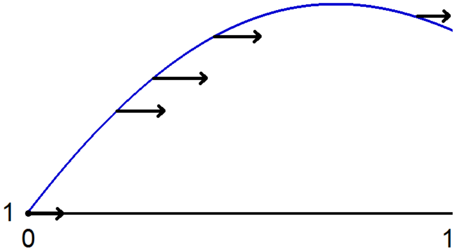
- **Solution**:

$$y(x) = e^{-x^2/2}\left(1 + \sqrt{2}\int_0^{x/\sqrt{2}} e^{t^2}\,dt\right)$$

- **Plot**:



Inductive reasoning for real numbers



"**Base case**" : $y(0) \geq 1$ holds

"**Step**" : Let $x_0 \in [0, 1)$. Assume that $y(x_0) \geq 1$.

- if $y(x_0) > 1$, then (by continuity) $y(x) > 1$ for all $x \in [x_0, x + \epsilon)$ for a sufficiently small $\epsilon > 0$.
- if $y(x_0) = 1$: $y'(x_0) = 1 - x_0 \cdot (1) > 0$ and $y(x) \geq 1$ in some right neighborhood of x_0.

How to complete the proof ?

Real induction principle

One of the variants[7]:

A subset $S \subseteq [a, b]$ (where $a < b$ are real) is *inductive*, if:

1. $a \in S$
2. if $a \leq x < b$ and $x \in S$, then $[x, y] \subseteq S$ for some $y > x$
3. if $a < x \leq b$ and $[a, x) \subseteq S$, then $x \in S$.

Induction principle:

$$S \subseteq [a, b] \text{ is inductive iff } S = [a, b]$$

In the example above one can take

$$a = 0, b = 1, S = \{x \in [0, 1] \mid y(x) \geq 1\}$$

The principle can be used in **ODE invariance axiomatization**[8].

[7] P. Clark. *The Instructor's Guide to Real Induction*, 2012

[8] A. Platzer, Yong Kiam Tan. *Differential Equation Invariance Axiomatization: The Impressive Power of Differential Ghosts*, LICS'18, 2018.

Open induction

Let $(X, \leq)$ be a poset. A predicate ("property") $P : X \to Bool$ is

- *inductive*, if

$$\forall y\ ((\forall x(y < x \Rightarrow P(x)) \Rightarrow P(y))$$

- *open*, if for each nonempty chain $C \subseteq X$,
  if C has a least upper bound x^* and $P(x^*) = true$
  then $P(x)$ for some $x \in C$.

Theorem[9] (Raoult). In a cpo, an inductive and open property is true everywhere.

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[9] J.C. Raoult. *Proving open properties by induction*. Information processing letters, 29(1):19–23, 1988

Topology on ARS

Let $(X, \to)$ be an ARS.

Definition

1. A set $A \subseteq X$ is **closed**, if
   for every nonempty chain C in the preordered set $(X, \to^*)$,

   if $C \subseteq A$ and C has a least upper bound $x \in X$ in the sense of the preordered set $(X, \to^*)$,
   then $x \in A$

2. A set $A \subseteq X$ is **open**, if $X \backslash A$ is closed.

Proposition

*If $(X, \to)$ is a **terminating** ARS, then every set $A \subseteq X$ is **open** (and **closed**).*

Examples of closed and open sets

Consider an ARS $(X, \to)$, where

- $X = [0, 1]$ (**real unit interval**)
- $\to$ is the **standard order** $\leq$ on real numbers restricted to X

Then:

- every subset $A \subseteq X$ is a **chain**
- $\{1\}$ is **closed**: every nonempty chain in $\{1\}$ contains only 1, so its least upper bound is 1, and $1 \in \{1\}$
- $\{1\}$ is ***not* open**: $1 \in \{1\}$ can be approached from below using a nonempty chain of elements outside $\{1\}$, e.g.

$$0.9,\ 0.99,\ 0.999, \ldots$$

- the set $(0, 1)$ is **open**: no number in $(0, 1)$ can be approached from below using a nonempty chain that has no elements in $(0, 1)$.

4. Main results*

*Based on the work: I. Ivanov. *Generalized Newman's Lemma for Discrete and Continuous Systems*, FSCD 2023

Open induction for ARS

Definition

An ARS $(X, \to)$ **has sound open induction principle (sound OIP)**, if for every open $S \subseteq X$,

$$\text{if} \quad \forall x \in X \, ((\forall y \in X \, (x \to^{+} y \Rightarrow y \in S)) \Rightarrow x \in S),$$
$$\text{then} \quad S = X$$

Proposition

For any ARS $(X, \to)$ the following conditions are ***equivalent****:*

- $(X, \to)$ *is* ***strictly inductive*** *and* ***acyclic***
- $(X, \to)$ *has* ***sound open induction principle****.*

Some examples of strictly inductive ARS

- **finite** ARS
- **terminating** ARS
- **dcpos** (for a dcpo $(X, \leq)$, either $\leq$, or a corresponding strict order $<$ can be considered as reduction relation), e.g.

  $([0, 1], <)$, where $[0, 1]$ is the real unit interval and $<$ is the standard strict order on real numbers, restricted to $[0, 1]$

Open normalization

Definition

An ARS $(X, \to)$ is **openly normalizing**, if $(X, \to)$ is normalizing and for each $x, x' \in X$ such that x' is a normal form of x, the set

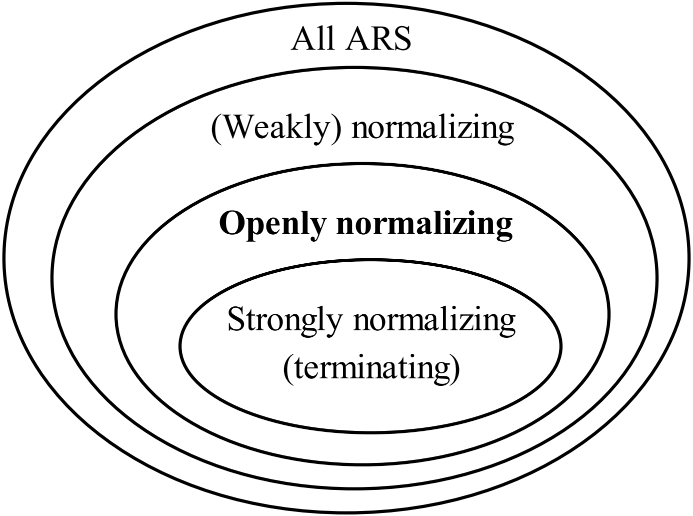
$$\{y \in X \mid x \to^* y \text{ and } x' \text{ is the } \textbf{only normal form} \text{ of } y\}$$

is **open** in $\{y \in X \mid x \to^* y\}$, considered as an induced preordered subset of $(X, \to^*)$.

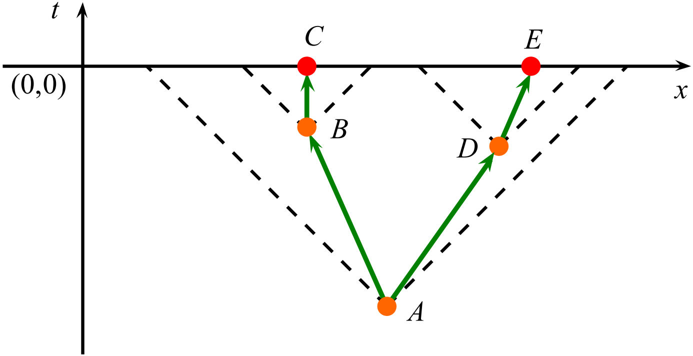
Proposition

*A **confluent** ARS is (weakly) **normalizing** if and only if it is **openly normalizing**.*

Normalization notions



Recall: “Spacetime ARS”




examples of reducible elements


examples of irreducible elements


examples of reductions

boundaries of sets
of direct successors

Confluence for ARS with sound open induction

Theorem

*For any ARS $(X, \to)$ with **sound open induction principle** the following conditions are **equivalent**:*

- *$(X, \to)$ is **confluent***
- *$(X, \to)$ is **locally confluent** and **openly normalizing**.*

Theorem

*For any **countable** ARS $(X, \to)$ with **sound open induction principle** the following conditions are **equivalent**:*

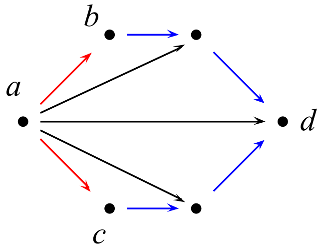
- *$(X, \to)$ is **confluent***
- *$(X, \to)$ is **locally confluent**.*

Further generalization to all strictly inductive ARS

Issue: a confluence criterion based on **local confluence** does ***not* handle well** cases when a reduction relation $\rightarrow$

1. has **cycles** ($x \rightarrow^+ x$)
2. is **transitive**, or includes a "large" transitive (sub)relation:

$$r \subseteq \rightarrow \;\wedge\; r^+ = r$$



$$(a \rightarrow b) \wedge (a \rightarrow c) \Rightarrow \exists d\, (b \rightarrow^* d \wedge c \rightarrow^* d)$$

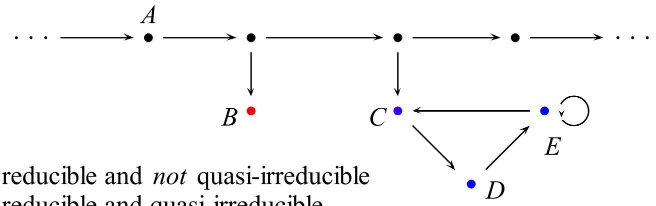
Quasi-irreducibility

Let ARS $(X, \to)$ be an ARS and $x, x' \in X$.

- x is **reducible**, if $\exists x' \in X \; x \to x'$
- x is **irreducible**, if x is not reducible.

Definition

1. x is **quasi-irreducible**, if $\forall y \in X \; (x \to^* y \Rightarrow y \to^* x)$
2. x' is a **quasi-normal form (QNF)** of x, if
   $x \to^* x'$ and x' is quasi-irreducible
3. x, x' **are QNF-equivalent**, if
   $\{y \in X \mid y \text{ is a QNF of } x\} = \{y \in X \mid y \text{ is a QNF of } x'\}$



- reducible and *not* quasi-irreducible
- reducible and quasi-irreducible
- irreducible (and quasi-irreducible)

Generalization of open normalization

Definition

An ARS $(X, \to)$ is

1. **quasi-normalizing**, if for each $x \in X$ there exists $x' \in X$ such that x' is a quasi-normal form of x.
2. **openly quasi-normalizing**, if $(X, \to)$ is quasi-normalizing and for each $x, x' \in X$ such that x' is a quasi-normal form of x, the set

$$\{y \in X \mid x \to^* y \ \wedge \ y \text{ and } x' \text{ are QNF-equivalent } \}$$

is **open** in $\{y \in X \mid x \to^* y\}$, considered as an induced preordered subset of $(X, \to^*)$.

Proposition

*An **acyclic** ARS is **openly quasi-normalizing** if and only if it is **openly normalizing**.*

Main result

Definition

An ARS $(X, \to)$ is **quasi-locally confluent**, if for each $a \in X$ there exists $S \subseteq \{x \in X \mid a \to^+ x\}$ such that :

1. **two-consistency condition**:

$$\forall b, c \in S\ \exists d \in X\ (b \to^* d \wedge c \to^* d)$$

2. **coinitiality condition**:

$$\forall x \in X\ (a \to^+ x \ \Rightarrow\ (x \to^* a) \vee \\ \vee (\exists b \in S\ \ b \to^* x \wedge \neg(b \to^* a))\ )$$

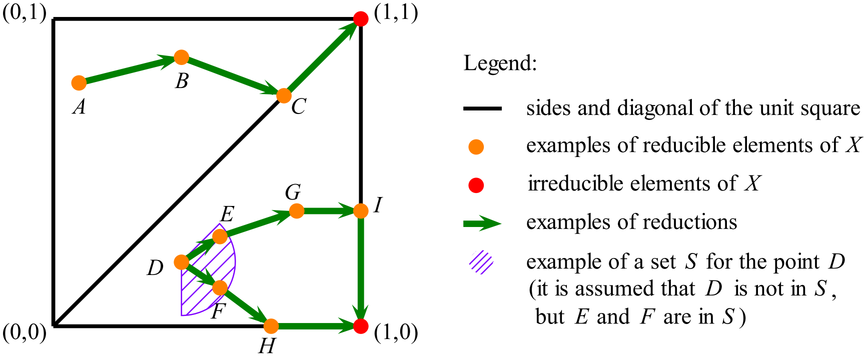
Theorem (Generalized Newman's lemma)

*A **strictly inductive** ARS is **confluent** if and only if it is **openly quasi-normalizing** and **quasi-locally confluent**.*

Example

Consider an ARS $(X, \to)$, where

- $X = [0,1] \times [0,1]$ (where $[0,1]$ denotes a real interval)
- $(x,y) \to (x',y') \Leftrightarrow$
  $$(x < x' \vee (x = x' \wedge y' < y \wedge (x < 1 \vee y < 1)))\wedge$$
  $$(x - y \leq x' - y') \wedge (x = y \Rightarrow x' = y') \wedge (x \leq y \Rightarrow x' \leq y').$$



Ordinary Newman's lemma from generalized one

1. A **strictly inductive** ARS is **confluent** if and only if it is **openly quasi-normalizing** and **quasi-locally confluent** (*Generalized Newman's lemma*)
2. any **terminating** ARS is **strictly inductive** and **acyclic**
3. any **terminating** ARS is **openly quasi-normalizing**
4. any **locally confluent acyclic** ARS is **quasi-locally confluent**

So any **locally confluent** and **terminating** ARS is **confluent.**

Importance of conditions of the main result

- a ~~strictly~~ **inductive** ARS is **confluent** if and only if it is **openly quasi-normalizing** and **quasi-locally confluent** – **wrong** (Newman's counterexample[11])
- a **strictly inductive** ARS is **confluent** if and only if it is ~~openly~~ **quasi-normalizing** and **quasi-locally confluent** – **wrong** ("Spacetime ARS")
- a **strictly inductive** ARS is **confluent** if and only if it is **openly quasi-normalizing** and ~~quasi-~~**locally confluent** – **wrong** (Hindley's counterexample[11])

[11] G. Huet. *Confluent reductions: Abstract properties and applications to term rewriting systems*, JACM, 1980

Formalization and machine-checked proofs

- We formalized the main result in **Isabelle 2022** proof assistant using **HOL** logic.
- Formalization includes **definitions**, **auxiliary facts**, and a **formal proof** of generalized Newman's lemma which can be checked automatically.
- Isabelle theory file (**.thy**):

  I. Ivanov. *Formalization of Generalized Newman's Lemma*, 2023, http://doi.org/10.5281/zenodo.7855691