AC-KBO Revisited*

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Abstract

We consider various definitions of AC-compatible Knuth-Bendix orders. The orders of Steinbach and of Korovin and Voronkov are revisited. The former is enhanced to a more powerful AC-compatible order and we modify the latter to amend its lack of monotonicity on non-ground terms. An extension reflecting the recent proposal of subterm coefficients in standard Knuth-Bendix orders is also given. The various orders are compared on problems in termination and completion.

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1 Introduction

Associative and commutative (AC) operators appear in many applications. We are interested in proving termination of term rewrite systems with AC symbols. AC termination is important when deciding validity in equational theories with AC operators by means of completion.

Several termination methods for plain rewriting have been extended to deal with AC symbols. Ben Cherifa and Lescanne [1] presented a sufficient restriction on polynomial interpretations to ensure compatibility with the AC axioms. There have been numerous papers on extending the recursive path order of Dershowitz [2] to deal with AC symbols, culminating in the fully syntactic AC recursive path order of Rubio [3]. Marché and Urbain [4] and, independently, Kusakari and Toyama [5], adapted the influential dependency pair method of Arts and Giesl [6] to AC rewriting.

We are aware of only two papers on AC extensions of the order of Knuth and Bendix [7]. In this paper we revisit these orders and present yet another AC-compatible Knuth-Bendix order. Steinbach [8] presented a first version, which comes with the restriction that AC symbols are minimal in the precedence. By incorporating ideas of [3], Korovin and Voronkov [9] presented a version without this restriction. Actually, they present two versions. One is defined on ground terms and another one on arbitrary terms. For (automatically) proving AC termination of rewrite systems, a (recursive) definition of an AC-compatible order on arbitrary terms is required.[1] We show that the second order of [9] lacks the crucial monotonicity property. Nevertheless we prove that the order is sound for proving termination by extending it to an AC-compatible simplification order.

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[1] Any AC-compatible reduction order $>_{\mathrm{g}}$ on ground terms can trivially be extended to an AC-compatible reduction order $>$ on arbitrary terms by defining $s > t$ if and only if $s\sigma >_{\mathrm{g}} t\sigma$ for all grounding substitutions σ. This is, however, only of (mild) theoretical interest.

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We furthermore present a simpler variant of this latter order which properly extends the order of [8]. In particular, Steinbach's order is a correct AC-compatible simplification order, contrary to what is claimed in [9]. Apart from these theoretical contributions, we implemented the various AC-compatible Knuth-Bendix orders to compare them also experimentally.

The remainder of this paper is organized as follows. After recalling basic concepts of rewriting modulo AC and orders, we revisit Steinbach's order in Section 3. Section 4 is devoted to the two orders of Korovin and Voronkov. We present a first version of our AC-compatible Knuth-Bendix order in Section 5, where we also give the non-trivial proof that it has the required properties. (The proofs in [9] are limited to the order on ground terms.) We strengthen our order in Section 6 with subterm coefficients. In Section 7 we relate the various orders to each other and provide experimental results. Due to lack of space, some of the proofs can be found in the Appendix.

2 Preliminaries

We assume familiarity with rewriting and termination. Throughout this paper we deal with rewrite systems over a *finite* signature $\mathcal{F}$ together with a designated subset $\mathcal{F}_{\mathsf{AC}}$ of binary AC symbols. The congruence relation induced by the equations $f(x,y) \approx f(y,x)$ and $f(f(x,y),z) \approx f(x,f(y,z))$ for all $f \in \mathcal{F}_{\mathsf{AC}}$ is denoted by $=_{\mathsf{AC}}$. A term rewrite system (TRS for short) $\mathcal{R}$ is AC terminating if the relation $=_{\mathsf{AC}} \cdot \rightarrow_{\mathcal{R}} \cdot =_{\mathsf{AC}}$ is well-founded. In this paper AC termination is established by *AC-compatible simplification orders*, i.e., proper orders $>$ closed under contexts and substitutions that have the subterm property $f(t_1,\dots,t_n) > t_i$ for all $1 \leqslant i \leqslant n$ and satisfy $=_{\mathsf{AC}} \cdot > \cdot =_{\mathsf{AC}} \subseteq >$. A proper order $>$ is *AC-total* if $s > t$, $t > s$ or $s =_{\mathsf{AC}} t$, for all ground terms s and t. A pair $(\gtrsim, >)$ consisting of a preorder $\gtrsim$ and a proper order $>$ is said to be an *order pair* if the *compatibility* condition $\gtrsim \cdot > \cdot \gtrsim \subseteq >$ holds.

▶ **Definition 1.** Let $>$ be a proper order and $\gtrsim$ be a preorder on a set A. The *lexicographic extensions* $>^{\mathsf{lex}}$ and $\gtrsim^{\mathsf{lex}}$ are defined on finite sequences over A as follows:

- $\vec{x} \gtrsim^{\mathsf{lex}} \vec{y}$ if $\vec{x} \sqsupset_k \vec{y}$ for some $1 \leqslant k \leqslant n$,
- $\vec{x} >^{\mathsf{lex}} \vec{y}$ if $\vec{x} \sqsupset_k \vec{y}$ for some $1 \leqslant k < n$.

Here $\vec{x} = (x_1,\dots,x_n)$, $\vec{y} = (y_1,\dots,y_n)$, and $\vec{x} \sqsupset_k \vec{y}$ denotes the following condition: $x_i \gtrsim y_i$ for all $i \leqslant k$ and either $k < n$ and $x_{k+1} > y_{k+1}$ or $k = n$. The *multiset extensions* $>^{\mathsf{mul}}$ and $\gtrsim^{\mathsf{mul}}$ are defined on finite multisets over A as follows:

- $M \gtrsim^{\mathsf{mul}} N$ if $M \sqsupset^k N$ for some $0 \leqslant k \leqslant \min(m,n)$,
- $M >^{\mathsf{mul}} N$ if $M \sqsupset^k N$ for some $0 \leqslant k \leqslant \min(m-1,n)$.

Here $M = \{x_1,\dots,x_m\}$, $N = \{y_1,\dots,y_n\}$, and $M \sqsupset^k N$ denotes the following condition: $x_j \gtrsim y_j$ for all $j \leqslant k$, and for every $k < j \leqslant n$ there is some $k < i \leqslant m$ with $x_i > y_j$. The *lexicographic product* $(\gtrsim_1, >_1) \otimes \cdots \otimes (\gtrsim_n, >_n)$ is the pair $(\gtrsim, >)$ defined on sequences of length n as follows:

- $(x_1,\dots,x_n) \gtrsim (y_1,\dots,y_n)$ if $x_i \gtrsim_i y_i$ for all $i \leqslant n$,
- $(x_1,\dots,x_n) > (y_1,\dots,y_n)$ if there is some $i \leqslant n$ such that $x_i >_i y_i$ and $x_j \gtrsim_j y_j$ for all $1 \leqslant j < i$.

Note that these extended relations depend on both $\gtrsim$ and $>$.

The following result is folklore; a recent formalization of multiset extensions in Isabelle/HOL is presented in [10].

▶ **Theorem 2.** *The class of order pairs is closed under lexicographic and multiset extensions as well as lexicographic products.* ◀

3 Steinbach's Order

In this section we recall the AC-compatible KBO $>_{\mathsf{S}}$ of Steinbach [8].[2] Just like standard KBO, $>_{\mathsf{S}}$ depends on a precedence and an admissible weight function. A *precedence* $>$ is a proper order on $\mathcal{F}$. A *weight function* (w, w_0) for a signature $\mathcal{F}$ consists of a mapping $w\colon \mathcal{F} \to \mathbb{N}$ and a constant $w_0 > 0$ such that $w(c) \geqslant w_0$ for every constant $c \in \mathcal{F}$. The *weight* of a term t is recursively computed as follows: $w(t) = w_0$ if $t \in \mathcal{V}$ and $w(f(t_1,\dots,t_n)) = w(f) + w(t_1) + \cdots + w(t_n)$. A weight function (w, w_0) is *admissible* for $>$ if every unary f with $w(f) = 0$ satisfies $f > g$ for all function symbols g different from f. Throughout this paper we assume admissibility.

The *top-flattening* of a term t with respect to an AC-symbol f is the multiset $\triangledown_f(t)$ defined inductively as follows: $\triangledown_f(t) = \{t\}$ if $\mathsf{root}(t) \neq f$ and $\triangledown_f(f(t_1,t_2)) = \triangledown_f(t_1) \uplus \triangledown_f(t_2)$.

▶ **Definition 3.** Let $>$ be a precedence and (w, w_0) a weight function. The order $>_{\mathsf{S}}$ is inductively defined as follows: $s >_{\mathsf{S}} t$ if $|s|_x \geqslant |t|_x$ for all $x \in \mathcal{V}$ and either $w(s) > w(t)$, or $w(s) = w(t)$ and one of the following alternatives holds:

(0) $s = f^k(t)$ and $t \in \mathcal{V}$ for some $k > 0$,

(1) $s = f(s_1,\dots,s_n)$, $t = g(t_1,\dots,t_m)$, and $f > g$,

(2) $s = f(s_1,\dots,s_n)$, $t = f(t_1,\dots,t_n)$, $f \notin \mathcal{F}_{\mathsf{AC}}$, $(s_1,\dots,s_n) >_{\mathsf{S}}^{\mathsf{lex}} (t_1,\dots,t_n)$,

(3) $s = f(s_1,s_2)$, $t = f(t_1,t_2)$, $f \in \mathcal{F}_{\mathsf{AC}}$, and $\triangledown_f(s) >_{\mathsf{S}}^{\mathsf{mul}} \triangledown_f(t)$.

The relation $=_{\mathsf{AC}}$ is used as preorder in $>_{\mathsf{S}}^{\mathsf{lex}}$ and $>_{\mathsf{S}}^{\mathsf{mul}}$.

Cases (0)–(2) are the same as in the classical Knuth-Bendix order. In case (3) terms rooted by the same AC symbol f are treated by comparing their top-flattenings in the multiset extension of $>_{\mathsf{S}}$.

▶ **Example 4.** Consider the signature $\mathcal{F} = \{\mathsf{a},\mathsf{f},\mathsf{g}\}$ with $\mathsf{f} \in \mathcal{F}_{\mathsf{AC}}$, precedence $\mathsf{g} > \mathsf{f} > \mathsf{a}$ and admissible weight function (w_0, w) with $w(\mathsf{f}) = w(\mathsf{g}) = 0$ and $w_0 = w(\mathsf{a}) = 1$. Let $\mathcal{R}_1$ be the following ground TRS:

$$\mathsf{g}(\mathsf{f}(\mathsf{a},\mathsf{a})) \to \mathsf{f}(\mathsf{g}(\mathsf{a}),\mathsf{g}(\mathsf{a})) \qquad (1) \qquad\qquad \mathsf{f}(\mathsf{a},\mathsf{g}(\mathsf{g}(\mathsf{a}))) \to \mathsf{f}(\mathsf{g}(\mathsf{a}),\mathsf{g}(\mathsf{a})) \qquad (2)$$

For $1 \leqslant i \leqslant 2$, let ℓ_i and r_i be the left-hand side and right-hand side of rule (i), $S_i = \triangledown_{\mathsf{f}}(\ell_i)$ and $T_i = \triangledown_{\mathsf{f}}(r_i)$. Both rules vacuously satisfy the variable condition. We have $w(\ell_1) = 2 = w(r_1)$ and $\mathsf{g} > \mathsf{f}$, so $\ell_1 >_{\mathsf{S}} r_1$ holds by case (1). We have $w(\ell_2) = 2 = w(r_2)$, $S_2 = \{\mathsf{a}, \mathsf{g}(\mathsf{g}(\mathsf{a}))\}$, and $T_2 = \{\mathsf{g}(\mathsf{a}), \mathsf{g}(\mathsf{a})\}$. Since $\mathsf{g}(\mathsf{a}) >_{\mathsf{S}} \mathsf{a}$ holds by case (1), $\mathsf{g}(\mathsf{g}(\mathsf{a})) >_{\mathsf{S}} \mathsf{g}(\mathsf{a})$ holds by case (2), and therefore $\ell_2 >_{\mathsf{S}} r_2$ by case (3).

▶ **Theorem 5** ([8])**.** *If every symbol in $\mathcal{F}_{\mathsf{AC}}$ is minimal with respect to $>$*[3] *then $>_{\mathsf{S}}$ is an AC-compatible simplification order.* ◀

In Section 7 we reprove[4] Theorem 5 by showing that $>_{\mathsf{S}}$ is a special case of our new AC-compatible Knuth-Bendix order. Due to the restriction on AC symbols, partial precedences increase the power of $>_{\mathsf{S}}$ in that multiple AC symbols are allowed. This is confirmed by the experimental results in Section 7.

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[2] Non-AC function symbols in [8] can have arbitrary status. To simplify the presentation, we do not consider status in this paper.

[3] In [8] AC symbols are further required to have weight 0 because terms are flattened and, unlike [9], the weight function is not modified to take flattening into account.

[4] The counterexample in [9] against the monotonicity of $>_{\mathsf{S}}$ is invalid as the condition that AC symbols are *minimal* in the precedence is not satisfied.

4 Korovin and Voronkov's Orders

In this section we recall the orders of [9]. The first one is defined on ground terms. The difference with $>_{\mathsf{S}}$ is that in case (3) of the definition a further case analysis is performed based on a decomposition of the multisets S and T into terms that are rooted by a function symbol greater than f in the precedence and variables. Rather than recursively comparing these multisets with the order being defined, a lighter non-recursive version is used in which the weights and root symbols of the terms involved are considered. This is formally defined below. Throughout this section $>$ is a *total* precedence.

▶ **Definition 6.** Given a weight function (w, w_0) and a precedence $>$, the relations $=_{\langle w,>\rangle}$ and $>_{\langle w,>\rangle}$ are defined as follows:

- $s =_{\langle w,>\rangle} t$ if $w(s) = w(t)$ and $\mathsf{root}(s) = \mathsf{root}(t)$,
- $s >_{\langle w,>\rangle} t$ if either $w(s) > w(t)$ or both $w(s) = w(t)$ and $\mathsf{root}(s) > \mathsf{root}(t)$.

Note that $=_{\langle w,>\rangle}$ is an equivalence relation that contains $=_{\mathsf{AC}}$. Given a multiset T of terms, a function symbol f, and a binary relation R on function symbols, we define the following submultisets of T: $T\restriction_{\mathcal{V}} = \{x \in T \mid x \in \mathcal{V}\}$ and $T\restriction_f^R = \{t \in T \setminus \mathcal{V} \mid \mathsf{root}(t)\ R\ f\}$. In the following definition we use the precedence for R in $T\restriction_f^R$.

▶ **Definition 7.** Let $>$ be a precedence and (w, w_0) a weight function. The order $>_{\mathsf{KV}}$ is inductively defined on ground terms as follows: $s >_{\mathsf{KV}} t$ if either $w(s) > w(t)$, or $w(s) = w(t)$ and one of the following alternatives holds:

(0) $s = f(s_1, \ldots, s_n)$, $t = g(t_1, \ldots, t_m)$, and $f > g$,

(1) $s = f(s_1, \ldots, s_n)$, $t = f(t_1, \ldots, t_n)$, $f \notin \mathcal{F}_{\mathsf{AC}}$, $(s_1, \ldots, s_n) >_{\mathsf{KV}}^{\mathsf{lex}} (t_1, \ldots, t_n)$,

(2) $s = f(s_1, s_2)$, $t = f(t_1, t_2)$, $f \in \mathcal{F}_{\mathsf{AC}}$, and for $S = \nabla_f(s)$ and $T = \nabla_f(t)$

  (a) $S\restriction_f^> >_{\langle w,>\rangle}^{\mathsf{mul}} T\restriction_f^>$, or

  (b) $S\restriction_f^> =_{\langle w,>\rangle}^{\mathsf{mul}} T\restriction_f^>$ and $|S| > |T|$, or

  (c) $S\restriction_f^> =_{\langle w,>\rangle}^{\mathsf{mul}} T\restriction_f^>$, $|S| = |T|$, and $S >_{\mathsf{KV}}^{\mathsf{mul}} T$.

The relation $=_{\mathsf{AC}}$ is used as preorder in $>_{\mathsf{KV}}^{\mathsf{lex}}$ and $>_{\mathsf{KV}}^{\mathsf{mul}}$ whereas $=_{\langle w,>\rangle}$ is used in $>_{\langle w,>\rangle}^{\mathsf{mul}}$.

Only in cases (2) and (3c) the order $>_{\mathsf{KV}}$ is used recursively. In case (3) terms rooted by the same AC symbol f are compared by extracting from the top-flattenings S and T the multisets ($S\restriction_f^>$ and $T\restriction_f^>$) consisting of all terms rooted by a function symbol greater than f in the precedence. If $S\restriction_f^>$ is larger than $T\restriction_f^>$ in the multiset extension of $>_{\langle w,>\rangle}$, we conclude in case (3a). Otherwise the multisets must be equal (with respect to $=_{\langle w,>\rangle}^{\mathsf{mul}}$). If S has more terms than T, we conclude in case (3b). In the final case (3c) S and T have the same number of terms and we compare S and T in the multiset extension of $>_{\mathsf{KV}}$.

▶ **Theorem 8** ([9])**.** *$>_{\mathsf{KV}}$ is an AC-compatible AC-total simplification order on ground terms.*

The following example shows that $>_{\mathsf{KV}}$ does not subsume $>_{\mathsf{S}}$ for ground TRSs.

▶ **Example 9.** Consider again the ground TRS $\mathcal{R}_1$ of Example 4. To orient rule (1) with $>_{\mathsf{KV}}$, the weight of the unary function symbol g must be 0 and admissibility demands $\mathsf{g} > \mathsf{a}$ and $\mathsf{g} > \mathsf{f}$. Hence rule (1) is handled by case (1) of the definition. For rule (2), the multisets $S = \{\mathsf{a}, \mathsf{g}(\mathsf{g}(\mathsf{a}))\}$ and $T = \{\mathsf{g}(\mathsf{a}), \mathsf{g}(\mathsf{a})\}$ are compared in case (3). There are two possibilities to make $>$ total. If $\mathsf{a} > \mathsf{f}$ then $S\restriction_f^> = S$ and if $\mathsf{f} > \mathsf{a}$ then $S\restriction_f^> = \{\mathsf{g}(\mathsf{g}(\mathsf{a}))\}$. In both cases we have $T\restriction_f^> = T$. Note that neither $\mathsf{a} >_{\langle w,>\rangle} \mathsf{g}(\mathsf{a})$ nor $\mathsf{g}(\mathsf{g}(\mathsf{a})) >_{\langle w,>\rangle} \mathsf{g}(\mathsf{a})$ holds. Hence case

(3a) does not apply. But also cases (3b) and (3c) are not applicable as $\mathsf{g}(\mathsf{g}(\mathsf{a})) =_{\langle w,>\rangle} \mathsf{g}(\mathsf{a})$ and $\mathsf{a} \neq_{\langle w,>\rangle} \mathsf{g}(\mathsf{a})$. Hence, independent of the choice of $>$, $\mathcal{R}_1$ cannot be proved terminating by $>_{\mathsf{KV}}$.

Next we present the second order of [9], the extension of $>_{\mathsf{KV}}$ to non-ground terms. Since it coincides with $>_{\mathsf{KV}}$ on ground terms, we use the same notation for the order. First we extend the relations $=_{\langle w,>\rangle}$ and $>_{\langle w,>\rangle}$ of Definition 6 to non-ground terms.

▶ **Definition 10.** Given a weight function (w, w_0) and a precedence $>$, the relations $=_{\langle w,>\rangle}$ and $>_{\langle w,>\rangle}$ are defined as follows:

- $s =_{\langle w,>\rangle} t$ if $|s|_x = |t|_x$ for all $x \in \mathcal{V}$, $w(s) = w(t)$ and $\mathsf{root}(s) = \mathsf{root}(t)$,
- $s >_{\langle w,>\rangle} t$ if $|s|_x \geqslant |t|_x$ for all $x \in \mathcal{V}$ and either $w(s) > w(t)$ or both $w(s) = w(t)$ and $\mathsf{root}(s) > \mathsf{root}(t)$.

In case (3) of the following definition, also variables appearing in the top-flattenings of S and T are taken into account in the first multiset comparison.

▶ **Definition 11.** Let $>$ be a precedence and (w, w_0) a weight function. The order $>_{\mathsf{KV}}$ is inductively defined as follows: $s >_{\mathsf{KV}} t$ if $|s|_x \geqslant |t|_x$ for all $x \in \mathcal{V}$ and either $w(s) > w(t)$, or $w(s) = w(t)$ and one of the following alternatives holds:

(0) $s = f^k(t)$ and $t \in \mathcal{V}$ for some $k > 0$,

(1) $s = f(s_1, \dots, s_n)$, $t = g(t_1, \dots, t_m)$, and $f > g$,

(2) $s = f(s_1, \dots, s_n)$, $t = f(t_1, \dots, t_n)$, $f \notin \mathcal{F}_{\mathsf{AC}}$, $(s_1, \dots, s_n) >^{\mathsf{lex}}_{\mathsf{KV}} (t_1, \dots, t_n)$,

(3) $s = f(s_1, s_2)$, $t = f(t_1, t_2)$, $f \in \mathcal{F}_{\mathsf{AC}}$, and for $S = \nabla_f(s)$ and $T = \nabla_f(t)$

  (a) $S{\restriction}^{>}_{f} >^{\mathsf{mul}}_{\langle w,>\rangle} T{\restriction}^{>}_{f} \uplus T{\restriction}_{\mathcal{V}} - S{\restriction}_{\mathcal{V}}$, or

  (b) $S{\restriction}^{>}_{f} =^{\mathsf{mul}}_{\langle w,>\rangle} T{\restriction}^{>}_{f} \uplus T{\restriction}_{\mathcal{V}} - S{\restriction}_{\mathcal{V}}$ and $|S| > |T|$, or

  (c) $S{\restriction}^{>}_{f} =^{\mathsf{mul}}_{\langle w,>\rangle} T{\restriction}^{>}_{f} \uplus T{\restriction}_{\mathcal{V}} - S{\restriction}_{\mathcal{V}}$, $|S| = |T|$, and $S >^{\mathsf{mul}}_{\mathsf{KV}} T$.

The relation $=_{\mathsf{AC}}$ is used as preorder in $>^{\mathsf{lex}}_{\mathsf{KV}}$ and $>^{\mathsf{mul}}_{\mathsf{KV}}$ whereas $=_{\langle w,>\rangle}$ is used in $>^{\mathsf{mul}}_{\langle w,>\rangle}$.

Contrary to what is claimed in [9], the order $>_{\mathsf{KV}}$ of Definition 11 is not a simplification order because it lacks the monotonicity property (i.e., $>_{\mathsf{KV}}$ is not closed under contexts), as shown in the following example.

▶ **Example 12.** Let f be an AC symbol and g a unary function symbol with $w(\mathsf{g}) = 0$ and $\mathsf{g} > \mathsf{f}$.[5] We obviously have $\mathsf{g}(x) >_{\mathsf{KV}} x$. However, $\mathsf{f}(\mathsf{g}(x), y) >_{\mathsf{KV}} \mathsf{f}(x, y)$ does not hold. Let $S = \nabla_{\mathsf{f}}(s) = \{\mathsf{g}(x), y\}$ and $T = \nabla_{\mathsf{f}}(t) = \{x, y\}$. We have $S{\restriction}^{>}_{\mathsf{f}} = \{\mathsf{g}(x)\}$, $S{\restriction}_{\mathcal{V}} = \{y\}$, $T{\restriction}^{>}_{\mathsf{f}} = \varnothing$, and $T{\restriction}_{\mathcal{V}} = \{x, y\}$. Note that $\mathsf{g}(x) >_{\langle w,>\rangle} x$ does not hold since $\mathsf{g} \not> x$. Hence case (3a) in Definition 11 does not apply. But also $\mathsf{g}(x) =_{\langle w,>\rangle} x$ does not hold, excluding cases (3b) and (3c).

Despite the lack of monotonicity, the example does not disprove the soundness of $>_{\mathsf{KV}}$ for proving AC termination of rewrite systems; note that also $\mathsf{f}(x, y) >_{\mathsf{KV}} \mathsf{f}(\mathsf{g}(x), y)$ does not hold. We now present an extension $>_{\mathsf{KV}'}$ of $>_{\mathsf{KV}}$ which has all desired properties. Inspecting Example 12, a natural idea is to extend the relation $>_{\langle w,>\rangle}$ of Definition 10 such that a non-variable term is larger than a variable contained in it. But then closure under substitutions will be lost. Instead we extend the equivalence relation $=_{\langle w,>\rangle}$ to a suitable preorder $\geqslant_{\langle w,>\rangle}$.

[5] The use of a unary function of weight 0 is not crucial, cf. Example 42 in the appendix.

▶ **Definition 13.** Given a weight function (w, w_0) and a precedence $>$, the relation $\geqslant_{\langle w,>\rangle}$ is defined as follows:

- $s \geqslant_{\langle w,>\rangle} t$ if $|s|_x \geqslant |t|_x$ for all $x \in \mathcal{V}$ and either $w(s) > w(t)$, or $w(s) = w(t)$ and either $\mathsf{root}(s) \geqslant \mathsf{root}(t)$ or $t \in \mathcal{V}$.

The order $>_{\mathsf{KV'}}$ is obtained as in Definition 11 after replacing $=^{\mathsf{mul}}_{\langle w,>\rangle}$ by $\geqslant^{\mathsf{mul}}_{\langle w,>\rangle}$ in cases (3b) and (3c), and using $\geqslant_{\langle w,>\rangle}$ as preorder in $>^{\mathsf{mul}}_{\langle w,>\rangle}$ in case (3a).

Note that $\geqslant_{\langle w,>\rangle}$ is a preorder that contains $=_{\mathsf{AC}}$.

▶ **Example 14.** Consider again Example 12. We have $\mathsf{f}(\mathsf{g}(x), y) >_{\mathsf{KV'}} \mathsf{f}(x, y)$ because now case (3c) applies: $S{\restriction}^{>}_{\mathsf{f}} = \{\mathsf{g}(x)\} \geqslant^{\mathsf{mul}}_{\langle w,>\rangle} \{x\} = T{\restriction}^{>}_{\mathsf{f}} \uplus T{\restriction}_{\mathcal{V}} - S{\restriction}_{\mathcal{V}}$, $|S| = 2 = |T|$, and $S = \{\mathsf{g}(x), y\} >^{\mathsf{mul}}_{\mathsf{KV'}} \{x, y\} = T$ because $\mathsf{g}(x) >_{\mathsf{KV'}} x$.

The following result is obvious from the definitions.

▶ **Lemma 15.** *For every precedence and weight function,* $>_{\mathsf{KV}} \subseteq >_{\mathsf{KV'}}$. ◀

In the appendix we prove that $>_{\mathsf{KV'}}$ is an AC-compatible simplification order. In combination with the above lemma this entails that, despite the lack of monotonicity, $>_{\mathsf{KV}}$ is a sound method for establishing AC termination.

▶ **Theorem 16.** $>_{\mathsf{KV'}}$ *is an AC-compatible simplification order.*

5 AC-KBO

In this section we present another AC-compatible simplification order. In contrast to $>_{\mathsf{KV'}}$, $>_{\mathsf{ACKBO}}$ contains $>_{\mathsf{S}}$. Moreover, its definition is simpler than $>_{\mathsf{KV'}}$ since we avoid the use of an auxiliary order in case (3). Hence it will be used as the basis for the extension discussed in Section 6.

▶ **Definition 17.** Let $>$ be a precedence and (w, w_0) a weight function. We define $>_{\mathsf{ACKBO}}$ inductively as follows: $s >_{\mathsf{ACKBO}} t$ if $|s|_x \geqslant |t|_x$ for all $x \in \mathcal{V}$ and either $w(s) > w(t)$, or $w(s) = w(t)$ and one of the following alternatives holds:

(0) $s = f^k(t)$ and $t \in \mathcal{V}$ for some $k > 0$,

(1) $s = f(s_1, \dots, s_n)$, $t = g(t_1, \dots, t_m)$, and $f > g$,

(2) $s = f(s_1, \dots, s_n)$, $t = f(t_1, \dots, t_n)$, $f \notin \mathcal{F}_{\mathsf{AC}}$, $(s_1, \dots, s_n) >^{\mathsf{lex}}_{\mathsf{ACKBO}} (t_1, \dots, t_n)$,

(3) $s = f(s_1, s_2)$, $t = f(t_1, t_2)$, $f \in \mathcal{F}_{\mathsf{AC}}$, and for $S = \nabla_f(s)$ and $T = \nabla_f(t)$

  (a) $S{\restriction}^{\not<}_{f} >^{\mathsf{mul}}_{\mathsf{ACKBO}} T{\restriction}^{\not<}_{f} \uplus T{\restriction}_{\mathcal{V}} - S{\restriction}_{\mathcal{V}}$, or

  (b) $S{\restriction}^{\not<}_{f} =^{\mathsf{mul}}_{\mathsf{AC}} T{\restriction}^{\not<}_{f} \uplus T{\restriction}_{\mathcal{V}} - S{\restriction}_{\mathcal{V}}$ and $|S| > |T|$, or

  (c) $S{\restriction}^{\not<}_{f} =^{\mathsf{mul}}_{\mathsf{AC}} T{\restriction}^{\not<}_{f} \uplus T{\restriction}_{\mathcal{V}} - S{\restriction}_{\mathcal{V}}$, $|S| = |T|$, and $S{\restriction}^{<}_{f} >^{\mathsf{mul}}_{\mathsf{ACKBO}} T{\restriction}^{<}_{f}$.

The relation $=_{\mathsf{AC}}$ is used as preorder in $>^{\mathsf{lex}}_{\mathsf{ACKBO}}$ and $>^{\mathsf{mul}}_{\mathsf{ACKBO}}$.

Note that in case (3c) we compare the multisets $S{\restriction}^{<}_{f}$ and $T{\restriction}^{<}_{f}$ rather than S and T in the multiset extension of $>_{\mathsf{ACKBO}}$.

We prove that $>_{\mathsf{ACKBO}}$ is an AC-compatible simplification order. First we show that $(=_{\mathsf{AC}}, >_{\mathsf{ACKBO}})$ is an order pair. To facilitate the proof, we decompose $>_{\mathsf{ACKBO}}$ into several orders. We write

- $s >_{01} t$ if $|s|_x \geqslant |t|_x$ for all $x \in \mathcal{V}$ and either $w(s) > w(t)$ or $w(s) = w(t)$ and case (0) or case (1) of Definition 17 applies,

- $s >_{23,k} t$ if $|s|, |t| \leqslant k$, $|s|_x \geqslant |t|_x$ for all $x \in \mathcal{V}$, $w(s) = w(t)$, and case (2) or case (3) applies.

The union of $>_{01}$ and $>_{23,k}$ is denoted by $>_k$. The next lemma states straightforward properties.

▶ **Lemma 18.** *(1)* $>_{\mathsf{ACKBO}} = \bigcup\{>_k \mid k \in \mathbb{N}\}$, *(2)* $(=_{\mathsf{AC}}, >_{01})$ *is an order pair, (3)* $(>_{01} \cdot >_k) \cup (>_k \cdot >_{01}) \subseteq >_{01}$.

Let $(\gtrsim, >)$ be an order pair on terms. Let $\sqsupset \in \{\gtrsim, >\}$ and $f \in \mathcal{F}$. We define the relation $\sqsupset^f$ on multisets as follows: $S \sqsupset^f T$ if $S{\upharpoonright}_f^{\not<} \sqsupset^{\mathsf{mul}} T{\upharpoonright}_f^{\not<} \uplus T{\upharpoonright}_{\mathcal{V}} - S{\upharpoonright}_{\mathcal{V}}$.

▶ **Lemma 19.** *If* $(\gtrsim, >)$ *is an order pair then* $(\gtrsim^f, >^f)$ *is an order pair.*

We employ the following simple criterion to construct order pairs, which enables us to prove correctness in a modular way.

▶ **Lemma 20.** *Let* $(\gtrsim, >_k)$ *be order pairs for* $k \in \mathbb{N}$ *with* $>_k \subseteq >_{k+1}$. *If* $>$ *is the union of all* $>_k$ *then* $(\gtrsim, >)$ *is an order pair.*

Let $(\gtrsim_{abc}, >_{abc})$ abbreviate $(=^f_{\mathsf{AC}}, >^f_{\mathsf{ACKBO}}) \otimes (=, >) \otimes (=^{\mathsf{mul}}_{\mathsf{AC}}, >^{\mathsf{mul}}_{\mathsf{ACKBO}})$. Case (3) of Definition 17 is equivalent to $(S, |S|, S{\upharpoonright}_f^{<}) >_{abc} (T, |T|, T{\upharpoonright}_f^{<})$ where $s = f(s_1, s_2)$, $t = f(t_1, t_2)$, $f \in \mathcal{F}_{\mathsf{AC}}$, $S = \nabla_f(s)$, and $T = \nabla_f(t)$.

▶ **Lemma 21.** $(=_{\mathsf{AC}}, >_{\mathsf{ACKBO}})$ *is an order pair.*

Proof. According to Lemmata 20 and 18(1), it is sufficient to prove that $(=_{\mathsf{AC}}, >_k)$ is an order pair for all $k \in \mathbb{N}$. Due to Lemma 18(2,3) it suffices to prove that $(=_{\mathsf{AC}}, >_{23,k})$ is an order pair. We perform induction on k.

- If $k \leqslant 1$ then $>_{23,k}$ is empty, hence $(=_{\mathsf{AC}}, >_{23,k})$ trivially forms a order pair.
- Suppose $k > 1$. By the induction hypothesis $(=_{\mathsf{AC}}, >_{23,k-1})$ is an order pair, and thus $(=_{\mathsf{AC}}, >_{k-1})$ is so. Let $t = f(t_1, \ldots, t_n)$ be an arbitrary term with $|t| \leqslant k$. We have $|t_i| \leqslant k - 1$ for all $1 \leqslant i \leqslant n$, and if $f \in \mathcal{F}_{\mathsf{AC}}$ then $|u| \leqslant k - 1$ for all $u \in \nabla_f(t)$. Therefore, Theorem 2 implies that $>_{23,k}$ is a proper order, as $>_{abc}$ characterizes case (3). The compatibility requirement $=_{\mathsf{AC}} \cdot >_{23,k} \cdot =_{\mathsf{AC}} \subseteq >_{23,k}$ follows from the induction hypothesis and the fact that $\nabla_f(s) =^{\mathsf{mul}}_{\mathsf{AC}} \nabla_f(t)$ whenever $s =_{\mathsf{AC}} t$ and $f \in \mathcal{F}_{\mathsf{AC}}$. ◀

The subterm property is an easy consequence of transitivity and admissibility.

▶ **Lemma 22.** $>_{\mathsf{ACKBO}}$ *has the subterm property.*

Proof. Using the fact that $>_{\mathsf{ACKBO}}$ is transitive, we only need to compare a term $s = f(s_1, \ldots, s_n)$ with its direct arguments $s_1, \ldots, s_n$. We prove that $s >_{\mathsf{ACKBO}} s_i$ by induction on s. If $n \geqslant 2$ or if $w(f) > 0$ then $w(s) > w(s_i)$. In the remaining case f is a unary function symbol with $w(f) = 0$. We distinguish three further cases. If s_1 is a variable then case (0) of Definition 17 applies. If $s_1 = f(t_1)$ then $s_1 >_{\mathsf{ACKBO}} t_1$ by the induction hypothesis and thus $s >_{\mathsf{ACKBO}} t$ by case (2). In the remaining case we have $s_1 = g(t_1, \ldots, t_m)$ with $g \neq f$. Admissibility yields $f > g$ and thus case (1) applies. ◀

Next we prove that $>_{\mathsf{ACKBO}}$ is closed under contexts. The following lemma is an auxiliary result needed for its proof. In order to reuse this lemma for the correctness proof of $>_{\mathsf{KV'}}$ in the appendix, we prove it using a condition that is weaker than the subterm property.

▶ **Lemma 23.** *Let* $(\gtrsim, >)$ *be an order pair and* $f \in \mathcal{F}_{\mathsf{AC}}$ *with* $f(u, v) > u, v$ *for all terms* u *and* v. *If* $s \gtrsim t$ *then* $\{s\} \gtrsim^{\mathsf{mul}} \nabla_f(t)$ *or* $\{s\} >^{\mathsf{mul}} \nabla_f(t)$. *If* $s > t$ *then* $\{s\} >^{\mathsf{mul}} \nabla_f(t)$. ◀

In the following proof of closure under contexts, admissibility is essential. This is in contrast to the corresponding result for standard KBO.

▶ **Lemma 24.** *If (w, w_0) is admissible for $>$ then $>_{\mathsf{ACKBO}}$ is closed under contexts.*

Proof. Suppose $s >_{\mathsf{ACKBO}} t$. We consider the context $h(\Box, u)$ with $h \in \mathcal{F}_{\mathsf{AC}}$ and u an arbitrary term, and prove that $s' = h(s, u) >_{\mathsf{ACKBO}} h(t, u) = t'$. Closure under contexts of $>_{\mathsf{ACKBO}}$ follows then by induction; contexts rooted by a non-AC symbol are handled as in the proof for standard KBO.

If $w(s) > w(t)$ then obviously $w(s') > w(t')$. So we assume $w(s) = w(t)$. Let $S = \nabla_h(s)$, $T = \nabla_h(t)$, and $U = \nabla_h(u)$. Note that $\nabla_h(s') = S \uplus U$ and $\nabla_h(t') = T \uplus U$. Because $>^{\mathsf{mul}}_{\mathsf{ACKBO}}$ is closed under multiset sum, it suffices to show that one of the cases (3a)–(3c) of Definition 17 holds for S and T. Let $f = \mathsf{root}(s)$ and $g = \mathsf{root}(t)$. We distinguish the following cases.

- Suppose $f \not\leqslant h$. We have $S = S{\upharpoonright}^{\not\leqslant}_{h} = \{s\}$, and from Lemmata 22 and 23 we obtain $S >^{\mathsf{mul}}_{\mathsf{ACKBO}} T$. Since T is a superset of $T{\upharpoonright}^{\not\leqslant}_{h} \uplus T{\upharpoonright}_{\mathcal{V}} - S{\upharpoonright}_{\mathcal{V}}$, (3a) applies.
- Suppose $f = h > g$. We have $T{\upharpoonright}^{\not\leqslant}_{h} \uplus T{\upharpoonright}_{\mathcal{V}} = \varnothing$. If $S{\upharpoonright}^{\not\leqslant}_{h} \neq \varnothing$, then (3a) applies. Otherwise, since AC symbols are binary and $T = \{t\}$, $|S| \geqslant 2 > 1 = |T|$. Hence (3b) applies.
- If $f = g = h$ then $s >_{\mathsf{ACKBO}} t$ must be derived by one of the cases (3a)–(3c) for S and T.
- Suppose $f, g < h$. We have $S{\upharpoonright}^{\not\leqslant}_{h} = T{\upharpoonright}^{\not\leqslant}_{h} \uplus T{\upharpoonright}_{\mathcal{V}} = \varnothing$, $|S| = |T| = 1$, and $S{\upharpoonright}^{<}_{h} = \{s\} >^{\mathsf{mul}}_{\mathsf{ACKBO}} \{t\} = T{\upharpoonright}^{<}_{h}$. Hence (3c) holds.

Note that $f \geqslant g$ since $w(s) = w(t)$ and $s >_{\mathsf{ACKBO}} t$. Moreover, if $t \in \mathcal{V}$ then $s = f^k(t)$ for some $k > 0$ with $w(f) = 0$, which entails $f > h$ because of admissibility. ◀

Closure under substitutions is the most tricky part. The reason is that by substituting AC-rooted terms for variables that appear in the top-flattening of a term, the structure of the term changes. In the proof, the multisets $\{t \in T \mid t \notin \mathcal{V}\}$, $\{t\sigma \mid t \in T\}$, and $\{\nabla_f(t) \mid t \in T\}$ are denoted by $T{\upharpoonright}_{\mathcal{F}}$, $T\sigma$, and $\nabla_f(T)$, respectively.

▶ **Lemma 25.** *Let $(\gtrsim, >)$ be an order pair on terms such that $\gtrsim$ and $>$ are closed under substitutions and let $f \in \mathcal{F}_{\mathsf{AC}}$ with $f(x, y) > x, y$. Consider terms s and t such that $S = \nabla_f(s)$, $T = \nabla_f(t)$, $S' = \nabla_f(s\sigma)$, and $T' = \nabla_f(t\sigma)$.*

(1) If $S >^{f} T$ then $S' >^{f} T'$.
(2) If $S \gtrsim^{f} T$ then either $S' >^{f} T'$ or $S' \gtrsim^{f} T'$. In the latter case, $|S| - |T| \leqslant |S'| - |T'|$ and $S'{\upharpoonright}^{<}_{f} >^{\mathsf{mul}} T'{\upharpoonright}^{<}_{f}$ whenever $S{\upharpoonright}^{<}_{f} >^{\mathsf{mul}} T{\upharpoonright}^{<}_{f}$.

Proof. Let v be an arbitrary term. By the assumption on $>$ we have either $\{v\} = \nabla_f(v)$ or both $\{v\} >^{\mathsf{mul}} \nabla_f(v)$ and $1 < |\nabla_f(v)|$. Hence, for any set V of terms, either $V = \nabla_f(V)$ or both $V >^{\mathsf{mul}} \nabla_f(V)$ and $|V| < |\nabla_f(V)|$. Moreover, for $V = \nabla_f(v)$, the equalities

$$\nabla_f(v\sigma){\upharpoonright}^{\not\leqslant}_{f} = V{\upharpoonright}^{\not\leqslant}_{f}\sigma \uplus \nabla_f(V{\upharpoonright}_{\mathcal{V}}\sigma){\upharpoonright}^{\not\leqslant}_{f} \qquad\qquad \nabla_f(v\sigma){\upharpoonright}_{\mathcal{V}} = \nabla_f(V{\upharpoonright}_{\mathcal{V}}\sigma){\upharpoonright}_{\mathcal{V}}$$

hold. To prove the lemma, assume $S \sqsupset^{f} T$ for $\sqsupset \in \{\gtrsim, >\}$. We have $S{\upharpoonright}^{\not\leqslant}_{f} \sqsupset^{\mathsf{mul}} T{\upharpoonright}^{\not\leqslant}_{f} \uplus U$ where $U = (T - S){\upharpoonright}_{\mathcal{V}}$. Since multiset extensions preserve closure under substitutions, $S{\upharpoonright}^{\not\leqslant}_{f}\sigma \sqsupset^{\mathsf{mul}} T{\upharpoonright}^{\not\leqslant}_{f}\sigma \uplus U\sigma$ follows. Using the above (in)equalities, we obtain

$$\begin{aligned} S'{\upharpoonright}^{\not\leqslant}_{f} = {} & S{\upharpoonright}^{\not\leqslant}_{f}\sigma \uplus \nabla_f(S{\upharpoonright}_{\mathcal{V}}\sigma){\upharpoonright}^{\not\leqslant}_{f} \\ \sqsupset^{\mathsf{mul}} & T{\upharpoonright}^{\not\leqslant}_{f}\sigma \uplus U\sigma \uplus \nabla_f(S{\upharpoonright}_{\mathcal{V}}\sigma){\upharpoonright}^{\not\leqslant}_{f} \\ O \quad & T{\upharpoonright}^{\not\leqslant}_{f}\sigma \uplus \nabla_f(U\sigma) \uplus \nabla_f(S{\upharpoonright}_{\mathcal{V}}\sigma){\upharpoonright}^{\not\leqslant}_{f} \end{aligned}$$

$$
\begin{array}{ll}
= & T{\restriction}_f^{\not<}\sigma \uplus \nabla_f(U\sigma){\restriction}_{\mathcal{V}} \uplus \nabla_f(U\sigma){\restriction}_f^{\not<} \uplus \nabla_f(U\sigma){\restriction}_f^{<} \uplus \nabla_f(S{\restriction}_{\mathcal{V}}\sigma){\restriction}_f^{\not<} \\
P & T{\restriction}_f^{\not<}\sigma \uplus \nabla_f(U\sigma){\restriction}_{\mathcal{V}} \uplus \nabla_f(T{\restriction}_{\mathcal{V}}\sigma){\restriction}_f^{\not<} \\
= & T{\restriction}_f^{\not<}\sigma \uplus \nabla_f(T{\restriction}_{\mathcal{V}}\sigma){\restriction}_f^{\not<} \uplus \nabla_f(T{\restriction}_{\mathcal{V}}\sigma){\restriction}_{\mathcal{V}} - \nabla_f(S{\restriction}_{\mathcal{V}}\sigma){\restriction}_{\mathcal{V}} \\
= & T'{\restriction}_f^{\not<} \uplus T'{\restriction}_{\mathcal{V}} - S'{\restriction}_{\mathcal{V}}
\end{array}
$$

Here O denotes $=$ if $U\sigma = \nabla_f(U\sigma)$ and $>^{\mathsf{mul}}$ if $|U\sigma| < |\nabla_f(U\sigma)|$, while P denotes $=$ if $U\sigma{\restriction}_f^{<} = \varnothing$ and $\supsetneq$ otherwise. Since $(\gtrsim^{\mathsf{mul}}, >^{\mathsf{mul}})$ is an order pair with $\supseteq \subseteq \gtrsim^{\mathsf{mul}}$ and $\supsetneq \subseteq >^{\mathsf{mul}}$, we obtain $S' \sqsupset^f T'$.

It remains to show (2). If $S' \not\succ^f T'$ then O and P are both $=$ and thus $U\sigma = \nabla_f(U\sigma)$ and $U\sigma{\restriction}_f^{<} = \varnothing$. Let $X = S{\restriction}_{\mathcal{V}} \cap T{\restriction}_{\mathcal{V}}$. We have $U = T{\restriction}_{\mathcal{V}} - X$.

- Since $|W{\restriction}_{\mathcal{F}}\sigma| = |W{\restriction}_{\mathcal{F}}|$ and $|W| \leqslant |\nabla_f(W)|$ for an arbitrary set W of terms, we have $|S'| \geqslant |S| - |X| + |\nabla_f(X\sigma)|$. From $|U\sigma| = |U| = |T{\restriction}_{\mathcal{V}}| - |X|$ we obtain $|T'| = |T{\restriction}_{\mathcal{F}}\sigma| + |\nabla_f(U\sigma)| + |\nabla_f(X\sigma)| = |T| - |X| + |\nabla_f(X\sigma)|$. Hence $|S| - |T| \leqslant |S'| - |T'|$ as desired.
- Suppose $S{\restriction}_f^{<} >^{\mathsf{mul}} T{\restriction}_f^{<}$. From $U\sigma{\restriction}_f^{<} = \varnothing$ we infer $T{\restriction}_{\mathcal{V}}\sigma{\restriction}_f^{<} \subseteq S{\restriction}_{\mathcal{V}}\sigma{\restriction}_f^{<}$. Because $S'{\restriction}_f^{<} = S{\restriction}_f^{<}\sigma \uplus S{\restriction}_{\mathcal{V}}\sigma{\restriction}_f^{<}$ and $T'{\restriction}_f^{<} = T{\restriction}_f^{<}\sigma \uplus T{\restriction}_{\mathcal{V}}\sigma{\restriction}_f^{<}$, closure under substitutions of $>^{\mathsf{mul}}$ (which it inherits from $>$ and $\gtrsim$) yields the desired $S'{\restriction}_f^{<} >^{\mathsf{mul}} T'{\restriction}_f^{<}$. ◀

▶ **Lemma 26.** $>_{\mathsf{ACKBO}}$ *is closed under substitutions.*

Proof. By induction on $|s|$ we verify that $s >_{\mathsf{ACKBO}} t$ implies $s\sigma >_{\mathsf{ACKBO}} t\sigma$. If $s >_{\mathsf{ACKBO}} t$ is obtained by cases (0)–(1) in Definition 17, the proof for standard KBO goes through. If (3a) or (3b) is used to obtain $s >_{\mathsf{ACKBO}} t$, according to Lemma 25 one of these cases also applies to $s\sigma >_{\mathsf{ACKBO}} t\sigma$. The final case is (3c). So $\nabla_f(s){\restriction}_f^{<} >^{\mathsf{mul}}_{\mathsf{ACKBO}} \nabla_f(t){\restriction}_f^{<}$. Suppose $s\sigma >_{\mathsf{ACKBO}} t\sigma$ cannot be obtained by (3a) or (3b). Lemma 25(2) yields $|\nabla_f(s\sigma)| = |\nabla_f(t\sigma)|$ and $\nabla_f(s\sigma){\restriction}_f^{<} >^{\mathsf{mul}}_{\mathsf{ACKBO}} \nabla_f(t\sigma){\restriction}_f^{<}$. Hence case (3c) is applicable to obtain $s\sigma >_{\mathsf{ACKBO}} t\sigma$. ◀

We arrive at the main theorem of this section.

▶ **Theorem 27.** $>_{\mathsf{ACKBO}}$ *is an AC-compatible simplification order.* ◀

Since we deal with finite non-variadic signatures, simplification orders are well-founded. The following example shows that AC-KBO is not *incremental*. This is in contrast to the AC-compatible recursive path order of Rubio [3]. However, this is not necessarily a disadvantage; actually, the example shows that by allowing partial precedences more TRSs can be proved to be AC terminating using AC-KBO.

▶ **Example 28.** Consider the TRS $\mathcal{R}$ consisting of the rules

$$\mathsf{a} \circ (\mathsf{b} \bullet \mathsf{c}) \to \mathsf{b} \circ \mathsf{f}(\mathsf{a} \bullet \mathsf{c}) \qquad\qquad \mathsf{a} \bullet (\mathsf{b} \circ \mathsf{c}) \to \mathsf{b} \bullet \mathsf{f}(\mathsf{a} \circ \mathsf{c})$$

over the signature $\{\mathsf{a}, \mathsf{b}, \mathsf{c}, \circ, \bullet\}$ with $\circ, \bullet \in \mathcal{F}_{\mathsf{AC}}$. By taking the precedence $\mathsf{f} > \mathsf{a}, \mathsf{b}, \mathsf{c}, \circ, \bullet$ and admissible weight function (w, w_0) with $w(\mathsf{f}) = w(\circ) = w(\bullet) = 0$, $w_0 = w(\mathsf{a}) = w(\mathsf{c}) = 1$, and $w(\mathsf{b}) = 2$, the resulting $>_{\mathsf{ACKBO}}$ orients both rules from left to right. It is essential that $\circ$ and $\bullet$ are incomparable in the precedence: We must have $w(\mathsf{f}) = 0$, so $\mathsf{f} > \mathsf{a}, \mathsf{b}, \mathsf{c}, \circ, \bullet$ is enforced by admissibility. If $\circ > \bullet$ then the first rule can only be oriented from left to right if $\mathsf{a} >_{\mathsf{ACKBO}} \mathsf{f}(\mathsf{a} \bullet \mathsf{c})$ holds, which contradicts the subterm property. If $\bullet > \circ$ then we use the second rule to obtain the impossible $\mathsf{a} >_{\mathsf{ACKBO}} \mathsf{f}(\mathsf{a} \bullet \mathsf{c})$.

The final theorem in this section is proved as in the case of the standard KBO.

▶ **Theorem 29.** *If* $>$ *is total then* $>_{\mathsf{ACKBO}}$ *is an AC-total order on ground terms.* ◀

6 Subterm Coefficients

Subterm coefficients were introduced in [11] in order to cope with rewrite rules like $\mathsf{f}(x) \to \mathsf{g}(x,x)$ which violate the variable condition. A *subterm coefficient function* is a partial mapping $s\colon \mathcal{F} \times \mathbb{N} \to \mathbb{N}$ such that for a function symbol f of arity n we have $s(f,i) > 0$ for all $1 \leqslant i \leqslant n$. Given a weight function (w, w_0) and a subterm coefficient function s, the weight of a term is inductively defined as follows: $w(t) = w_0$ if $t \in \mathcal{V}$ and $w(f(t_1,\dots,t_n)) = w(f) + s(f,1)\cdot w(t_1) + \cdots + s(f,n)\cdot w(t_n)$. The *variable coefficient* $\mathsf{vc}(x,t)$ of a variable x in a term t is inductively defined as follows: $\mathsf{vc}(x,t) = 1$ if $t = x$, $\mathsf{vc}(x,t) = 0$ if $t \in \mathcal{V} \setminus \{x\}$, and $\mathsf{vc}(x, f(t_1,\dots,t_n)) = s(f,1)\cdot \mathsf{vc}(x,t_1) + \cdots + s(f,n)\cdot \mathsf{vc}(x,t_n)$.

▶ **Definition 30.** The order $>^s_{\mathsf{ACKBO}}$ is obtained from Definition 17 by replacing the condition "$|s|_x \geqslant |t|_x$ for all $x \in \mathcal{V}$" with "$\mathsf{vc}(x,s) \geqslant \mathsf{vc}(x,t)$ for all $x \in \mathcal{V}$" and using the modified weight function introduced above.

In order to guarantee AC-compatibility of $>^s_{\mathsf{ACKBO}}$, the subterm coefficient function s has to assign the value 1 to arguments of AC-symbols. This follows by considering the terms $t + (u + v)$ and $(t + u) + v$ for an AC-symbol $+$ with $s(+,1) = m$ and $s(+,2) = n$. We have

$$w(t + (u + v)) = 2\cdot w(+) + m\cdot w(t) + mn\cdot w(u) + n^2\cdot w(v)$$
$$w((t + u) + v) = 2\cdot w(+) + m^2\cdot w(t) + mn\cdot w(u) + n\cdot w(v)$$

Since $w(t + (u + v)) = w((t + u) + v)$ must hold for all possible terms t, u, and v, it follows that $m = m^2$ and $n^2 = n$, implying $m = n = 1$.

The proof of the following theorem is very similar to the one of Theorem 27 and hence omitted.

▶ **Theorem 31.** *If $s(f,1) = s(f,2) = 1$ for every function symbol $f \in \mathcal{F}_{\mathsf{AC}}$ then $>^s_{\mathsf{ACKBO}}$ is an AC-compatible simplification order.* ◀

▶ **Example 32.** Consider a signature $\mathcal{F} = \{\mathsf{f},\mathsf{g},\mathsf{s},\mathsf{0}\}$ with $\mathsf{f} \in \mathcal{F}_{\mathsf{AC}}$, precedence $\mathsf{g} > \mathsf{f} > \mathsf{s} > \mathsf{0}$, and weights and subterm coefficients given by the following interpretation $\mathcal{A}$:

$$\mathsf{s}_{\mathcal{A}}(x) = x + 6 \qquad \mathsf{g}_{\mathcal{A}}(x,y) = 4x + 4y + 5 \qquad \mathsf{f}_{\mathcal{A}}(x,y) = x + y + 3 \qquad \mathsf{0}_{\mathcal{A}} = 1$$

Termination of the TRS $\mathcal{R}$

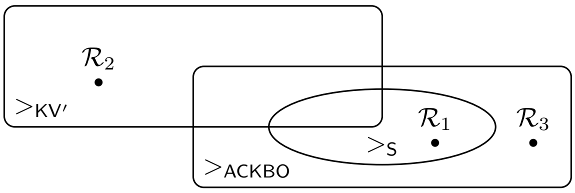
$$\mathsf{g}(\mathsf{0},\mathsf{f}(x,x)) \to x \quad (1) \qquad \mathsf{g}(\mathsf{s}(x),y) \to \mathsf{g}(\mathsf{f}(x,y),\mathsf{0}) \quad (3)$$
$$\mathsf{g}(x,\mathsf{s}(y)) \to \mathsf{g}(\mathsf{f}(x,y),\mathsf{0}) \quad (2) \qquad \mathsf{g}(\mathsf{f}(x,y),\mathsf{0}) \to \mathsf{f}(\mathsf{g}(x,\mathsf{0}),\mathsf{g}(y,\mathsf{0})) \quad (4)$$

was shown using AC dependency pairs in [12, Example 4.2.30]. It is easy to check that the first three rules result in a weight decrease. The left- and right-hand side of rule (4) are both interpreted as $4x + 4y + 21$, so both terms have weight 30, but since $\mathsf{g} > \mathsf{f}$ we conclude termination of $\mathcal{R}$ from case (1) in Definition 17 (30). Note that termination of $\mathcal{R}$ cannot be shown by AC-RPO or any of the previously considered versions of AC-KBO.

7 Comparison and Experiments

Fig. 1 summarizes the relationships between the orders introduced in Sections 3–5. Note that Steinbach's AC-KBO is a special case of the order defined in Section 5. This entails that the former is a correct AC-compatible simplification order.

▶ **Theorem 33.** *If every symbol in $\mathcal{F}_{\mathsf{AC}}$ is minimal with respect to $>$ then $>_{\mathsf{S}} = >_{\mathsf{ACKBO}}$.*



$\mathcal{R}_1$	Example 4 (9)
$\mathcal{R}_2$	Example 40 (Appendix)
$\mathcal{R}_3$	Example 41 (Appendix)

Figure 1 Comparison.

We ran experiments on a server equipped with eight dual-core AMD Opteron® processors 885 running at a clock rate of 2.6GHz with 64GB of main memory. The different versions of AC-KBO considered in this paper as well as AC-RPO [3] were implemented on top of $\mathsf{T_TT_2}$ using encodings in SAT/SMT. These encodings resemble those for standard KBO [13] and transfinite KBO [14], although the AC case additionally requires a careful implementation of multiset extensions based on different order pairs.

For termination experiments, our test set comprises all AC problems in TPDB,[6] all examples in this paper, some further problems harvested from the literature, and constraint systems produced by the completion tool mkbtt [15] (126 TRSs in total). The timeout was set to 60 seconds. The results are summarized in Table 1, where we list for each order the number of successful termination proofs, the total time, and the number of timeouts (column ∞). Although AC-RPO succeeds on more input problems, termination of several TRSs could only be established by (variants of) AC-KBO. We found that our definition of AC-KBO is about equally powerful as Korovin and Voronkov's order, but both are considerably more useful than Steinbach's version. When it comes to proving termination, we did not observe a difference between Definitions 11 and 13. Subterm coefficients clearly increase the success rate. The distinction t/p indicates whether total precedences were enforced or partial precedences allowed. The latter are obviously beneficial for Steinbach's order since multiple AC symbols are possible but AC-KBO gains only one additional problem (Example 28).

For completion experiments, we ran the normalized completion tool mkbtt with AC-RPO and (some versions of) AC-KBO for termination checks on 67 equational systems collected from the literature. The overall timeout was set to 200 seconds, the timeout for each termination check to 1.5 seconds. Table 1 summarizes our results, listing for each order the number of successful completions, the average time, and the number of timeouts.

All experimental details, source code, as well as $\mathsf{T_TT_2}$ binaries are available online.[7] The interested reader will also find experimental results for AC-dependency pairs combined with reduction pairs based on AC-RPO and AC-KBO (with argument filterings), as well as more sophisticated strategies involving dependency graphs, different reduction pairs, and usable rules.

▶ **Example 34.** Consider the following TRS $\mathcal{R}$ [4] for addition of binary numbers:

$$\# + \mathsf{0} \to \# \qquad x\mathsf{0} + y\mathsf{0} \to (x + y)\mathsf{0} \qquad x\mathsf{1} + y\mathsf{1} \to (x + y + \#\mathsf{1})\mathsf{0}$$
$$x + \# \to x \qquad x\mathsf{0} + y\mathsf{1} \to (x + y)\mathsf{1}$$

Here $+ \in \mathcal{F}_{\mathsf{AC}}$, $\mathsf{0}$ and $\mathsf{1}$ are unary operators in postfix notation, and $\#$ denotes the empty bit sequence. For example, $\#\mathsf{100}$ represents the number 4. This TRS is not compatible

6 http://termination-portal.org/wiki/TPDB
7 http://cl-informatik.uibk.ac.at/software/ackbo

Table 1 Termination and Normalized Completion

	termination			normalized completion		
	126 TRSs			67 problems		
method	yes	time	∞	yes	$\varnothing$ time	∞
AC-KBO (t/p)	31/32	2.1/1.7	0/0	25/25	0.36/0.33	37/37
Steinbach (t/p)	17/23	3.1/3.2	0/0	17/24	0.23/0.22	27/36
Korovin & Voronkov	30	2.1	0	25	0.36	37
subterm coefficients (t/p)	36/37	90.6/47.2	1	28/28	1.2/1.2	26/26
AC-RPO	63	2.8	0	28	0.4	26
$\sum$	70			31		

with AC-RPO but AC termination can easily be shown by AC-KBO, for instance with the weight function (w, w_0) with $w(+) = 0$, $w_0 = w(0) = w(\#) = 1$, and $w(1) = 3$. The system can be completed into an AC convergent TRS using AC-KBO but not with AC-RPO.

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A Omitted Proofs

We first present the omitted proofs concerning the correctness of AC-KBO.

Proof of Lemma 18.

(1) The inclusion from right to left is obvious from the definition. For the inclusion from left to right, suppose $s >_{\mathsf{ACKBO}} t$. If either $w(s) > w(t)$, or $w(s) = w(t)$ and case (0) or case (1) of Definition 17 applies, then trivially $s >_{01} t$. If case (2) or case (3) applies, then $s >_{23,k} t$ for any k with $k \geqslant \max(|s|, |t|)$.

(2) First we show that $>_{01}$ is transitive. Suppose $s >_{01} t >_{01} u$. If $w(s) > w(t)$ or $w(t) > w(u)$, then $w(s) > w(u)$ and $s >_{01} u$. Hence suppose $w(s) = w(t) = w(u)$. Since $s, t \notin \mathcal{V}$, we may write $s = f(s_1, \dots, s_n)$ and $t = g(t_1, \dots, t_m)$ with $f > g$. Because of admissibility, g is not a unary symbol with $w(g) = 0$. Thus $u \notin \mathcal{V}$, and we may write $u = h(u_1, \dots, u_l)$ with $g > h$. By the transitivity of $>$ we obtain $s >_{01} u$. The irreflexivity of $>_{01}$ is obvious from the definition. It remains to show the compatibility condition $=_{\mathsf{AC}} \cdot >_{01} \cdot =_{\mathsf{AC}} \subseteq >_{01}$. This easily follows from the fact that $w(s) = w(t)$ and $\mathsf{root}(s) = \mathsf{root}(t)$ whenever $s =_{\mathsf{AC}} t$.

(3) Suppose $s = f(s_1, \dots, s_n) >_{01} t = g(t_1, \dots, t_m) >_k u$. If $t >_{01} u$ then $s >_{01} u$ follows from the transitivity of $>_{01}$. Suppose $t >_{23,k} u$. So $w(t) = w(u)$. Thus $w(s) > w(u)$ if $w(s) > w(t)$, and case (1) applies if $w(s) = w(t)$. The inclusion $>_k \cdot >_{01} \subseteq >_k$ is proved in exactly the same way. ◀

Proof of Lemma 19. We first prove compatibility. Suppose $S \gtrsim^f T >^f U$. From $T >^f U$ we infer that $T{\restriction}_f^{\not\leq} \uplus T{\restriction}_{\mathcal{V}} >^{\mathsf{mul}} U{\restriction}_f^{\not\leq} \uplus U{\restriction}_{\mathcal{V}}$. Hence $S{\restriction}_f^{\not\leq} >^{\mathsf{mul}} U{\restriction}_f^{\not\leq} \uplus U{\restriction}_{\mathcal{V}} - S{\restriction}_{\mathcal{V}}$ follows from $S \gtrsim^f T$. Hence also $S\ (\gtrsim \cdot >)^f\ U$. We obtain the desired $S >^f U$ from the compatibility of $\gtrsim$ and $>$. The transitivity $\gtrsim^f$ and $>^f$ is obtained in a very similar way. Reflexivity of $\gtrsim^f$ and irreflexivity of $>^f$ are obvious. ◀

Proof of Lemma 20. The relation $\gtrsim$ is a preorder by assumption. Suppose $s > t > u$. According to Lemma 18(1), there exist k and l such that $s >_k t >_l u$. Let $m = \max(k, l)$. We obtain $s >_m t >_m u$ from the assumptions of the lemma and hence $s >_m u$ follows from the fact that $(\gtrsim, >_m)$ is an order pair. Compatibility is an immediate consequence of the assumptions and the irreflexivity of $>$ is obtained by an easy induction proof. ◀

Proof of Lemma 23. Let $\triangledown_f(t) = \{t_1, \dots, t_m\}$. If $m = 1$ then $\triangledown_f(t) = \{t\}$ and the lemma holds trivially. Otherwise we get $t > t_j$ for all $j = 1, \dots, m$ by recursively applying the assumption. Hence $s > t_j$ by the transitivity of $>$ or the compatibility of $>$ and $\gtrsim$. We conclude that $\{s\} >^{\mathsf{mul}} \triangledown_f(t)$. ◀

We now prove that $>_{\mathsf{KV'}}$ is an AC-compatible simplification order. The proof mimics the one given in Section 5 for $>_{\mathsf{ACKBO}}$, but there are some subtle differences. The easy proof of the following lemma is omitted.

▶ **Lemma 35.** *$(=_{\mathsf{AC}}, >_{\langle w,>\rangle})$ and $(\geqslant_{\langle w,>\rangle}, >_{\langle w,>\rangle})$ are order pairs.* ◀

▶ **Lemma 36.** *$(=_{\mathsf{AC}}, >_{\mathsf{KV'}})$ is an order pair.*

Proof. Similar to the proof of Lemma 21, except for case (3) of Definition 13, where we need Lemma 35 and Theorem 2. ◀

The subterm property follows exactly as in the proof of Lemma 22; note that the proof of Lemma 22 shows that the relation $>_{01}$ has the subterm property, and we obviously have $>_{01} \subseteq >_{\mathsf{KV'}}$.

▶ **Lemma 37.** *If w is admissible for $>$ then $>_{\mathsf{KV'}}$ has the subterm property.* ◀

▶ **Lemma 38.** *If w is admissible for $>$ then $>_{\mathsf{KV}'}$ is closed under contexts.*

Proof. Suppose $s >_{\mathsf{KV}'} t$. We follow the proof for $>_{\mathsf{ACKBO}}$ in Lemma 24 and consider here the case that $w(s) = w(t)$. We will show that one of the cases (3a)–(3c) in Definition 13 (11) is applicable to $S = \nabla_h(s)$ and $T = \nabla_h(t)$. Let $f = \mathsf{root}(s)$ and $g = \mathsf{root}(t)$. The proof proceeds by case splitting according to the derivation of $s >_{\mathsf{KV}'} t$.

- Suppose $s = f^k(t)$ with $k > 0$ and $t \in \mathcal{V}$. By admissibility, f is maximal in the precedence. Hence, $S{\restriction}_h^{>} = \{s\} \geqslant^{\mathsf{mul}}_{\langle w,>\rangle} \{t\}$. We have $|S| = |T| = 1$ and $S >^{\mathsf{mul}}_{\mathsf{KV}'} T$. Hence (3c) applies. (This case breaks down for $>_{\mathsf{KV}}$.)
- Suppose $f = g \notin \mathcal{F}_{\mathsf{AC}}$. We have $S \geqslant^{\mathsf{mul}}_{\langle w,>\rangle} T$, $|S| = |T| = 1$, and $S = \{s\} >^{\mathsf{mul}}_{\mathsf{KV}'} \{t\} = T$. Hence (3c) applies.
- The remaining cases are similar to the proof of Lemma 24, except that we use Lemma 23 with $(\geqslant_{\langle w,>\rangle}, >_{\langle w,>\rangle})$. ◀

For closure under substitutions we need to extend Lemma 25 with the following case:

(3) If $S \gtrsim^f T$ and $S' \not>^f T'$ then $S' - T' \supseteq S\sigma - T\sigma$ and $T\sigma - S\sigma \supseteq T' - S'$.

Proof. We continue the proof of Lemma 25. From $\nabla_f(U\sigma) = U\sigma$ we infer that $T' = T{\restriction}_{\mathcal{F}}\sigma \uplus U\sigma \uplus \nabla_f(X\sigma)$. On the other hand, $S' = S{\restriction}_{\mathcal{F}}\sigma \uplus \nabla_f(Y\sigma) \uplus \nabla_f(X\sigma)$ with $Y = S{\restriction}_{\mathcal{V}} - X$. Hence

$$\begin{aligned} T' - S' &\subseteq T{\restriction}_{\mathcal{F}}\sigma \uplus U\sigma - S{\restriction}_{\mathcal{F}}\sigma \\ &= T{\restriction}_{\mathcal{F}}\sigma \uplus U\sigma \uplus X\sigma - (S{\restriction}_{\mathcal{F}} \uplus X\sigma) \\ &\subseteq T\sigma - S\sigma \end{aligned}$$

and

$$\begin{aligned} S' - T' &\supseteq S{\restriction}_{\mathcal{F}}\sigma - T{\restriction}_{\mathcal{F}}\sigma - U\sigma \\ &= S{\restriction}_{\mathcal{F}}\sigma \uplus X\sigma - (T{\restriction}_{\mathcal{F}} \uplus U\sigma \uplus X\sigma) \\ &\supseteq S\sigma - T\sigma \end{aligned}$$

◀

▶ **Lemma 39.** *$>_{\mathsf{KV}'}$ is closed under substitutions.*

Proof. By induction on $|s|$ we verify that $s >_{\mathsf{KV}'} t$ implies $s\sigma >_{\mathsf{KV}'} t\sigma$. If $s >_{\mathsf{KV}'} t$ is derived by one of the cases (0), (1), (2), (3a) or (3b) in Definition 13 (11), the proof of Lemma 24 goes through. So suppose that $s >_{\mathsf{KV}'} t$ is derived by case (3c) and further suppose that $s\sigma >_{\mathsf{KV}'} t\sigma$ can be derived neither by case (3a) nor (3b). By definition we have $\nabla_f(s) >^{\mathsf{mul}}_{\mathsf{KV}'} \nabla_f(t)$. This is equivalent[8] to

$$\nabla_f(s) - \nabla_f(t) >^{\mathsf{mul}}_{\mathsf{KV}'} \nabla_f(t) - \nabla_f(s)$$

We obtain $\nabla_f(s)\sigma - \nabla_f(t)\sigma >^{\mathsf{mul}}_{\mathsf{KV}'} \nabla_f(t)\sigma - \nabla_f(s)\sigma$ from the induction hypothesis and thus $\nabla_f(s\sigma) - \nabla_f(t\sigma) >^{\mathsf{mul}}_{\mathsf{KV}'} \nabla_f(t\sigma) - \nabla_f(s\sigma)$ by Lemma 25(3). Using the earlier equivalence, we infer $\nabla_f(s\sigma) >^{\mathsf{mul}}_{\mathsf{KV}'} \nabla_f(t\sigma)$ and hence case (3c) applies to obtain the desired $s\sigma >_{\mathsf{KV}'} t\sigma$. ◀

The combination of the above results proves Theorem 16.

[8] This property is well-known for standard multiset extensions (involving a single proper order). It is also not difficult to prove for the multiset extension defined in Definition 1.

Proof of Theorem 33. Suppose that every function symbol in $\mathcal{F}_{\mathsf{AC}}$ is minimal with respect to $>$. We show that $s >_{\mathsf{S}} t$ if and only if $s >_{\mathsf{ACKBO}} t$ by induction on s. It is clearly sufficient to consider case (3) in Definition 3 and cases (3a)–(3c) in Definition 17. So let $s = f(s_1, s_2)$ and $t = f(t_1, t_2)$ such that $w(s) = w(t)$, $f \in \mathcal{F}_{\mathsf{AC}}$.

- Let $s >_{\mathsf{S}} t$ by case (3). We have $S >_{\mathsf{S}}^{\mathsf{mul}} T$ for $S = \nabla_f(s)$ and $T = \nabla_f(t)$. Since $S >_{\mathsf{S}}^{\mathsf{mul}} T$ involves only comparisons $s' >_{\mathsf{S}} t'$ for subterms s' of s, the induction hypothesis yields $S >_{\mathsf{ACKBO}}^{\mathsf{mul}} T$. Because f is minimal in $>$, $S = S{\restriction}_f^{\not<} \uplus S{\restriction}_{\mathcal{V}}$ and $T = T{\restriction}_f^{\not<} \uplus T{\restriction}_{\mathcal{V}}$. For no elements $u \in S{\restriction}_{\mathcal{V}}$ and $v \in T{\restriction}_f^{\not<}$, $u >_{\mathsf{ACKBO}} v$ or $u =_{\mathsf{AC}} v$ holds. Hence $S >_{\mathsf{ACKBO}}^{\mathsf{mul}} T$ implies $S >_{\mathsf{ACKBO}}^{f} T$ or both $S =_{\mathsf{AC}}^{f} T$ and $|S{\restriction}_{\mathcal{V}}| > |T{\restriction}_{\mathcal{V}}|$. In the former case $s >_{\mathsf{ACKBO}} t$ is due to case (3a) in Definition 17. In the latter case we have $|S| > |T|$ and $s >_{\mathsf{ACKBO}} t$ follows by case (3b).
- Let $s >_{\mathsf{ACKBO}} t$ by applying one of the cases (3a)–(3c) in Definition 17. Let $S = \nabla_f(s)$ and $T = \nabla_f(t)$.
  - Suppose (3a) applies. Then we have $S >_{\mathsf{ACKBO}}^{f} T$. Since f is minimal in $>$, $S{\restriction}_f^{\not<} = S - S{\restriction}_{\mathcal{V}}$ and $T{\restriction}_f^{\not<} \uplus T{\restriction}_{\mathcal{V}} = T$. Hence $S >_{\mathsf{ACKBO}}^{\mathsf{mul}} (T - S{\restriction}_{\mathcal{V}}) \uplus S{\restriction}_{\mathcal{V}} \supseteq T$. We obtain $S >_{\mathsf{S}}^{\mathsf{mul}} T$ from the induction hypothesis and thus case (3) in Definition 3 applies.
  - Suppose (3b) applies. Analogous to the previous case, we have $S =_{\mathsf{AC}}^{\mathsf{mul}} (T - S{\restriction}_{\mathcal{V}}) \uplus S{\restriction}_{\mathcal{V}} \supseteq T$. Since $|S| > |T|$, $S =_{\mathsf{AC}}^{\mathsf{mul}} T$ is not possible. Thus $(T - S{\restriction}_{\mathcal{V}}) \uplus S{\restriction}_{\mathcal{V}} \supset T$ and hence $S >_{\mathsf{S}}^{\mathsf{mul}} T$.
  - If case (3c) applies then $S{\restriction}_f^{<} >_{\mathsf{ACKBO}}^{\mathsf{mul}} T{\restriction}_f^{<}$. This is impossible since both sides are empty as f is minimal in $>$. ◀

In Example 9 we have seen that $>_{\mathsf{KV}}$ does not subsume $>_{\mathsf{S}}$ for ground TRSs. The following example shows that the two orders are incomparable on ground TRSs.

▶ **Example 40.** Consider the ground TRS $\mathcal{R}_2$ consisting of the rules

$$\mathsf{g}(\mathsf{f}(\mathsf{a},\mathsf{a})) \to \mathsf{f}(\mathsf{g}(\mathsf{a}),\mathsf{g}(\mathsf{a})) \qquad \mathsf{f}(\mathsf{g}(\mathsf{a}),\mathsf{g}(\mathsf{a})) \to \mathsf{f}(\mathsf{a},\mathsf{g}(\mathsf{g}(\mathsf{a})))$$

over the signature $\{\mathsf{a},\mathsf{f},\mathsf{g}\}$ with $\mathsf{f} \in \mathcal{F}_{\mathsf{AC}}$. Consider the precedence $\mathsf{g} > \mathsf{f} > \mathsf{a}$ and admissible weight function (w, w_0) with $w(\mathsf{f}) = w(\mathsf{g}) = 0$ and $w_0 = w(\mathsf{a}) = 1$. We have $\mathsf{g}(\mathsf{f}(\mathsf{a},\mathsf{a})) >_{\mathsf{KV}} \mathsf{f}(\mathsf{g}(\mathsf{a}),\mathsf{g}(\mathsf{a}))$ by case (1) and $\mathsf{f}(\mathsf{g}(\mathsf{a}),\mathsf{g}(\mathsf{a})) >_{\mathsf{KV}} \mathsf{f}(\mathsf{a},\mathsf{g}(\mathsf{g}(\mathsf{a})))$ by case (3a) of Definition 7; note that $\mathsf{g}(\mathsf{a}) =_{\langle w,>\rangle} \mathsf{g}(\mathsf{g}(\mathsf{a}))$. Hence $\mathcal{R}_2$ is compatible with $>_{\mathsf{KV}}$. AC termination of $\mathcal{R}_2$ cannot be shown by $>_{\mathsf{S}}$. This can be seen as follows. The first rule demands $w(\mathsf{g}) = 0$ and thus $\mathsf{g} > \mathsf{f}$ and $\mathsf{g} > \mathsf{a}$ by admissibility. The second rule requires $\{\mathsf{g}(\mathsf{a}),\mathsf{g}(\mathsf{a})\} >_{\mathsf{S}}^{\mathsf{mul}} \{\mathsf{a},\mathsf{g}(\mathsf{g}(\mathsf{a}))\}$ which is impossible due to $\mathsf{g}(\mathsf{g}(\mathsf{a})) >_{\mathsf{S}} \mathsf{g}(\mathsf{a})$.

Note that also $>_{\mathsf{ACKBO}}$ cannot prove AC termination of the TRS of Example 40. The following example shows that $>_{\mathsf{ACKBO}}$ and $>_{\mathsf{KV'}}$ are incomparable.

▶ **Example 41.** Consider the TRS $\mathcal{R}_3$ consisting of the rules

$$\begin{array}{lll} \mathsf{f}(x+y) \to \mathsf{f}(x)+y & \mathsf{h}(\mathsf{a},\mathsf{b}) \to \mathsf{h}(\mathsf{b},\mathsf{a}) & \mathsf{h}(\mathsf{g}(\mathsf{a}),\mathsf{a}) \to \mathsf{h}(\mathsf{a},\mathsf{g}(\mathsf{b})) \\ \mathsf{g}(x)+y \to \mathsf{g}(x+y) & \mathsf{h}(\mathsf{a},\mathsf{g}(\mathsf{g}(\mathsf{a}))) \to \mathsf{h}(\mathsf{g}(\mathsf{a}),\mathsf{f}(\mathsf{a})) & \mathsf{h}(\mathsf{g}(\mathsf{a}),\mathsf{b}) \to \mathsf{h}(\mathsf{a},\mathsf{g}(\mathsf{a})) \\ \mathsf{f}(\mathsf{a})+\mathsf{g}(\mathsf{b}) \to \mathsf{f}(\mathsf{b})+\mathsf{g}(\mathsf{a}) & & \end{array}$$

over the signature $\{+,\mathsf{f},\mathsf{g},\mathsf{h},\mathsf{a},\mathsf{b}\}$ with $+ \in \mathcal{F}_{\mathsf{AC}}$. Consider the precedence $\mathsf{f} > + > \mathsf{g} > \mathsf{a} > \mathsf{b} > \mathsf{h}$ together with the admissible weight function (w, w_0) with $w(+) = w(\mathsf{h}) = 0$, $w(\mathsf{f}) = w(\mathsf{a}) = w(\mathsf{b}) = w_0 = 1$ and $w(\mathsf{g}) = 2$. The interesting rule is $\mathsf{f}(\mathsf{a})+\mathsf{g}(\mathsf{b}) \to \mathsf{f}(\mathsf{b})+\mathsf{g}(\mathsf{a})$. For $S = \nabla_+(\mathsf{f}(\mathsf{a})+\mathsf{g}(\mathsf{b}))$ and $T = \nabla_+(\mathsf{f}(\mathsf{b})+\mathsf{g}(\mathsf{a}))$ the multisets $S' = S{\restriction}_+^{\not<} = \{\mathsf{f}(\mathsf{a})\}$ and

$T' = T{\restriction}_{+}^{\nless} \uplus T{\restriction}_{\mathcal{V}} - S{\restriction}_{\mathcal{V}} = \{\mathsf{f}(\mathsf{b})\}$ satisfy $S' >_{\mathsf{ACKBO}}^{\mathsf{mul}} T'$ as $\mathsf{f}(\mathsf{a}) >_{\mathsf{ACKBO}} \mathsf{f}(\mathsf{b})$, so that case (3a) of Definition 17 applies. All other rules are oriented from left to right by both $>_{\mathsf{KV'}}$ and $>_{\mathsf{ACKBO}}$, and they enforce a precedence and weight function which are identical (or very similar) to the one given above. Since $>_{\mathsf{KV'}}$ orients the rule $\mathsf{f}(\mathsf{a}) + \mathsf{g}(\mathsf{b}) \to \mathsf{f}(\mathsf{b}) + \mathsf{g}(\mathsf{a})$ from right to left, $\mathcal{R}_3$ cannot be compatible with $>_{\mathsf{KV'}}$.

The following example illustrates that the use of the preorder $\geqslant_{\langle w,>\rangle}$ is essential for the non-ground version of $>_{\mathsf{KV}}$ to be closed under context, even if there is no unary symbol of weight zero.

▶ **Example 42.** Let $+$ be an AC symbol, c a constant, f a unary function symbol and g a non-AC binary symbol such that $w(\mathsf{c}) = w_0$, $w(\mathsf{f}) > 0$, and $\mathsf{g} > + > \mathsf{c}$. We have $\ell = \mathsf{g}(\mathsf{f}(\mathsf{c}), x) >_{\mathsf{KV}} \mathsf{g}(\mathsf{c}, \mathsf{f}(\mathsf{c})) = r$ by case (2). However, $s = \ell + \mathsf{c} >_{\mathsf{KV}} r + \mathsf{c} = t$ does not hold. Let $S = \nabla_{+}(s) = \{\ell, \mathsf{c}\}$ and $T = \nabla_{+}(t) = \{r, \mathsf{c}\}$. We have $S{\restriction}_{+}^{>} = \{\ell\}$, $T{\restriction}_{+}^{>} = \{r\}$, and $S{\restriction}_{\mathcal{V}} = T{\restriction}_{\mathcal{V}} = \varnothing$. Note that $\ell >_{\langle w,>\rangle} r$ does not hold since $w(\ell) = w(r)$ and $\mathsf{root}(\ell) = \mathsf{g} = \mathsf{root}(r)$. Hence case (3a) in Definition 11 does not apply. But also $\ell =_{\langle w,>\rangle} r$ does not hold since $|\ell|_x = 1 \neq 0 = |r|_x$, excluding cases (3b) and (3c).