First Exam (A) Algorithm Theory, LVA 703608 Institute of Computer Science
June 29, 2007 University of Innsbruck

Name: **Matr.-Nr.**: **Points**:

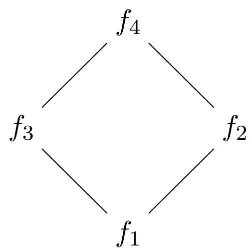
Good luck!

Big $\mathcal{O}$ notation

Which of the following functions are big $\mathcal{O}$ of which others? (Recall that we are only interested in functions $f\colon \mathbb{N} \to \mathbb{N}$. Hence when we write $f(n)$, we really mean $\max(\{\lceil f(n) \rceil, 0\})$.) [5]

(a)
- $h(n) = \begin{cases} n \cdot \log_{10}(n) & \text{if } n \text{ is even} \\ \frac{n^3}{\log_2(n)} & \text{if } n \text{ is odd} \end{cases}$
- 4711^{2n}
- n^n
- $n \cdot \log_2(n)$

Here the answer suffices in the form of a diagram:



(Meaning that $f_1(n) = \mathcal{O}(f_2(n))$, $f_1(n) = \mathcal{O}(f_3(n))$, $f_2(n) = \mathcal{O}(f_4(n))$ and $f_3(n) = \mathcal{O}(f_4(n))$ and neither $f_2(n) = \mathcal{O}(f_3(n))$ nor $f_3(n) = \mathcal{O}(f_2(n))$.)

(b) Explain your answers to (a) convincingly. [5]

Turing Machines

Consider the following Turing machine $\mathsf{M} = (K, \Sigma, \delta, s)$ with $K = \{s, s_a, s_b, s_c, s_d, s_e, yes, no\}$, $\Sigma = \{0, \grave{0}, \acute{0}, 1, \grave{1}, \acute{1}, \triangleright, \sqcup\}$ and δ defined as follows:

$p \in K$	$\sigma \in \Sigma$	$\delta_M(p, \sigma)$
s	$\triangleright$	$(s_a, \triangleright, \rightarrow)$
s_a	0	$(s_b, \grave{0}, \rightarrow)$
s_a	1	$(s_b, \grave{1}, \rightarrow)$
s_a	$\acute{0}$	$(s_e, \acute{0}, \leftarrow)$
s_a	$\acute{1}$	$(s_e, \acute{1}, \leftarrow)$
s_b	0	$(s_b, 0, \rightarrow)$
s_b	1	$(s_b, 1, \rightarrow)$
s_b	$\sqcup$	$(s_c, \sqcup, \leftarrow)$
s_b	$\acute{0}$	$(s_c, \acute{0}, \leftarrow)$
s_b	$\acute{1}$	$(s_c, \acute{1}, \leftarrow)$

$p \in K$	$\sigma \in \Sigma$	$\delta_M(p, \sigma)$
s_c	0	$(s_d, \acute{0}, \leftarrow)$
s_c	1	$(s_d, \acute{1}, \leftarrow)$
s_c	$\grave{0}$	$(\mathsf{no}, \grave{0}, -)$
s_c	$\grave{1}$	$(\mathsf{no}, \grave{1}, -)$
s_d	0	$(s_d, 0, \leftarrow)$
s_d	1	$(s_d, 1, \leftarrow)$
s_d	$\grave{0}$	$(s_a, \grave{0}, \rightarrow)$
s_d	$\grave{1}$	$(s_a, \grave{1}, \rightarrow)$
s_e	$\grave{0}$	$(s_e, \grave{0}, \leftarrow)$
s_e	$\grave{1}$	$(s_e, \grave{1}, \leftarrow)$
s_e	$\triangleright$	$(\mathsf{yes}, \triangleright, -)$

(a) For arbitrary strings $x \in \{0, 1\}^*$, what is $\mathsf{M}(x)$? [5]

(b) Define the notion of space complexity of 1-string Turing machines formally and give an upper-bound on the space complexity of M. [5]

(c) Explain informally how to extend M to decide the language $\mathrm{L} = \{ww \mid w \in \{0, 1\}^*\}$. [4]

Complexity Classes

Consider the following tasks.

(a) Define the complexity class **NL** and state a language L such that $\mathrm{L} \in$ **NL**. [4]

(b) Prove the following assertion: **PSPACE** = **NPSPACE**. (If you use a theorem to prove this assertion, please give a brief explanation of how the theorem was proven or give the name of the proof method used.) [4]

Reductions and NP-completeness

Let $G = (V, E)$ be an undirected graph, $I \subseteq V$. Recall that I is *independent* if for $i, j \in I$, $\{i, j\} \notin E$.

$$\text{INDEPENDENT SET} = \left\{ (G, K) \colon \begin{array}{l} \text{there exists an independent} \\ \text{set } I \text{ for } G \text{ with } |I| = K \end{array} \right\}$$

(a) Define a reduction R from 3SAT to INDEPENDENT SET. [5]

(b) Show that $\varphi \in$ 3SAT if and only if $R(\varphi) \in$ INDEPENDENT SET. [6]

(c) Show that R is logspace computable. [3]

(d) Show that INDEPENDENT SET is **NP**-complete. (Under the assumption that (a)–(c) have been established.) [4]