First Exam (B) — Algorithm Theory, LVA 703608 — Institute of Computer Science
June 29, 2007 — University of Innsbruck

Name: **Matr.-Nr.**: **Points**:

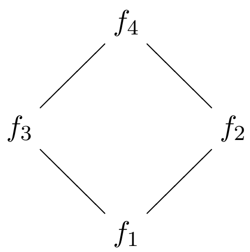
Good luck!

Big $\mathcal{O}$ notation

Which of the following functions are big $\mathcal{O}$ of which others? (Recall that we are only interested in functions $f\colon \mathbb{N} \to \mathbb{N}$. Hence when we write $f(n)$, we really mean $\max(\{\lceil f(n) \rceil, 0\})$.) [5]

(a)
- $h(n) = \begin{cases} n \cdot \log_2(n) & \text{if } n \text{ is even} \\ \frac{n^3}{\log_2(n)} & \text{if } n \text{ is odd} \end{cases}$
- $n^{\log_2(n)}$
- n^{4711}
- $n \cdot \sqrt{n}$

Here the answer suffices in the form of a diagram:



(Meaning that $f_1(n) = \mathcal{O}(f_2(n))$, $f_1(n) = \mathcal{O}(f_3(n))$, $f_2(n) = \mathcal{O}(f_4(n))$ and $f_3(n) = \mathcal{O}(f_4(n))$ and neither $f_2(n) = \mathcal{O}(f_3(n))$ nor $f_3(n) = \mathcal{O}(f_2(n))$.)

(b) Explain your answers to (a) convincingly. [5]

Turing Machines

Consider the following Turing machine $\mathsf{M} = (K, \Sigma, \delta, s)$ with $K = \{s, s_1, s_2, s_3, s_4, s_5, yes, no\}$, $\Sigma = \{\mathsf{a}, \grave{\mathsf{a}}, \acute{\mathsf{a}}, \mathsf{b}, \grave{\mathsf{b}}, \acute{\mathsf{b}}, \triangleright, \sqcup\}$ and δ defined as follows:

$p \in K$	$\sigma \in \Sigma$	$\delta_M(p, \sigma)$
s	$\triangleright$	$(s_1, \triangleright, \rightarrow)$
s_1	a	$(s_2, \grave{\mathsf{a}}, \rightarrow)$
s_1	b	$(s_2, \grave{\mathsf{b}}, \rightarrow)$
s_1	á	$(s_5, \acute{\mathsf{a}}, \leftarrow)$
s_1	$\acute{\mathsf{b}}$	$(s_5, \acute{\mathsf{b}}, \leftarrow)$
s_2	a	$(s_2, \mathsf{a}, \rightarrow)$
s_2	b	$(s_2, \mathsf{b}, \rightarrow)$
s_2	$\sqcup$	$(s_3, \sqcup, \leftarrow)$
s_2	á	$(s_3, \acute{\mathsf{a}}, \leftarrow)$
s_2	$\acute{\mathsf{b}}$	$(s_3, \acute{\mathsf{b}}, \leftarrow)$

$p \in K$	$\sigma \in \Sigma$	$\delta_M(p, \sigma)$
s_3	a	$(s_4, \acute{\mathsf{a}}, \leftarrow)$
s_3	b	$(s_4, \acute{\mathsf{b}}, \leftarrow)$
s_3	à	$(\mathsf{no}, \grave{\mathsf{a}}, -)$
s_3	$\grave{\mathsf{b}}$	$(\mathsf{no}, \grave{\mathsf{b}}, -)$
s_4	a	$(s_4, \mathsf{a}, \leftarrow)$
s_4	b	$(s_4, \mathsf{b}, \leftarrow)$
s_4	à	$(s_1, \grave{\mathsf{a}}, \rightarrow)$
s_4	$\grave{\mathsf{b}}$	$(s_1, \grave{\mathsf{b}}, \rightarrow)$
s_5	à	$(s_5, \grave{\mathsf{a}}, \leftarrow)$
s_5	$\grave{\mathsf{b}}$	$(s_5, \grave{\mathsf{b}}, \leftarrow)$
s_5	$\triangleright$	$(\mathsf{yes}, \triangleright, -)$

(a) For arbitrary strings $x \in \{\mathsf{a}, \mathsf{b}\}^*$, what is $\mathsf{M}(x)$? [5]

(b) Define the notion of space complexity of 1-string Turing machines formally and give an upper-bound on the space complexity of M. [5]

(c) Explain informally how to extend M to decide the language $\mathrm{L} = \{ww \mid w \in \{\mathsf{a}, \mathsf{b}\}^*\}$. [4]

Complexity Classes

Consider the following tasks.

(a) Define the complexity class **P** and state a language L such that $\mathrm{L} \in \mathbf{P}$. [4]

(b) Prove or disprove **coPSPACE** = **NPSPACE**. (If you use a theorem to prove this assertion, please give a brief explanation of how the theorem was proven or give the name of the proof method used.) [4]

Reductions and NP-completeness

Let $G = (V, E)$ be an undirected graph, $I \subseteq V$. Recall that I is *independent* if for $i, j \in I$, $(i, j) \notin E$.

$$\text{INDEPENDENT SET} = \left\{ (G, K) \colon \begin{array}{l} \text{there exists an independent} \\ \text{set } I \text{ for } G \text{ with } |I| = K \end{array} \right\}$$

(a) Define a reduction R from 3SAT to INDEPENDENT SET. [5]

(b) Show that $\varphi \in$ 3SAT if and only if $R(\varphi) \in$ INDEPENDENT SET. [6]

(c) Show that R is logspace computable. [3]

(d) Show that INDEPENDENT SET is **NP**-complete. (Under the assumption that (a)–(c) have been established.) [4]