Algorithm Theory

Georg Moser   Mircea Dan Hernest

Institute of Computer Science @ UIBK

Summer 2007

NAESAT | CLIQUE & INDEPENDENT SET | 3SAT$_3$

Content

Definition NAESAT

$$\text{NAESAT} = \left\{ \varphi \colon \begin{array}{l} \varphi \text{ is a satisfiable 3CNF-formula and in no clause} \\ \text{all literals get the truth values } \textbf{true} \end{array} \right\}$$

Theorem

NAESAT is **NP**-complete

Proof

we reduce from CIRCUIT SAT

Definition construction

1. recall reduction from CIRCUIT SAT to 3SAT
2. instead duplicating literals, we add the new variable z to all small clauses

$R(C) \in$ NAESAT implies $C \in$ CIRCUIT SAT

- ➡ suppose $R(C) \in$ NAESAT with assignment T
- ➡ the complement $\overline{T}$ satisfies $R(C)$ too
- ➡ wlog $\exists$ assignment such that z is false
- ➡ this assignment satisfies C

$C \in$ CIRCUIT SAT implies $R(C) \in$ NAESAT

- ➡ suppose C is satisfied; $\exists$ CNF formula φ such that for each clause $C_i \in \varphi$, $|C_i| \leqslant 3$
- ➡ $\exists$ assignment T satisfying φ
- ➡ extend T such that $T(z) =$ **false**
- ➡ we show that literals in the extended clauses $C_i' \in R(C)$ get alternating truth-values:
  (i) one- or two-literal clauses easy
  (ii) $g \leftrightarrow (h \wedge h')$, (iii) $g \leftrightarrow (h \vee h')$

➡ consider $g \leftrightarrow (h \wedge h')$
this becomes

$$\neg g \vee h \vee z \qquad \neg g \vee h' \vee z \qquad \neg h \vee \neg h' \vee g$$

due to z the first clause have alternating truth values
assume $T(\neg h) = T(\neg h) = T(g) = \textbf{true}$
then $T(\neg g \vee h \vee z) = \textbf{false}$

finally, log-space computablility is easily shown □

Definition — k-clique

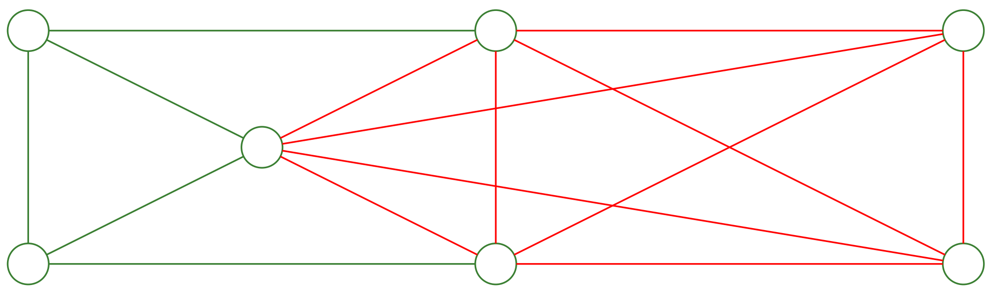
a clique in an undirected graph G is

1. a subgraph C
2. every two nodes in C are connected by an edge — C is complete

a k-clique is a clique that contains k-nodes

Definition — CLIQUE

$$\text{CLIQUE} = \{(G, K) \mid G \text{ is an undirected graph with a } K\text{-clique}\}$$



Definition — INDEPENDENT SET

$G = (V, E)$ be an undirected graph, $I \subseteq V$

➡ I is independent if for $i, j \in I$, $(i, j) \notin E$

$$\text{INDEPENDENT SET} = \left\{ (G, K) \mid \begin{array}{l} \exists \text{ independent set } I \text{ of } G \\ |I| = K \end{array} \right\}$$

Theorem

INDEPENDENT SET is **NP**-complete.

Proof

reduce from 3SAT to restricted graphs
where nodes can be partitioned in m disjoint triangles

- the main gadgets are triangles
- the only independent sets of triangles are single nodes

Definition

construction of R

1. for each clause in φ (assume that there are m)
   we create a triangle
2. each node corresponds to a literal in the clause
3. connect dual literals
   respectively the nodes representing literals
4. set $K = m$

$\varphi \in$ 3SAT implies $R(\varphi) \in$ INDEPENDENT SET

- assume $\varphi \in$ 3CNF is satisfiable with assignment T
- define I by picking a true literal-node for each triangle

$R(\varphi) \in$ INDEPENDENT SET implies $\varphi \in$ 3SAT

- assume $R(\varphi) \in$ INDEPENDENT SET with independet set I
- define assignment T by setting the literals in I to **true**
- for each triangle there has to be such a literal-node
- all clauses are satisfied

R is log-space computable

triangles and the connecting edges have to be constructed one after one, saving space; we need only store the indices of variables in the clauses □

Definition — $3SAT_3$

$$3SAT_3 = \left\{ \varphi \;\middle|\; \begin{array}{l}\varphi \text{ is a satisfiable 3CNF-formula, each variable occurs at} \\ \text{most 3 times and each literal at most twice in the for-} \\ \text{mula}\end{array} \right\}$$

Theorem

$3SAT_3$ is in **P**

Proof

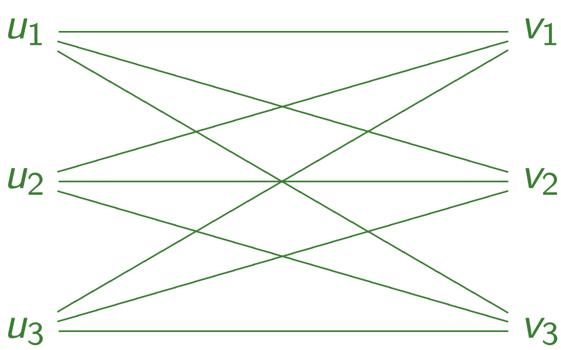
reduction to bipartite matching problem:

- ➡ given a bipartite graph such that
  for each set B of boys, $\exists$ adjacent set $g(B)$ of girls
  such that $|g(B)| \geqslant |B|$
- ➡ show there exists a perfect matching

Definition — bipartite graph

a bipartite graph is a tripe $B = (U, V, E)$ where

1. $U = \{u_1, \ldots, u_n\}$ — boys
2. $V = \{v_1, \ldots, v_n\}$ — girls
3. $E =$ pairs $\{i, j\}$ of nodes with $i \in U$, $j \in V$ — edges



Definition — matching

a (perfect) matching $M \subseteq E$ in a bipartite graph is

1. set of n edges
2. $\{u, v\}$ and $\{u', v'\}$ in M implies $u \neq u'$ and $v \neq v'$

Problem MATCHING

- ➡ given a bipartite graph B
- ➡ does B have a matching

Theorem

- ➡ MATCHING $\in$ **P**
- ➡ bipartite matching problem $\in$ **P**

Proof Sketch

reduce to MAX FLOW □

Definition construction of R

let $\varphi \in$ 3CNF

1. consider the set of clause in φ as the set U of boys
2. the variables as the set V of girls
3. $\{C, x\} \in E$ if variable x occurs in clause C

Theorem

$3SAT_3$ is in **P**

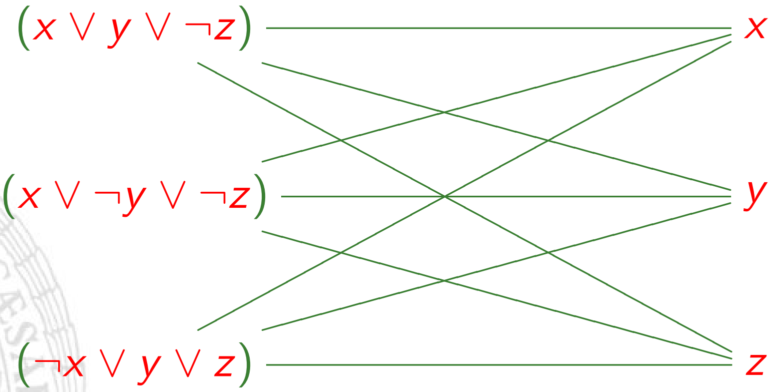
Proof by example

reduce to bipartite matching problem

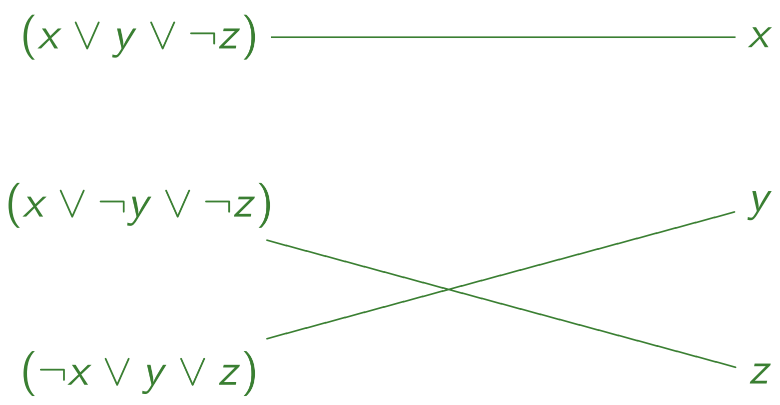
- ➡ given a subset of m clauses
- ➡ the clauses connect to at least m variables

the formula $(x \vee y \vee \neg z) \wedge (x \vee \neg y \vee \neg z) \wedge (\neg x \vee y \vee z)$

yields the (extended) bipartite (sub)graph B



B contains the matching



giving rise to assignment T:

$T(x) = \textbf{true}$ $\quad T(y) = \textbf{true}$ $\quad T(z) = \textbf{false}$

$\square$