Algorithm Theory

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Content

Hamilton Path

Definition

HAMILTON PATH

- given a directed graph G
- does there exists a path (starting in any node) through G such that every node in G is visited exactly once?

Theorem

HAMILTON PATH is **NP**-complete

Proof

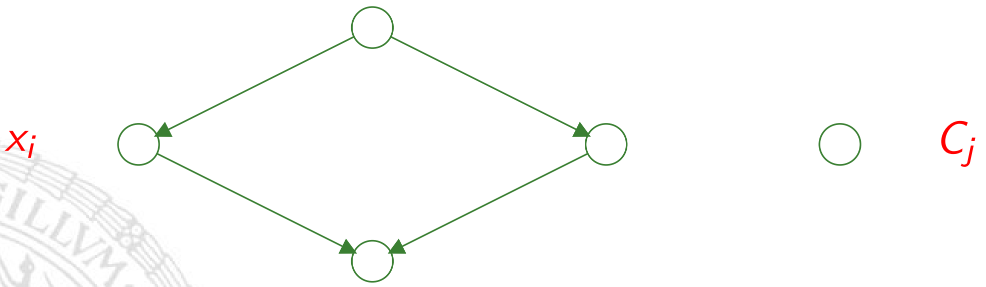
we have already shown that HAMILTON PATH $\in$ **NP**; it suffices to show that 3SAT reduces to HAMILTON PATH

let $\varphi \in$ 3CNF such that

$$\varphi = (\alpha_1 \vee \beta_1 \vee \gamma_1) \wedge \cdots \wedge (\alpha_m \vee \beta_m \vee \gamma_m)$$

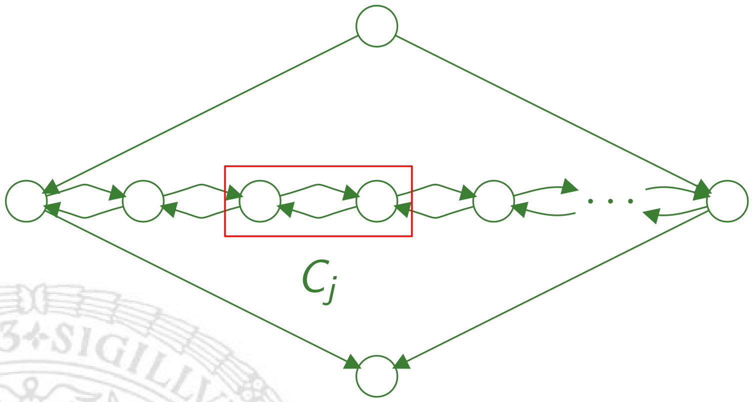
how to convert φ to a graph?

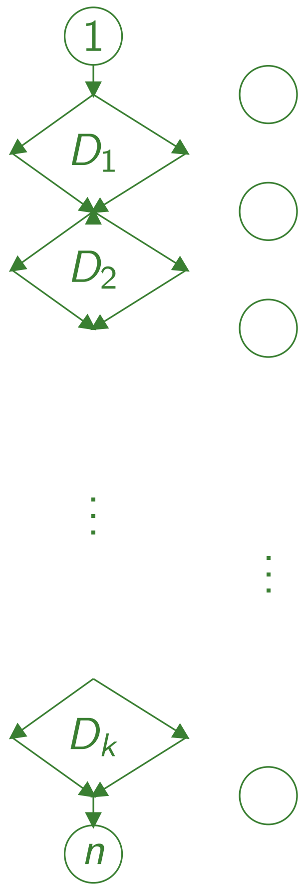
let k denote the number of variables in φ



variable gadget or (diamond) clause gadget inner structure of diamonds D:

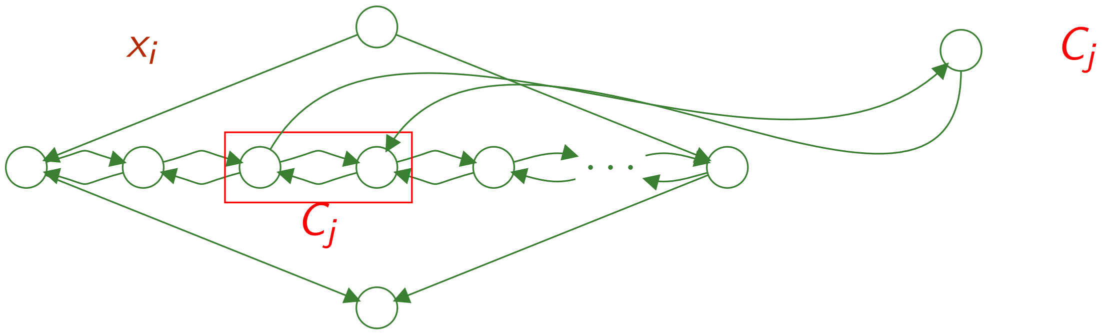
the row contains $3m + 3$ nodes, conceived as representations for the clauses plus separator nodes



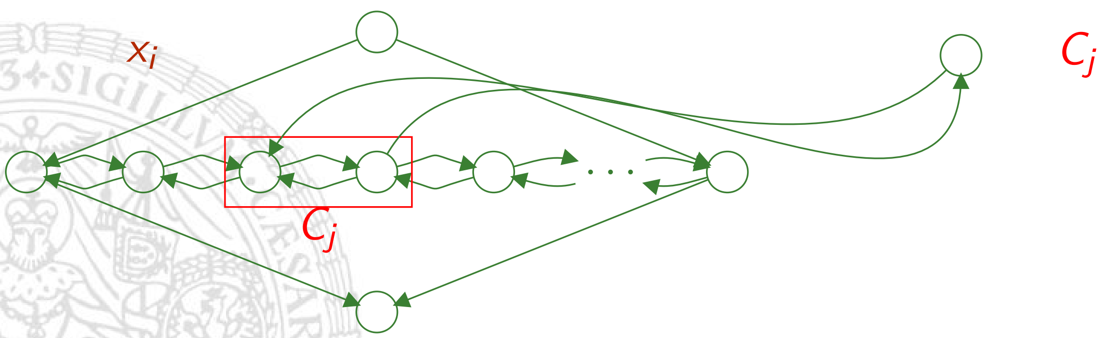


higher-level structure

x_i appears in C_j



$\neg x_i$ appears in C_j



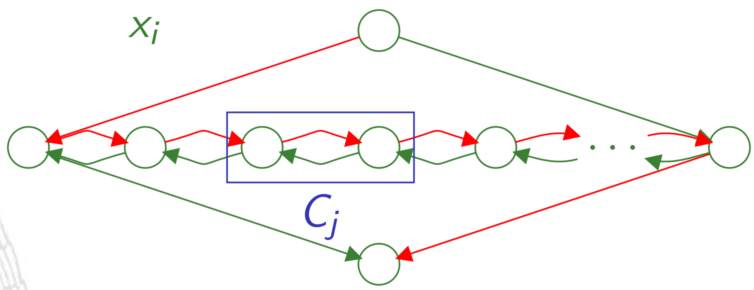
Lemma

if $\varphi \in$ 3SAT, then $R(\varphi) \in$ HAMILTON PATH

Proof

suppose φ is satisfiable by T

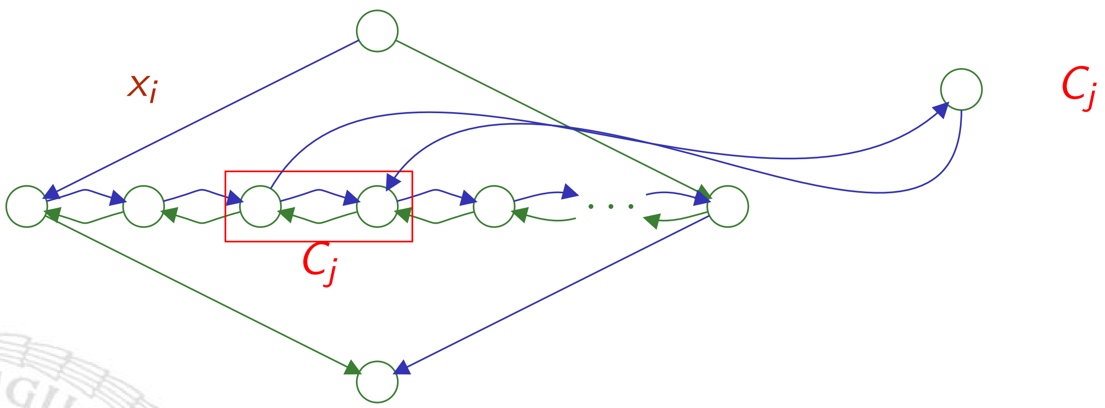
- the path begins at 1, goes through the diamonds, ends in n
- to hit the inner nodes of each diamond the path either traverses from left to right (zig-zag), or from right to left (zag-zig)
- if x_i is true (in T) we zig-zag through the diamond otherwise we zag-zig:



- to cover the clause nodes, the path has to detour

Detour

1. select one of the satisfied literals.
2. if x_i is selected in C_j
   detour at the inner nodes corresponding to C_j



we have constructed a Hamilton path through $R(\varphi)$ □

Observation

- any Hamilton path through $R(\varphi)$ has to start in 1 and end in n

Proof

otherwise: indegree of 1 is 0, outdegree of n is 0

the assumed path cannot include 1 and n at all

Contradiction □

Definition

we call a Hamilton path normal, if

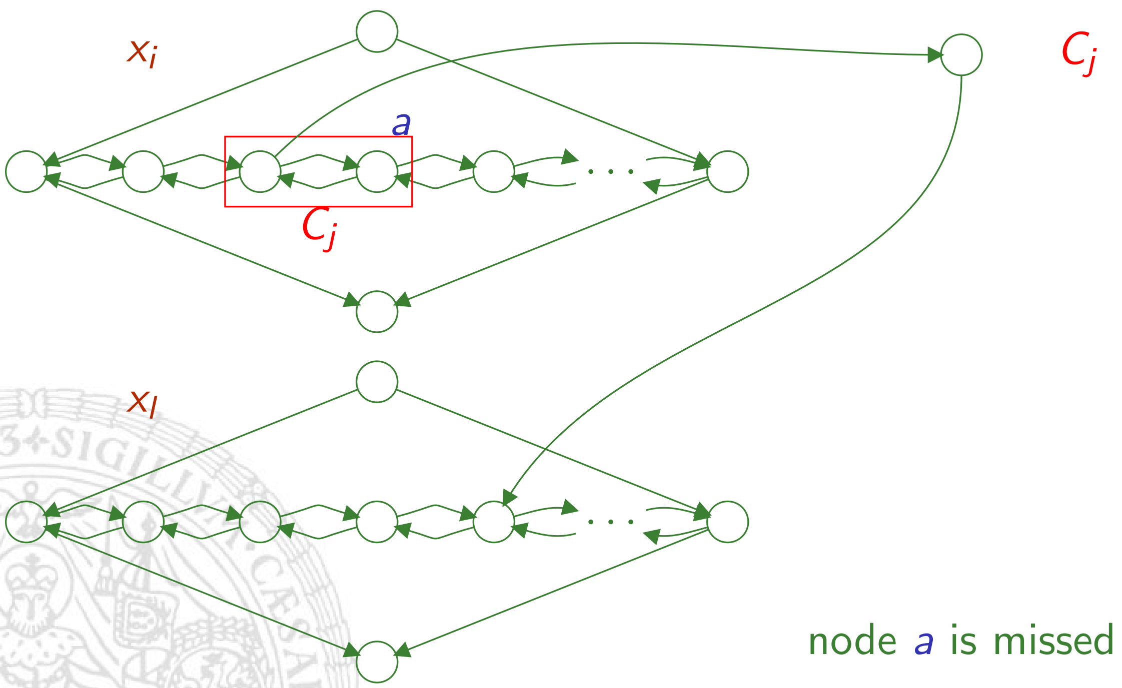
1. it either zig-zags or zag-zigs through the diamonds
2. processes the diamonds in order
3. except for possible detours through the clause-nodes

Lemma

any Hamilton path has to be normal

Proof

any deviation would have an edge from diamond D_i to clause-node C_j and then to a diamond D_l, $i < l$



Lemma
if $R(\varphi) \in$ HAMILTON PATH, then $\varphi \in$ 3SAT

Proof
assume

1. the existence of a normal Hamilton path through $R(\varphi)$
2. starting in 1, ending in n

Definition
assignment T

1. if the path zig-zags through diamond D_i set $T(x_i) = $ **true**
2. if the path zag-zigs, set $T(x_i) = $ **false**

for each clause, the path chooses a (true) literal,
hence φ is satisfied □

Lemma
R is log-space computable

- easy

Travelling Sales Person

Definition
TSP(D)

given

1. n cities $1, \ldots, n$
2. a nonnegative integer distance d_{ij} between cities i and j
3. distance is symmetric
4. budget B

is it possible to visit all n cities (and return) with the budget B so that every city is visited at most once?

Fact
HAMILTON PATH is defined for directed graphs, but is definable for undirected graphs as well; the latter problem is **NP**-complete, too

Theorem
TSP(D) is **NP**-complete

Proof

reduction from HAMILTON PATH on undirected graphs

Definition

construction of R

1. given a graph G with n nodes
2. consider n cities
3. set $d_{ij} = 1$, if $(i, j) \in G$, otherwise 2
4. set $B = n + 1$

$G \in$ HAMILTON PATH iff $R(G) \in \mathrm{TSP}(D)$

- easy

R is log-space computable

- trivial

□