Algorithm Theory

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Time Complexity Revisited

Reminder

- $\mathrm{L} \in \mathbf{TIME}(f(n))$,
  if L is decided by a multi-string M operating in time $f(n)$

Fact

PALINDROMES $\in \mathbf{TIME}(\mathcal{O}(n))$

"Theorem" Linear Speed-Up Theorem

- Assume $\exists$ TM M deciding L within time-bound $f(n)$
- Then $\forall\, d > 0$, $\exists\, n_0 \in \mathbb{N}$, $\exists$ TM M′ deciding L in time-bound $d \cdot f(n)$ for all $n \geqslant n_0$

Corollary

PALINDROMES $\in \mathbf{TIME}(n)$

Theorem

- Let $\mathrm{L} \in \mathbf{TIME}(f(n))$
- $\exists\, C \geqslant 0$, $\forall\, c > 0$ such that
- $\mathrm{L} \in \mathbf{TIME}(f'(n))$ with $f'(n) = c \cdot f(n) + n + \lceil \frac{c}{6} \rceil \cdot n + C$

Proof

Let $\mathrm{M} = (K, \Sigma, \delta, a)$ be a k-string TM that decides L, $m \in \mathbb{N}$

Define $\mathrm{M}' = (K', \Sigma', \delta', s)$ with M′

$$\Sigma' = \Sigma \cup \Sigma^m$$ Σ'

Initialisation

1. Read blocks of input $(a_1, \ldots, a_m)$ of length m
2. write them as one symbol on the second string
3. If necessary "padd" blocks

Fact

We use extra states of form $\{s\} \times \Sigma^i$ for all $i \in [1, m-1]$ K'

Initialisation needs $|x| + 1$ steps

Simulation

- M′ simulates m steps of M by at most six steps

The simulation proceeds in stages

Encoding of States in k-string TM M

State of M in the current stage is described by K'

1. $(q, j_1, \ldots, j_k)$
2. $j_i \leqslant m$ is the position of the cursor on the i^{th} tape within the scanned m-tuple

Stage

- M′ checks the surrounding m-blocks 4 steps
  by moving to the left and twice to the right, left once
- Based on this M′ simulates m moves of M 2 steps

Fact

We use extra states of form $K \times \{1, \ldots, m\}^k \times \Sigma^{3mk}$ K'

Each stage of simulation needs 6 steps

Total Number of Steps

- Initial Phase $|x| + 1$ steps
- Prepare Simulation $\lceil \frac{|x|}{m} \rceil + 1$ steps
  moving the cursor to the left
- Simulation $6\lceil \frac{f(|x|)}{m} \rceil$
- Auxilliary Steps .

Hence in total:

$$6\lceil \frac{f(|x|)}{m} \rceil + |x| + \lceil \frac{|x|}{m} \rceil + 8$$

Setting $m = \frac{6}{c}$, $n = |x|$, $C = 8$ yields:

$$\begin{aligned} 6\lceil \frac{f(n)}{m} \rceil + |x| + \lceil \frac{n}{m} \rceil + 8 &= cf(n) + n + \frac{cn}{6} + 8 \\ &\leqslant cf(n) + n + \lceil \frac{c}{6} \rceil n + C \\ &=: f'(n) \end{aligned}$$

□

Linear Speed-Up Theorem

Theorem

- Assume $\exists$ TM M deciding L within time-bound $f(n)$, and $f(n) = \omega(n)$
- Then $\forall\ d > 0$, $\exists\ n_0 \in \mathbb{N}$, $\exists$ TM M′ deciding L in time-bound $d \cdot f(n)$ for all $n \geqslant n_0$

Small o-Notation

$f(n)$ is small-o of $g(n)$

- $\forall\ c > 0$, $\exists\ n_c \in \mathbb{N}$ such that
  $\forall\ n \geqslant n_c$: $f(n) < c \cdot g(n)$
  Then $f(n) = o(g(n))$
- Equivalent to
  $$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$$
- $f(n) = \omega(g(n))$, if $g(n) = o(f(n))$

Examples: Small o-Notation

Example

We have $k \cdot n = o(n \cdot \log n)$, i.e.

- let $c > 0$ be arbitrary
- $\exists\ n_c$ and $\forall\ n \geqslant n_c$:
  $$k \cdot n < c \cdot (n \cdot \log n)$$

Example

But $n \neq o(k \cdot n)$

Assume otherwise, i.e.,

- $\forall\ c > 0\ \exists\ n_c$
- $\forall\ n \geqslant n_c$
  $$n < c \cdot (kn)$$

Proof (of Linear Speed-Up Theorem)

Assume

- L is decided by a k-string TM M in time $f(n)$, and
- $f(n) = \omega(n)$, i.e., $n = o(f(n))$.

Then

- $\exists\ k'$-string TM M′ ($k' = k$ or $k' = 2$)
  deciding L in time $cf(n) + n + \lceil \frac{c}{6} \rceil n + 8$,
  where $c > 0$ arbitrary
  $$\begin{aligned} cf(n) + n + \lceil \tfrac{c}{6} \rceil n + 8 &= cf(n) + (1 + \lceil \tfrac{c}{6} \rceil)n + 8 \\ &\leqslant cf(n) + cf(n) + cf(n) \\ &\leqslant 3cf(n) \quad \text{(almost everywhere)} \end{aligned}$$
- Set $c = \frac{d}{3}$ to achieve the theorem

□

Corollary

We can freely use the $\mathcal{O}$-notation in time bounds

TM with input and output

Definition

A k-string TM with input and output is a k-string TM but if

$$\delta(q, \sigma_1, \dots, \sigma_k) = (p, \rho_1, D_1, \dots, \rho_k, D_k)$$

Then

1. $\rho_1 = \sigma_1$ — input read-only
2. $D_k \neq \leftarrow$ — output write-only
3. if $\sigma_1 = \sqcup$ then $D_1 = \leftarrow$

Theorem

- $\forall$ k-string TM M, operating within time bound $f(n) \geqslant n$
- $\exists$ $(k+2)$-string TM with input and output, operating in time bound $\mathcal{O}(f(n))$

Space Bounds

Suppose

$$(s, \triangleright, x, \dots, \triangleright, \epsilon) \xrightarrow{\text{M}^*} (H, w_1, u_1, \dots, w_k, u_k) \quad H \in \{h, \mathit{yes}, \mathit{no}\}$$

- Space required by M on input x is

$$\sum_{i=1}^{k} |w_i u_i|$$

- If M is TM with input and output, then space required by M on input x is

$$\sum_{i=2}^{k-1} |w_i u_i|$$

- M operates within space bound $f(n)$ if, $\forall$ input x, M requires at most space $f(|x|)$

Space-complexity Classes

Definition — **SPACE**$(f(n))$

- $\mathrm{L} \in$ **SPACE**$(f(n))$, if $\exists$ TM with input and output that requires at most space-bound $f(n)$, and decides L

Example

PALINDROMES $\in$ **SPACE**$(\log n)$

We construct a 3-string TM M′ with input and output. M′ uses the second string to count the number of iterations in binary

1. Move the first cursor to i-th position
2. Read the i-th symbol of input, Accept, if it is $\sqcup$, Store the symbol in a state, otherwise
3. Move the first cursor to the last i-th position
4. Compares the i-th symbol and the last i-th symbol, Reject, if they are different