Algorithm Theory

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Summer 2007

Review: Time/Space Constructibility

Definition

A function $f \colon \mathbb{N} \to \mathbb{N}$, where $f(n) \geqslant n \log n$ is called time constructible if the mapping:

$$1^n \mapsto \text{binary representation of } f(n)$$

is computable in time $\mathcal{O}(f(n))$

Definition

A function $f \colon \mathbb{N} \to \mathbb{N}$, where $f(n) \geqslant \log n$ is called space constructible if the mapping:

$$1^n \mapsto \text{binary representation of } f(n)$$

is computable on a TM with input and output in space $\mathcal{O}(f(n))$

Papadimitriou defines proper complexity functions, cf. Def. 7.1; these we ignore

Review: Relations between Complexity Classes

Theorem — The time hierarchy theorem

If $f(n)$ is a time constructible function, then the class $\mathbf{TIME}(f(n))$ is strictly contained in $\mathbf{TIME}(f(2n+1)^3)$.

Theorem — The space hierarchy theorem

If $f(n)$ is a space constructible function, then the class $\mathbf{SPACE}(f(n))$ is strictly contained in $\mathbf{SPACE}(f(n) \log f(n))$.

(Simple) Facts

1. $\mathbf{SPACE}(f(n)) \subseteq \mathbf{NSPACE}(f(n))$
2. $\mathbf{TIME}(f(n)) \subseteq \mathbf{NTIME}(f(n))$

Review: The hierarchy theorems

Definition

- ➡ $\mathrm{H} = \{M; x \colon M \text{ halts on } x\}$ — halting problem
- ➡ $\mathrm{H}_f = \{M; x \colon M \text{ accepts } x \text{ after at most } f(|x|) \text{ steps}\}$

Lemma

$\mathrm{H}_f \in \mathbf{TIME}(f(n)^3)$

Lemma

$\mathrm{H}_f \notin \mathbf{TIME}(f(\lfloor \frac{n}{2} \rfloor))$

Theorem

For any time constructible function $f(n)$:

$$\mathbf{TIME}(f(n)) \subsetneq \mathbf{TIME}(f(2n+1)^3)$$

Gap theorem

Theorem

There is a computable function $f: \mathbb{N} \to \mathbb{N}$ such that
$\mathbf{TIME}(f(n)) = \mathbf{TIME}(2^{f(n)})$

Proof Idea

- ➡ define a function f such that no TM, computing on input on length n, halts after more than $f(n)$ but less than $2^{f(n)}$ steps
- ➡ observe f is recursive, not time constructible
- ➡ assume some enumeration of all TMs
- ➡ let $T_i(x)$ denote the running time of TM M_i on input x

Define
$f(n) =$ the least m such that for all $i \leqslant n$
if $T_i(x) \leqslant 2^m$, for some input x with $|x| = n$, then also $T_i(x) \leqslant m$

□

Relations between Complexity Classes

Theorem **NTIME(.) → SPACE(.)**

Let $f(n)$ be a time and space constructible function; then
$\mathbf{NTIME}(f(n)) \subseteq \mathbf{SPACE}(f(n))$.

Proof Idea

- ➡ simulation of the computation tree needs at most $f(n)$ space

□

Theorem **NSPACE(.) → TIME(.)**

Suppose $f(n)$ is space constructible; then
$\mathbf{NSPACE}(f(n)) \subseteq \mathbf{TIME}(k^{\log n + f(n)})$ $(= \bigcup_{d>0} \mathbf{TIME}(d^{\log n + f(n)}))$

Proof Idea

- ➡ use reachability method

□

Review: Configurations

Consider the following TM M

$p \in K$	$\sigma \in \Sigma$	$\delta(p, \sigma)$
s	0	$(s, 0, \rightarrow)$
s	1	$(s, 1, \rightarrow)$
s	$\sqcup$	$(q, \sqcup, \leftarrow)$
s	$\triangleright$	$(s, \triangleright, \rightarrow)$
q	0	$(h, 1, -)$
q	1	$(q, 0, \leftarrow)$
q	$\triangleright$	$(h, \triangleright, \rightarrow)$

Then $(s, \triangleright 1, 1011)$ is a configuration and we have for example

$$(s, \triangleright 11, 1) \xrightarrow{M^3} (q, \triangleright 111, \sqcup)$$

Reachability Method

Proposition

Consider a TM (deterministic or nondeterministic) deciding a language L so that the number of possible configurations is $\leqslant f(n)$.
Assume further, we can decide in time $g(n)$ that one configuration yields another.
Then $\mathrm{L} \in \mathbf{TIME}(\mathcal{O}(f(n)^2 \cdot g(n)))$.

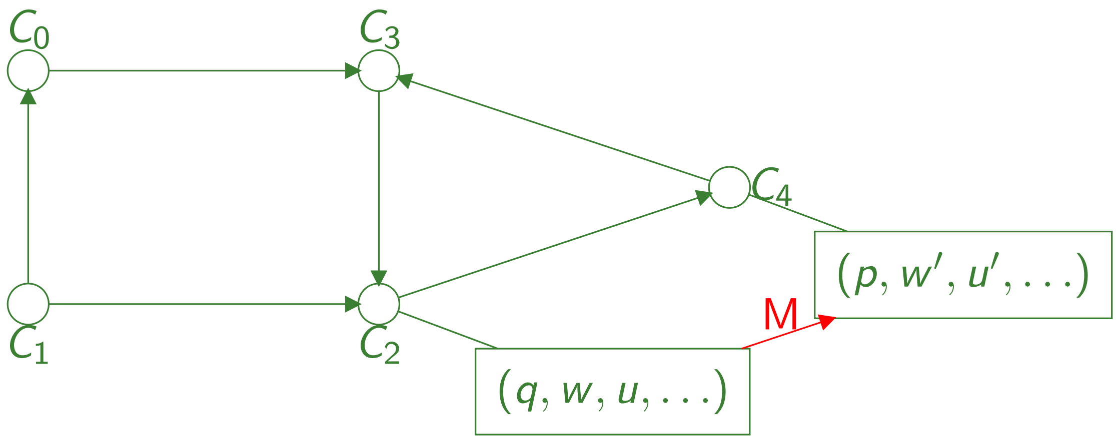
Definition — configuration graph

Let M be a TM; the configuration graph on input x, written $G(\mathrm{M}, x)$ is defined as follows:

1. the sets of nodes is the set of all possible configurations
2. let C_1, C_2 be two configurations

$$\exists \text{ edge between } C_1 \text{ and } C_2 \text{ in } G(\mathrm{M}, x) \text{ iff } C_1 \xrightarrow{\mathrm{M}} C_2$$

Configuration Graph



Representation of $G(\mathrm{M}, x)$:

- ➡ Represent $G(\mathrm{M}, x)$ by constructing the adjacency matrix explicitly, or
- ➡ Represent $G(\mathrm{M}, x)$ implicitly;
  look up the program of M to decide whether two configuration nodes are connected or not

Proof of the Proposition

Obviously $x \in \mathrm{L}$ iff there is a path from

$$C_0 = (s, \triangleright, x, \ldots)$$

to some configuration

$$C = (yes, w_1', u_1', \ldots)$$

- ➡ Represent $G(\mathrm{M}, x)$ implicitely
- ➡ Connection between C_1 and C_2 can be checked in time $g(n)$
- ➡ Apply an algorithm for REACHABILITY with quadratic time-complexity in the number of nodes

This yields the proposition. □

Theorem $\mathbf{NSPACE}(.) \to \mathbf{TIME}(.)$

Suppose $f(n)$ is space constructible; then

$\mathbf{NSPACE}(f(n)) \subseteq \mathbf{TIME}(k^{\log n + f(n)})$ $(= \bigcup_{d>0} \mathbf{TIME}(d^{\log n + f(n)}))$

Corollary

$\mathbf{NL} \subseteq \mathbf{P}$.

Proof

$$\begin{aligned}\mathbf{NL} = \mathbf{NSPACE}(\log n) &\subseteq \bigcup_{d>0} \mathbf{TIME}(d^{\log n + \log n}) \\ &= \bigcup_{d>0} \mathbf{TIME}((2^{\log d})^{\log n \cdot 2}) \\ &= \bigcup_{d>0} \mathbf{TIME}(n^{2 \cdot \log d}) \\ &\subseteq \bigcup_{d>0} \mathbf{TIME}(n^d) = \mathbf{P} \quad \square\end{aligned}$$

Proof of the Theorem

Assume $\mathrm{L} \in \mathbf{NSPACE}(f(n))$. Then there exists a NTM N with input and output deciding $x \in \mathrm{L}$ in space $f(n)$, $n = |x|$.

Representation of configurations

- let $\mathrm{N} = (K, \Sigma, \Delta, s)$ be a k-string TM; the configuration is defined as

$$(q, w_1, u_1, \ldots, w_k, u_k)$$

- a configuration of a TM with input and output that decides L can be represented as:

$$(q, i, w_2, u_2, \ldots, w_{k-1}, u_{k-1})$$

- pointer (i) to the position of cursor on input tape is kept
- output tape are dropped

Estimation

As N uses at most $f(n)$ space, the number of possible configurations is bounded by

$$|K| \cdot (n+1) \cdot |\Sigma|^{2(k-2)f(n)}$$

Total number of configurations is at most $nc_1^{f(n)} =: c_2^{\log n + f(n)}$

Reachability method:

- ➡ Whether configuration C_1 yields C_2 takes $\mathcal{O}(f(n))$ steps
- ➡ We have at most $c_2^{\log n + f(n)}$ many configurations
- ➡ We can decide L by a deterministic TM in time at most $(c_2^{\log n + f(n)})^2 \cdot c_3 \cdot f(n) =: c^{\log n + f(n)}$

Remark

The configuration graph $G(\mathrm{N}, x)$ can be implicitly represented:

- ➡ Use i to find the symbol scanned on the input tape of N

Summary

Corollary

$$\mathbf{L} \subseteq \mathbf{NL} \subseteq \mathbf{P} \subseteq \mathbf{NP} \subseteq \mathbf{PSPACE} \subseteq \mathbf{EXP}$$

Proof Idea

- ➡ $\mathbf{L} \subseteq \mathbf{NL}$ $\quad$ $\mathbf{SPACE}(f(n)) \subseteq \mathbf{NSPACE}(f(n))$
- ➡ $\mathbf{NL} \subseteq \mathbf{P}$ $\quad$ $\mathbf{NSPACE}(f(n)) \subseteq \mathbf{TIME}(k^{\log n + f(n)})$
- ➡ $\mathbf{P} \subseteq \mathbf{NP}$ $\quad$ $\mathbf{TIME}(f(n)) \subseteq \mathbf{NTIME}(f(n))$
- ➡ $\mathbf{NP} \subseteq \mathbf{PSPACE}$ $\quad$ $\mathbf{NTIME}(f(n)) \subseteq \mathbf{SPACE}(f(n))$
- ➡ $\mathbf{PSPACE} \subseteq \mathbf{EXP}$ $\quad$ $\mathbf{SPACE}(f(n)) \subseteq \mathbf{TIME}(k^{\log n + f(n)})$

Observation

space hierarchy theorem

$$\mathbf{L} \subsetneq \mathbf{PSPACE}$$

All inclusions are conjectured to be strict