

Algorithm Theory

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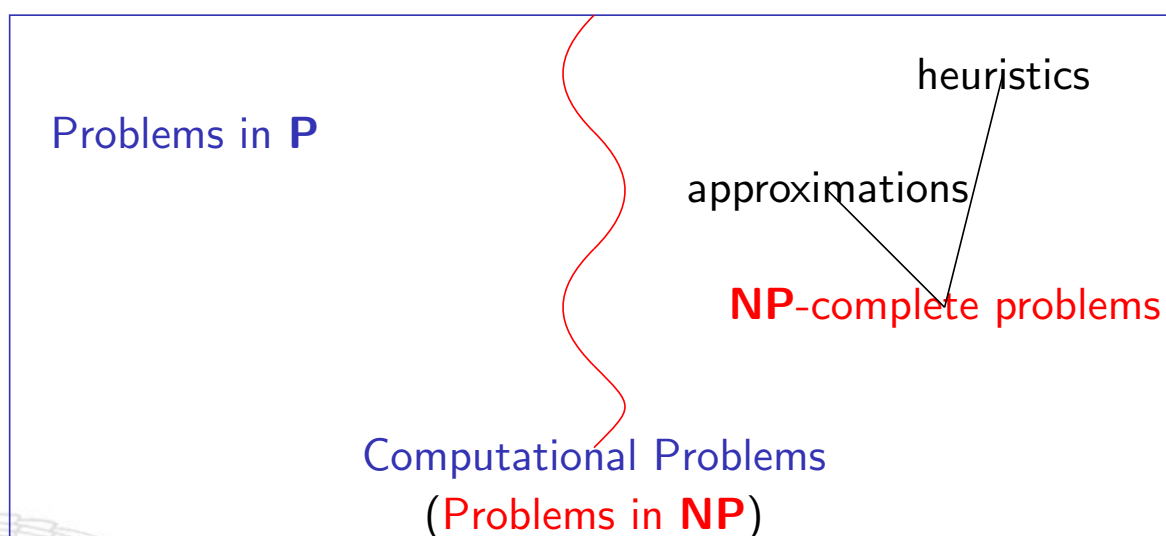
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3SAT

2SAT

MAX2SAT

The two faces of complexity theory



➔ “There is nothing wrong with trying to prove $\mathbf{P} = \mathbf{NP}$ by developing a **P**-algorithm for an **NP**-complete problem. However, without **NP**-completeness proof we wouldn't know”

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Theorem

CIRCUIT SAT reduces to SAT

Proof

- given a circuit C
- define a Boolean expression $R(C)$ with C is satisfiable iff $R(C)$ is satisfiable

Definition

construction

- variables of $R(C)$ contain all variables in C
+ for any gate g a variable g
- g a variable gate, corresponding to x add CNF of $(g \leftrightarrow x)$
- g a **true** gate add clause (g)
- g **false** gate add clause $(\neg g)$
- g a not gate add CNF of $(g \leftrightarrow \neg h)$

here h is the predecessor of g in C

- g an or gate add CNF of $(g \leftrightarrow (h \vee h'))$
- g an and gate add CNF of $(g \leftrightarrow (h \wedge h'))$
- g the output gate add (g)

here h, h' are the predecessors of g in C

□

Definition

3SAT

- a **3CNF-formula** is a CNF,
where each clause has exactly 3 literals
- **3SAT** = $\{\varphi : \varphi \text{ is a satisfiable 3CNF-formula}\}$

Theorem

3SAT is **NP**-complete

Proof

- reduction from CIRCUIT SAT produces clauses D with $|D| \leq 3$
- duplicate literals if necessary

□

Variants of 3SAT

Definition

3SAT₁

$$3SAT_1 = \left\{ \varphi \mid \varphi \text{ is a satisfiable 3CNF-formula and each clause contains distinct variables} \right\}$$

Theorem

3SAT₁ is **NP**-complete

Definition

3SAT₂

$$3SAT_2 = \left\{ \varphi \mid \begin{array}{l} \varphi \text{ is a satisfiable CNF-formula, for each clause } C, |C| \leq 3 \\ \text{and each variable occurs at most 3 times and each literal} \\ \text{at most twice in the formula} \end{array} \right\}$$

Theorem

3SAT₂ is **NP**-complete

Proof Idea

rewrite instances of 3SAT such that the restriction is fulfilled

Proof

➔ given $\varphi \in 3CNF$

suppose the variable x occurs $k > 3$ times in φ

Definition

construction

- 1 introduce k new variables x_1, \dots, x_k
- 2 replace distinct occurrences by distinct variables
- 3 add $(\neg x_1 \vee x_2) \wedge (\neg x_2 \vee x_3) \wedge \dots \wedge (\neg x_k \vee x_1)$
- 4 number of occurrences of other variables is unchanged
- 5 number of occurrences of literals in the new clauses is ≤ 2

repeat for all variables violating the restriction

□

Definition

2SAT

$$2SAT = \{\varphi : \varphi \text{ is a satisfiable 2CNF-formula}\}$$

Theorem

2SAT is in **P**, more precisely in **NL**.

Definition

 $G(\varphi)$

φ be a 2CNF formula

define a **graph** $G(\varphi)$:

- 1 nodes of $G(\varphi)$ are the variables (and their negations) of φ
- 2 for any pair $(\alpha, \beta) \in G$:
if exists clause $\neg\alpha \vee \beta$ in φ , add (α, β) to the set of edges

Fact

If (α, β) is an edge, then so is $(\neg\beta, \neg\alpha)$

Theorem

φ is unsatisfiable iff \exists variable x such that \exists paths from x to $\neg x$ and from $\neg x$ to x in $G(\varphi)$

Proof

if

- ➔ suppose there exists a path from x to $\neg x$ **and** from $\neg x$ to x ; further there is a satisfying assignment T
- ➔ we assume $T(x) = \mathbf{true}$
- ➔ (i) $T(x) = \mathbf{true}$ (ii) $T(\neg x) = \mathbf{false}$
(iii) \exists path between x and $\neg x$
- ➔ \exists an edge $(\alpha, \beta) \in G(\varphi)$: $T(\alpha) = \mathbf{true}$, $T(\beta) = \mathbf{false}$
- ➔ \exists clause in φ : $\neg\alpha \vee \beta$ and $T(\neg\alpha \vee \beta) = \mathbf{false}$
- ➔ contradiction

only-if

→ suppose no such paths exists; we define assignment T

Definition

- 1 pick a node α , not yet assigned a truth value
- 2 such that there is no path from α to $\neg\alpha$
- 3 set α and all successors to **true**
- 4 set $\neg\alpha$ and all predecessors to **false**

→ this is **well-defined**

→ only β or $\neg\beta$ can be successor of α

→ we cannot change existing truth-values

□

Theorem

2SAT is in $\mathbf{NL} \subseteq \mathbf{P}$

Proof

\mathbf{NL} is closed under complement, it suffices to recognise **unsatisfiable** expressions

Definition

algorithm

- 1 guess a variable x
- 2 check the existence of a path from x to $\neg x$ (and back)
- 3 instance of REACHABILITY which is decidable by a NTM operating in log-space

□

Definition

MAX2SAT

MAX2SAT = $\left\{ \varphi : \varphi \text{ is a 2CNF-formula and at least } K \text{ clauses are satisfiable.} \right\}$

Theorem

MAX2SAT is **NP**-complete

$$\begin{array}{cccc} (x) & (y) & (z) & (w) \\ (\neg x \vee \neg y) & (\neg y \vee \neg z) & (\neg z \vee \neg x) & \\ (x \vee \neg w) & (y \vee \neg w) & (z \vee \neg w) & \end{array}$$

if $T(x \vee y \vee z) = \mathbf{true}$

7 clauses can be satisfied; otherwise only 6

Proof

$$\begin{array}{cccc} (x) & (y) & (z) & (w) \\ (\neg x \vee \neg y) & (\neg y \vee \neg z) & (\neg z \vee \neg x) & \\ (x \vee \neg w) & (y \vee \neg w) & (z \vee \neg w) & \end{array}$$

block

MAX2SAT \in **NP**

easy

reduction from 3SAT

- ➔ let $\varphi \in 3\text{CNF}$
- ➔ let m denote the number of clauses in φ

Definition

construction

- 1 for any clause $C_i = (\alpha \vee \beta \vee \gamma)$ in φ
add the block with w_i fresh to $R(\varphi)$
- 2 set $K = 7m$

$\varphi \in 3\text{SAT}$ iff $R(\varphi) \in \text{MAX2SAT}$

→ suppose φ is satisfied

for each block 7 clauses are satisfied

thus $R(\varphi) \in \text{MAX2SAT}$

→ suppose in $R(\varphi)$ $7m$ clauses are satisfied

hence in each block at most 7 clauses can be satisfied

hence in any block $\alpha \vee \beta \vee \gamma$ is satisfied

hence any clause in φ is satisfied

logspace reduction

easy to check that computation of R needs only log-space

□

this form of approach to a reduction is called **gadget construction**;
the blocks form the gadget