Complexity Theory

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Outline

- Summary of Last Lecture: Savitch's Theorem
- Exercises
- Deterministic Separation Results
- Nondeterministic Separation Results
- A Space-Constructible Function $S(n) \leqslant \mathrm{O}(\log \log n)$

Theorem 1

Assume $S(n) \geqslant \log n$, then

$$
\begin{aligned}
①&: \mathrm{DTIME}(T(n)) \subseteq \mathrm{DSPACE}(T(n)) \\
②&: \mathrm{NTIME}(T(n)) \subseteq \mathrm{NSPACE}(T(n)) \\
③&: \mathrm{DSPACE}(S(n)) \subseteq \mathrm{DTIME}(2^{\mathrm{O}(S(n))}) \\
④&: \mathrm{NSPACE}(S(n)) \subseteq \mathrm{NTIME}(2^{\mathrm{O}(S(n))})
\end{aligned}
$$

Theorem 2

Assume $S(n) \geqslant \log n$, then

$$
\begin{aligned}
①&: \mathrm{NTIME}(T(n)) \subseteq \mathrm{DSPACE}(T(n)) \\
②&: \mathrm{NSPACE}(S(n)) \subseteq \mathrm{DTIME}(2^{\mathrm{O}(S(n))})
\end{aligned}
$$

Theorem (Savitch)

Let $S(n) \geqslant \log n$. Then

$$\mathrm{NSPACE}(S(n)) \subseteq \mathrm{DSPACE}(S(n)^2) \text{ ,}$$

in particular $\mathrm{PSPACE} = \mathrm{NPSPACE}$

Deterministic Separation Results

Theorem

Let $S(n)$ be space-constructible. Then there exists a set in DSPACE($S(n)$) that is not in DSPACE($S'(n)$) for any $S'(n) = o(S(n))$.

Proof

The idea of the proof is to use a diagonalisation argument; let

$$M_0, M_1, \ldots$$

be a list of all Turing machines with binary input alphabet.

- assume $(i)_2$ is the code machine M_i that allows simulation by an universal TM
- assume $\forall i$, $(i)_2$ encodes some Turing machine
- assume in the encoding leading zeros are ignored; i.e., we can pad the code with arbitrary zeros from the left
- hence any M_i has arbitrary long codes (by padding)

Proof (cont'd)

Let x be a binary string and $\#(x)$ denotes the number represented by x

Construction of machine M:

1. mark $S(n)$ many cells on the worktape
2. simulate M_i on input x, where $i = \#(x)$ not exceeding space $S(n)$
3. one of the following cases happens:
   (a) there is enough space to simulate M_i on x:

   M reverses the behaviour of M_i
   if M_i accepts, M rejects and vice versa

   (b) M_i loops:

   M loops as well

   (c) the simulation needs more space than $S(n)$:

   M halts and rejects

Proof (cont'd)

Let M_i be $o(S(n))$ space-bounded, then

- M will simulate M_i on x for all sufficiently large x with $\#(x) = i$
- by assumption the simulation doesn't need more space than $S(n)$
- hence $\mathrm{L}(M) \neq \mathrm{L}(M_i)$

In sum, the Turing machine M differs from any machine running in $o(S(n))$ space. ∎

Theorem

Let $T(n)$ be time-constructible, $T(n) \geqslant n$. Then there exists a set in $\mathrm{DTIME}(T(n))$ that is not in $\mathrm{DTIME}(T'(n))$ for any $T'(n)$ such that $T'(n) \log T'(n) = o(T(n))$

Proof

similar ∎

Nondeterministic Separation Results

Lemma

$$\mathrm{NSPACE}(n^3) \subsetneq \mathrm{NSPACE}(n^4)$$

Proof

Assume otherwise $\mathrm{NSPACE}(n^4) \subseteq \mathrm{NSPACE}(n^3)$
we use the following

Claim

$$\mathrm{NSPACE}(n^4) \subseteq \mathrm{NSPACE}(n^3) \Longrightarrow \mathrm{NSPACE}(n^5) \subseteq \mathrm{NSPACE}(n^4)$$

repeating the pattern of the proof of the claim and using the assumption, we obtain: $\mathrm{NSPACE}(n^7) \subseteq \mathrm{NSPACE}(n^3)$ hence

$$\begin{aligned}
\mathrm{NSPACE}(n^7) &\subseteq \mathrm{NSPACE}(n^3) \\
&\subseteq \mathrm{DSPACE}(n^6) && \text{by Savitch's theorem} \\
&\subsetneq \mathrm{DSPACE}(n^7) && \text{by the previous theorem} \\
&\subseteq \mathrm{NSPACE}(n^7) \, ,
\end{aligned}$$

which is the desired contradiction. ∎

Claim

$$\mathrm{NSPACE}(n^4) \subseteq \mathrm{NSPACE}(n^3) \implies \mathrm{NSPACE}(n^5) \subseteq \mathrm{NSPACE}(n^4)$$

Proof

- assume M is an arbitrary nondeterministic machine running in space n^5
- let $A = \mathrm{L}(M)$
- consider $A' := \{x\#^{|x|^{\frac{5}{4}}-|x|} \mid x \in A\}$

Construction of machine M':

1. M' gets $x\#^m$ as input
2. M' checks whether $m = |x|^{\frac{5}{4}} - |x|$
3. if yes M' runs M on x.

M' needs the same space as M, hence it runs in space

$$|x|^5 = (|x|^{\frac{5}{4}})^4 = |x\#^{|x|^{\frac{5}{4}}-|x|}|^4 ,$$

thus $A' = \mathrm{L}(M') \in \mathrm{NSPACE}(n^4) \subseteq \mathrm{NSPACE}(n^3)$

Proof (cont'd)

Assume the nondeterministic Turing machine M'' accepts A' in space n^3.

Construction of machine M''':

1. M''' works on input x
2. M''' appends a string of $\#$s of length $|x|^{\frac{5}{4}} - |x|$ to x
3. M''' runs M'' on the resulting string.

the space used by M'' is estimateable by the space used by M''', hence

$$|x\#^{|x|^{\frac{5}{4}}-|x|}|^3 = (|x|^{\frac{5}{4}})^3 = |x|^{\frac{15}{4}} \leqslant |x|^4 ,$$

and $\mathrm{L}(M''') = A$. ∎

A Space-Constructible Function $S(n) \leqslant O(\log \log n)$

Definition

1. let $\ell(n)$ be the least positive number not dividing n
2. note that $\ell(n)$ is a prime power
3. $\ell(n) = O(\log n)$, because for every ℓ:

$$\prod_{p \leqslant \ell \;\; p \text{ prime}} p \geqslant 2^{\Omega(\ell)}$$

Lemma

$\exists$ an unbounded space-constructible function $S(n)$ that is $O(\log \log n)$, namely the space needed to accept $A = \{a^n \mid \ell(n) \text{ is prime}\}$.

Lemma

The function $\lceil \log \log n \rceil$ is not space constructible.