Complexity Theory

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Outline

- Summary of Last Lecture: Separation Results
- Exercises
- The Immerman-Szelepcsényi Theorem

Separation Results

Theorem
Let $S(n)$ be space-constructible. Then there exists a set in DSPACE($S(n)$) that is not in DSPACE($S'(n)$) for any $S'(n) = o(S(n))$.

Lemma
$$\mathrm{NSPACE}(n^3) \subsetneq \mathrm{NSPACE}(n^4)$$

A Space-Constructible Function $S(n) \leqslant O(\log\log n)$

Definition
1. let $\ell(n)$ be the least positive number not dividing n
2. note that $\ell(n)$ is a prime power
3. $\ell(n) = O(\log n)$, because for every ℓ:

$$\prod_{p \leqslant \ell \;\; p \text{ prime}} p \geqslant 2^{\Omega(\ell)}$$

Lemma
$\exists$ an unbounded space-constructible function $S(n)$ that is $O(\log\log n)$, namely the space needed to accept $A = \{a^n \mid \ell(n) \text{ is prime}\}$.

Lemma
The function $\lceil \log\log n \rceil$ is not space constructible.

Homework

1. Miscellaneous Exercises 61 in Kozen, Automata and Computability
2. Homework 1.2
3. Homework 1.3
4. Homework 2.1
5. Homework 2.2
6. Homework 2.3

Theorem — Immerman-Szelepcsényi Theorem

For $S(n) \geqslant \log n$, $\mathrm{NSPACE}(S(n)) = co - \mathrm{NSPACE}(S(n))$.

Proof Idea

1. assume we have finite set A and a nondeterministic test for membership
2. assume we know the size of A
3. we can test for non-membership as follows:
   - given y
   - let $n := |A|$
   - guess n elements x, make sure $x \in A$ and $x \neq y$
   - hence $y \notin A$

Two cases:

① Assume $S(n)$ is space-constructible

② Assume $S(n) \geqslant \log n$

Proof (for ①)

1. let M be nondeterministic $S(n)$-space bounded TM; we wish to build a $S(n)$-space bounded TM N accepting the complement of $\mathrm{L}(M)$
2. suppose encoding of the configurations of M over Δ $|\Delta| = d$
3. configurations on inputs of length n become strings in $\Delta^{S(n)}$
4. assume there is a unique accept configuration $\texttt{accept} \in \Delta^{S(n)}$
5. let $\texttt{start} \in \Delta^{S(n)}$ denote the start configuration on input x $|x| = n$
6. if x is accepted, the computation has length $\leqslant d^{S(n)}$
7. define $A_m = \{\alpha \in \Delta^{S(n)} \mid \texttt{start} \xrightarrow{\leqslant m} \alpha\}$
8. thus $A_0 = \{\texttt{start}\}$ and M accepts x iff $\texttt{accept} \in A_{d^{S(n)}}$

Construction of N

- lay off $S(n)$ space on the worktape $S(n)$ is space-constructible
- compute sizes $|A_0|, |A_1|, \ldots$ inductively
- first $|A_0| = 1$
- suppose $|A_m|$ is computed and written on the worktape:
  1. write down $\beta \in \Delta^{S(n)}$
  2. determine whether $\beta \in A_{m+1}$
  3. increment counter
- suppose $|A_{d^{S(n)}}|$ is known
- guess $|A_{d^{S(n)}}|$ α's to verify that $\texttt{start} \xrightarrow{\leqslant d^{S(n)}} \alpha$ holds
- test $\alpha \neq \texttt{accept}$
- if successful: $\texttt{accept} \notin A_{d^{S(n)}}$ and N can accept

the above steps can be implemented such that N uses at most $S(n)$ space

Determine $\beta \in A_{m+1}$

- guess $|A_m|$ elements α of A_m in lexicographic order
- verify for each $\alpha \in A_m$ by guessing a path to test $\texttt{start} \xrightarrow{\leqslant m} \alpha$
- test $\alpha \xrightarrow{\leqslant 1} \beta$
- if $\exists$ α with $\alpha \xrightarrow{=1} \beta$: $\beta \in A_{m+1}$
- if $\forall$ α with $\alpha \not\xrightarrow{=1} \beta$: $\beta \notin A_{m+1}$

In sum: the machine N accepts complement of $\mathrm{L}(M)$ is space $S(n)$

Proof (for ②)

To remove the space-constructible assumption for $S(n)$ do the construction on the fly, increasing the available space if necessary

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