Complexity Theory

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Outline

- Summary of Last Lecture: Separation Results
- Exercises
- Logspace Computability
- Logspace Transducers
- Logspace Reducibility
- Completeness

The Immerman-Szelepcsényi Theorem

Theorem (Immerman-Szelepcsényi Theorem)

For $S(n) \geqslant \log n$, $\mathrm{NSPACE}(S(n)) = co - \mathrm{NSPACE}(S(n))$.

Proof Idea

1. assume we have finite set A and a nondeterministic test for membership
2. assume we know the size of A
3. we can test for non-membership as follows:
   - given y
   - let $n := |A|$
   - guess n elements x, make sure $x \in A$ and $x \neq y$
   - hence $y \notin A$

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Homework

1. Homework 3.1
2. Homework 3.3
3. Homework 6.1
4. Miscellaneous Exercise 9

Logspace Computability

The following resources are equally powerful:

1. logarithmic workspace
2. counting up to the length of the input n
3. "finite fingers"

the following automata describe the same class of languages:

1. logspace-bounded TMs
2. k-counter automaton with linearly bounded counters
   - automata with two-way read-only input head
   - k integer counters, tests for zero, add, subtract
   - counters can hold values between 0 and n
3. k-headed two-way finite automaton (k-FA for short)
   - k two-way read-only input heads
   - can only move on the input

Logspace Transducers

Definition — logspace transducer

- a total deterministic logspace-bounded TM with output is called logspace transducer
- total means: it halts on all inputs
- with output means: $\exists$ write-only output tape
- hence, a logspace transducer has:
  1. a two-way read-only input tape
  2. a two-way read/write logspace-bounded worktape initially blank
  3. a write-only left-to-right output tape initially blank
  4. Σ is the input alphabet
  5. Γ is the worktape alphabet
  6. Δ is the output alphabet

Definition — logspace computable

function $\sigma\colon \Sigma^* \to \Delta^*$ is logspace computable if $\exists$ logspace transducer computing (in the natural way) σ

Lemma

output of a logspace transducer is polynomially bounded in length, i.e.,

- $\forall$ logspace computable $\sigma\colon \Sigma^* \to \Delta^*$
- $\exists$ constant d
- $\forall\ x \in \Sigma^*$, $|\sigma(x)| \leqslant |x|^d$

Proof

1. each step can induce only one output symbol
2. and the transducer can run at most $2^{O(\log n)}$ many steps
3. otherwise configurations would be repeated

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Logspace Reducibility

Definition — logspace reducibility

- for $A \subseteq \Sigma^*$, $B \subseteq \Delta^*$
- we write $A \leqslant_m^{\log} B$ if $\exists$ logspace-computable function $\sigma\colon \Sigma^* \to \Delta^*$ with

$$x \in A \quad \text{if and only if} \quad \sigma(x) \in B$$

- we say A is logspace reducible to B

Lemma ①

the relation $\leqslant_m^{\log}$ is transitive; i.e., from $A \leqslant_m^{\log} B$ and $B \leqslant_m^{\log} C$ follows $A \leqslant_m^{\log} C$

Lemma ②

for $A \subseteq \Sigma^*$: $A \in \text{LOGSPACE}$ iff $A \leqslant_m^{\log} \{0, 1\}$

Lemma

if $A \leqslant_m^{\log} B$ and $B \in$ LOGSPACE, then $A \in$ LOGSPACE

Proof

1. assume $A \leqslant_m^{\log} B$ and $B \in$ LOGSPACE
2. by Lemma ② we have $B \leqslant_m^{\log} \{0,1\}$
3. by Lemma ① we have $A \leqslant_m^{\log} \{0,1\}$
4. hence $A \in$ LOGSPACE by Lemma ②

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Definition

we write $\leqslant_m^{\mathrm{p}}$ for Karp reducibility,
i.e., the sometimes studied polytime many-one reducibility

Lemma

if $A \leqslant_m^{\log} B$, then $A \leqslant_m^{\mathrm{p}} B$

Completeness

Definition — hardness

a set $A \subseteq \Sigma^*$ is $\leqslant_m^{\log}$-hard for a complexity class $\mathcal{C}$ if

- $\forall B \in \mathcal{C}$ we have $B \leqslant_m^{\log} A$

Definition — completeness

a set $A \subseteq \Sigma^*$ is complete for $\mathcal{C}$ with respect to $\leqslant_m^{\log}$ if

1. A is $\leqslant_m^{\log}$-hard for $\mathcal{C}$
2. $A \in \mathcal{C}$

Definition — MAZE

- given a directed graph $G = (V, E)$ and nodes $s, t \in V$
- determine whether there is a directed path from s to t in G
- this problem is called MAZE or directed graph reachability

Theorem — Jones, Lien, and Lasser (1976)

MAZE is $\leqslant_m^{\log}$-complete for NLOGSPACE

Proof

on the black board

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Corollary

MAZE $\in$ LOGSPACE if and only if LOGSPACE $=$ NLOGSPACE