Complexity Theory

Georg Moser

Institute of Computer Science @ UIBK

Summer 2008

Outline

- Summary of Last Lecture: Alternating Turing Machines
- Exercises
- Quantified Boolean Formula Problem
- Complexity of Games

Definition — ATM

- an alternating Turing machine is defined like a nondeterministic Turing machine, but includes a function

$$\text{type}\colon Q \to \{\wedge, \vee, \neg\}$$

- a configuration is called $\wedge$-, $\vee$-, or $\neg$-configuration, depending on the type of its state
- all $\neg$-configurations have exactly one successor
- accept and reject states are formalised implicitly

Definition — three valued logic

we write $\bot$ for the truth value "don't know"

$\vee$	1	$\bot$	0
1	1	1	1
$\bot$	1	$\bot$	$\bot$
0	1	$\bot$	0

$\wedge$	1	$\bot$	0
1	1	$\bot$	0
$\bot$	$\bot$	$\bot$	0
0	0	0	0

$\neg$	
1	0
$\bot$	$\bot$
0	1

Definition — information order

the information order is defined as $\bot \sqsubseteq 0$, $\bot \sqsubseteq 1$

Definition — labeling ℓ

let $\mathcal{C}$ a set of configuration, a labeling is a map $\ell\colon \mathcal{C} \to \{0, 1, \bot\}$; we define

$$\ell \sqsubseteq \ell' :\Longleftrightarrow \forall \alpha \in \mathcal{C}\ \ell(\alpha) \sqsubseteq \ell'(\alpha)$$

Lemma

the set of labelings together with $\sqsubseteq$ form a complete lattice, i.e., every set of labelings has a supremum

Definition — τ

we define an operator on labels

$$\tau(\ell)(\alpha) := \begin{cases} \bigwedge_{\alpha \to \beta} \ell(\beta) & \alpha \text{ an } \wedge\text{-configuration} \\ \bigvee_{\alpha \to \beta} \ell(\beta) & \alpha \text{ an } \vee\text{-configuration} \\ \neg \ell(\beta) & \alpha \text{ a } \neg\text{-configuration and } \alpha \to \beta \end{cases}$$

define ℓ_* as the $\sqsubseteq$-least fixpoint of τ

Alternating Complexity Classes

Definition

$$\text{ALOGSPACE} := \text{ASPACE}(\log n) \qquad \text{APTIME} := \text{ATIME}(n^{O(1)})$$

$$\text{APSPACE} := \text{ATIME}(n^{O(1)}) \qquad \text{AEXPTIME} := \text{ATIME}(2^{n^{O(1)}})$$

Theorem

let $T(n) \geqslant n$ and $S(n) \geqslant \log n$

1. $\text{ATIME}(T(n)) \subseteq \text{DSPACE}(T(n))$
2. $\text{DSPACE}(S(n)) \subseteq \text{ATIME}(S(n)^2)$
3. $\text{ASPACE}(S(n)) \subseteq \text{DTIME}(2^{O(S(n))})$
4. $\text{DTIME}(T(n)) \subseteq \text{ASPACE}(\log T(n))$

Corollary

for $T(n) \geqslant n$ and $S(n) \geqslant \log n$: $\text{ATIME}(T(n)^{O(1)}) = \text{DSPACE}(T(n)^{O(1)})$ and $\text{ASPACE}(S(n)) = \text{DTIME}(2^{O(S(n))})$

Homework

1. Extend the definition of complete lattice to a formal one and prove that for a complete lattice U, for any $A \subseteq U$, $\sup A$ is unique.
2. Miscellaneous Exercise 19
3. Miscellaneous Exercise 22
4. Miscellaneous Exercise 26

Definition QBF

the quantified Boolean formula problem is the problem of determining the truth of

$$Q_1 x_1 Q_2 x_2 \dots Q_n x_n B(x_1, \dots, x_n)$$

$B(x_1, \dots, x_n)$ is a Boolean formula, $Q_i \in \{\forall, \exists\}$ with quantification over $\{0, 1\}$

Observation

- $\exists x_1 \dots \exists x_n B(x_1, \dots, x_n)$-QBF $=$ SAT
- $\forall x_1 \dots \forall x_n B(x_1, \dots, x_n)$-QBF $=$ VALIDITY

Theorem

QBF is $\leqslant_m^{\log}$-complete for PSPACE

Definition information game

a 2-person perfect information game is a graph $G = (\text{BOARDS}, \text{MOVE})$ and a start state $s \in \text{BOARDS}$

- BOARDS defines the play field
- MOVE specifies the legal actions
- players alternate, Player I starts in s
- Player I chooses $s_1 \in$ BOARDS, such that MOVE(s, s_1)
- Player II chooses $s_2 \in$ BOARDS, such that MOVE(s_1, s_2) $\dots$
- a player wins, if the other cannot make a move

Definition

- $\text{CHECKMATE}(y) :\Leftrightarrow \forall z \neg \text{MOVE}(y, z)$
- $\text{WIN}(x) \Leftrightarrow \exists \text{MOVE}(x, y) \wedge (\text{CHECKMATE}(y) \vee \forall z(\text{MOVE}(y, z) \rightarrow \text{WIN}(z)))$
- for $x \in \text{BOARDS} \wedge \text{WIN}(x)$, x is forced win

Observation

monotone operator can be simplified

$$\mathsf{WIN}(x) \Leftrightarrow \exists \mathsf{MOVE}(x, y) \wedge \forall z(\mathsf{MOVE}(y, z) \rightarrow \mathsf{WIN}(z))$$

Definition

generalised geography

a triple (C, E, s) is called a generalised geography game if

- (C, E) directed graph
- $s \in C$
- Player I starts in $s = s_0$ and moves in even states from s_{2i} to adjacent vertex s_{2i+1}
- Player II moves from s_{2i+1} to s_{2i+2}
- vertices must not be revisited

Theorem

Generalised geography is $\leqslant_m^{\log}$-complete for PSPACE