

Experiments in Verification

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Sets in Isabelle

Type

(* A set is defined by its characteristic function. *)

types 'a set = ('a \Rightarrow bool)

Membership

- ▶ x is member of set S if characteristic function returns True
- ▶ **lemma** mem_def: " $x \in S \equiv S\ x$ "

This Time

Session 1

formal verification, Isabelle/HOL basics, functional programming in HOL

Session 2

simplification, function definitions, induction, calculational reasoning

Session 3

natural deduction, propositional logic, predicate logic

Session 4

sets, relations, inductively defined sets, advanced topics

Basic Operations on Sets

Intersection

- ▶ notation: $A \cap B$ (ASCII: $A \text{ Int } B$)
- ▶ IntI: $\llbracket c \in A; c \in B \rrbracket \Longrightarrow c \in A \cap B$
- ▶ IntD1: $c \in A \cap B \Longrightarrow c \in A$
- ▶ IntD2: $c \in A \cap B \Longrightarrow c \in B$

Union

- ▶ notation: $A \cup B$ (ASCII: $A \text{ Un } B$)
- ▶ UnI1: $c \in A \Longrightarrow c \in A \cup B$
- ▶ UnI2: $c \in B \Longrightarrow c \in A \cup B$
- ▶ UnE: $\llbracket c \in A \cup B; c \in A \Longrightarrow P; c \in B \Longrightarrow P \rrbracket \Longrightarrow P$

Basic Operations on Sets (cont'd)

Complement and Difference

- ▶ complement: $\neg A$
- ▶ Compl_iff: $(c \in \neg A) = (c \notin A)$
- ▶ difference: $A - B$

Subsets

- ▶ notation: $A \subseteq B$ (ASCII: $A \leq B$)
- ▶ subsetI: $(\bigwedge x. x \in A \implies x \in B) \implies A \subseteq B$
- ▶ equalityI: $\llbracket A \subseteq B; B \subseteq A \rrbracket \implies A = B$

An Example Proof

lemma

lemma " $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ "

Proof.

Isabelle

□

Set Notation

- ▶ the empty set: $\{\}$
- ▶ the universal set: UNIV
- ▶ a singleton set: $\{x\}$
- ▶ insertion (insert_is_Un): $\text{insert } x \ A = \{x\} \cup A$
- ▶ finite sets, e.g., $\{a, b, c, d\}$

A Shorter Proof

The Method blast

- ▶ applies introduction and elimination rules automatically
- ▶ suitable for many goals concerning logical and/or set operations

lemma " $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ "
by blast

Set Comprehension - By Example

Mathematics	Isabelle
$\{x \mid P(x)\}$	$\{x. P\ x\}$
$\{(x, y) \mid x \in A, y \in B\}$	$\{(x, y) \mid x \in A \wedge y \in B\}$

Relations

Basics

- ▶ a relation is a **set of pairs** ($(a \times b)$ set)
- ▶ **identity relation**: $\text{Id} \equiv \{p. \exists x. p = (x, x)\}$
- ▶ **composition**: $r \circ s \equiv \{(x, z). \exists y. (x, y) \in s \wedge (y, z) \in r\}$
- ▶ **converse**: $((a, b) \in r^{-1}) = ((b, a) \in r)$

Reflexive and Transitive Closure

- ▶ reflexive and transitive closure: r^*
- ▶ transitive closure: r^+
- ▶ `rtrancl_refl`: $(a, a) \in r^*$
- ▶ `r_into_rtrancl`: $p \in r \implies p \in r^*$
- ▶ `rtrancl_trans`: $[(a, b) \in r^*; (b, c) \in r^*] \implies (a, c) \in r^*$

Binding Operators for Sets

Operators

- ▶ bounded universal quantifier: $\forall x \in A. P\ x$
- ▶ bounded existential quantifier: $\exists x \in A. P\ x$

Useful Lemmas

- ▶ `ballI`: $(\bigwedge x. x \in A \implies P\ x) \implies \forall x \in A. P\ x$
- ▶ `bspec`: $[\bigwedge x \in A. P\ x; x \in A] \implies P\ x$
- ▶ `bexI`: $[P\ x; x \in A] \implies \exists x \in A. P\ x$
- ▶ `bexE`: $[\exists x \in A. P\ x; \bigwedge x. [x \in A; P\ x] \implies Q] \implies Q$

Example

Lemma

lemma " $(r \circ s)^{-1} = s^{-1} \circ r^{-1}$ "

Proof.

Isabelle □

An Introductory Definition

Example (Even Numbers)

```
inductive_set even :: "nat set"
where zero[intro!]: "0 ∈ even"
  | step[intro!]: "n ∈ even ⇒ Suc(Suc n) ∈ even"
```

Remarks

- ▶ `intro`: declares a lemma as introduction rule (for `blast/auto`)
- ▶ `elim`: declares a lemma as elimination rule (for `blast/auto`)
- ▶ adding a `!` tells the system that a rule is safe (i.e., it can always be applied without making the goal unprovable)
- ▶ `even` is the smallest set constructed by finitely many applications of the two rules `zero` and `step` (i.e., it contains only elements that can be added via the rules)

Even Numbers Are Divisible By 2

```
lemma even_imp_dvd: "n ∈ even ⇒ 2 dvd n"
proof (induct rule: even.induct)
  case zero show ?case by simp
next
  case (step n)
  hence IH: "2 dvd n" by simp
  then obtain k where "n = 2*k"
    unfolding dvd_def by (rule exE)
  hence "Suc(Suc n) = 2*(Suc k)" by simp
  thus ?case unfolding dvd_def by (rule exI)
qed
```

Advanced Inductive Sets

Arguments

- ▶ an inductive definition may take arguments
- ▶ hence it is possible to define functions yielding sets inductively
- ▶ the keyword `for` is used to introduce arguments

Example (Reflexive Transitive Closure)

```
inductive_set
  rtc :: "('a × 'a)set ⇒ ('a × 'a)set" ("_*" [1000] 999)
  for r :: "('a × 'a)set"
where rtc_refl[iff]: "(x,x) ∈ r*"
  | rtc_step: "(x,y) ∈ r ⇒ (y,z) ∈ r* ⇒ (x,z) ∈ r*"
```

rtc is Transitive

Lemma

```
lemma "(x,y) ∈ r* ⇒ (y,z) ∈ r* ⇒ (x,z) ∈ r*"
```

Proof.

Isabelle

□

Evaluation

LVA-Code

703523-0

Additional Questions

- (1) I can prove simple lemmas in Isabelle/HOL.
- (2) I would prefer having a final exam instead of a project.
- (3) The slides were generally helpful.
- (4) There was too little theory.

Projects

<http://isabelle.in.tum.de/exercises/advanced/sorting/ex.pdf>
<http://isabelle.in.tum.de/exercises/advanced/mergesort/ex.pdf>
<http://isabelle.in.tum.de/exercises/advanced/tries/ex.pdf>
<http://isabelle.in.tum.de/exercises/advanced/interval/ex.pdf>
<http://isabelle.in.tum.de/exercises/advanced/regmachine/ex.pdf>
<http://isabelle.in.tum.de/exercises/proj/hanoi/ex.pdf>
<http://isabelle.in.tum.de/exercises/proj/euclid/ex.pdf>
<http://isabelle.in.tum.de/exercises/proj/compSE/ex.pdf>
<http://isabelle.in.tum.de/exercises/proj/bignat/ex.pdf>
<http://isabelle.in.tum.de/exercises/proj/optComp/ex.pdf>