Seminar Report

A Reduction-Preserving Completion for Proving Confluence of Non-Terminating Term Rewriting Systems

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Abstract

We present three methods which prove confluence of term rewriting systems. In contrast to the famous Newman's Lemma, these methods work also with non-terminating aspects like commutative and associative rewrite rules. The methods split the term rewriting system into a terminating and a potentially non-terminating part. The proofs themselves are mainly based on the well–known critical pairs calculation. One method also includes the calculation of parallel critical pairs. For every method we present a step-by-step procedure which deterministically gives an answer about the confluence of a given term rewriting system. Additionally we present the relationship between and the limitations of the methods.

Contents

1 Introduction

Term rewriting systems (abbreviated TRSs) are an important formalism in logic, computer science and also in mathematics. By using TRSs one can model a system or a program algebraically and check its correctness. One of the most important properties of TRSs is confluence. Therefore, much effort is spent on developing new techniques which show confluence. The most common technique to show this property is Newman's Lemma [4, Theorem 3]. This lemma is based on termination, which is also a very important property of TRSs. As not every confluent TRS is terminating, Newman's Lemma is not always helpful. For example the following TRS R is not terminating:

$$R = \begin{cases} 1: & f(g(x), y) \to g(f(x, y)) & 2: & g(x) \to g(g(x)) \\ 3: & f(x, g(y)) \to g(f(x, y)) & 4: & g(g(x)) \to g(x) \\ 5: & f(x, y) \to f(y, x) \end{cases}$$

A TRS is non-termination if it contains associativity and commutativity rules (AC-rules). The TRS R is not terminating because of the commutativity rule 5.

In this report we present three methods which show confluence of TRSs containing non-terminating parts like AC-rules. The first method presented in Section 3 is the simplest method and is also a special case of the second method. The second method, which is presented in Section 4, uses parallel critical pairs. The last but not least method deals with linear TRSs instead of left-linear TRSs and is presented in Section 5. The full relation between the three methods is shown in Figure 1. If one method shows confluence, the TRS is confluent. In this report also the confluence of the TRS R shown above will be proved. This report is based mainly on [1] and [2]. For the definition of parallel critical pairs we also looked at [3].

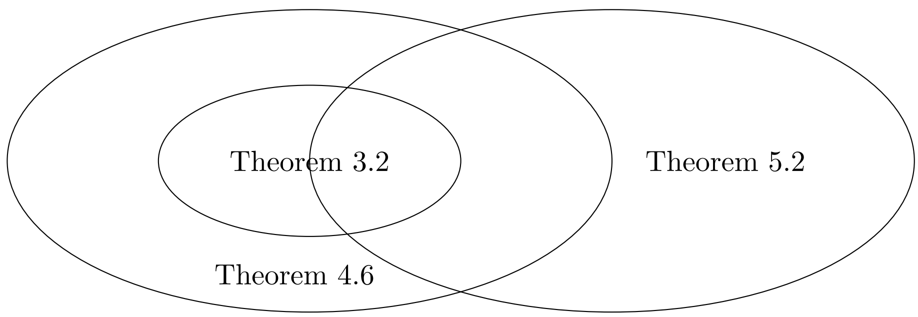

Figure 1: Relation between the three methods

2 Preliminaries

In this section we give some basic definitions, which we need for the following sections.

Definition 2.1. Let p be a position and t be a term. Then $t(p)$ denotes the symbol of term t at position p and $t|_p$ abbreviates the subterm of t at position p. The replacement of a subterm $t|_p$ by another term s is abbreviated by $t[s]_p$.

Definition 2.2. A term t of a TRS R is called a *normal form* if $t \to_R u$ holds for no term u. The set of all normal terms is denoted by $NF(R)$.

Definition 2.3. Let R be a TRS and s, t two terms of R. We use the notation

$$s \xrightarrow{!}_R t$$

if $s \xrightarrow{*}_R t$ and $t \in NF(R)$.

Definition 2.4. Let R be a TRS, s, t two terms in R and $p_1, \ldots, p_n$ parallel positions in term s. We write

$$s \mathrel{+\!\!\!\!\twoheadrightarrow}_{\{p_1,\ldots,p_n\},R}\ t$$

or simply $s \mathrel{+\!\!\!\!\twoheadrightarrow}_R t$ if there exists rewrite rules $l_1 \to r_1, \ldots, l_n \to r_n \in R$ and substitutions $\theta_1, \ldots, \theta_n$ such that $s|_{p_i} = l_i\theta_i$ for each i and $t = s[r_1\theta_1, \ldots, r_n\theta_n]_{p_1,\ldots,p_n}$. We call $s \mathrel{+\!\!\!\!\twoheadrightarrow}_R t$ a *parallel rewrite step.*

Note: The parallel rewrite step includes the identity relation.

Definition 2.5. A set of n equations is written in the form:

$$E = \{s_1 \approx t_1, \ldots, s_n \approx t_n\}$$

We call a set of equations E unifiable if a substition σ exists with $s_i\sigma = t_i\sigma$ for all i. The substitution σ is a unifier of E.

Note: The abbreviation E^{-1} denotes the same set of equations, but in reversed order: $E^{-1} = \{t_1 \approx s_1, \ldots, t_n \approx s_n\}$.

Definition 2.6. Let s, t be two terms without common variables and p a non-variable position in t. The term s *overlaps* on term t at position p, if the subterm $t|_p$ unifies with s.

Definition 2.7. A critical pair is generated when there exists an overlap between the left-hand sides of two rewrite rules. Let $l_1 \to r_1$ (from a TRS S) and $l_2 \to r_2$ (from a TRS T) be two rewrite rules without any common variables and suppose l_1 has an overlap on l_2 at position p. Let σ be a most general unfier for l_1 and $l_2|_p$. The *critical pair* is the following:

$$\langle l_2[r_1]_p\sigma, r_2\sigma\rangle$$

$CP(S,T)$ is the set of all critical pairs originating from overlaps with $l_1 \to r_1$ in S on $l_2 \to r_2$ in T. If no rule in S overlaps on a rule in T the set of critical pairs is empty.

Note: For further discussion the ordering will be important: $CP(S,T) \neq CP(T,S)$.

Definition 2.8. An *outer* (*inner*) critical pair denotes a critical pair where the overlap occurs at the position $p = \epsilon$ ($p > \epsilon$). The abbreviation CP_{out} (CP_{in}) is used for the set of outer (inner) critical pairs.

Definition 2.9. A TRS R is called *left-linear* (*linear*) if for every rewrite rule $l \to r$ in R the term(s) l (and r) is (are) linear, which means that no variable occurs more often than once in l (and r separately).

Each of the three following sections presents a theorem which will be used to show confluence of a TRS including AC-rules. Every theorem is followed by a step by step procedure in which the theorem is integrated completely (in [1, Definition 4.7] called "concrete reduction-preserving completion procedure"), and illustrated by an extensive example. The step by step procedure is needed because the theorems cannot solve every problem directly.

3 Confluence Criterion

Definition 3.1. A TRS P is called *reversible* if for all rewrite rules $l \to r$ of P the condition $r \xrightarrow{*}_P l$ holds.

Theorem 3.2. *[1, Theorem 3.7] Let S, P be TRSs such that S is left-linear and terminating and P is reversible. If*

(i) $CP(S,S) \subseteq \xrightarrow{*}_S \circ \mathrel{\leftarrow\!\!\shortparallel\!\!\rightarrow}_{P \cup P^{-1}} \circ \xleftarrow{*}_S$

(ii) $CP_{in}(P \cup P^{-1}, S) = \emptyset$

(iii) $CP(S, P \cup P^{-1}) \subseteq \xrightarrow{*}_S \circ \mathrel{\shortparallel\!\!\twoheadrightarrow}_{P \cup P^{-1}} \circ \xleftarrow{*}_S$

then $S \cup P$ is confluent.

Note: $\mathrel{\leftarrow\!\!\shortparallel\!\!\rightarrow}_{P \cup P^{-1}} = \mathrel{\shortparallel\!\!\twoheadrightarrow}_{P \cup P^{-1}}$

Before we present the step by step procedure we explain shortly the completion idea. In this procedure repeatedly rewrite rules are added to S and/or P such that the conditions of Theorem 3.2 are satisfied. The newly added rules can be simulated by the original rules, guaranteeing that the original TRS is confluent.

Lemma 3.3. *Let R be a TRS and s, t two terms. If $s \xrightarrow{*}_R t$ and the TRS $R' = R \cup \{s \to t\}$ is confluent, then R is confluent.*

Definition 3.4. (Step by step procedure)

Input: TRS R
Output: $Success$ or $Failure$

Step 1: Set $R_0 := R$ and $i := 0$.

Step 2: Choose $S_i \cup P_i = R_i$ such that S_i is left-linear and terminating, P_i is reversible and $CP_{in}(P_i \cup P_i^{-1}, S_i) = \emptyset$. If S_i and P_i do not exist return $Failure$.

Step 3: Let $U := \emptyset$. For each $\langle p, q \rangle \in CP(S_i, P_i \cup P_i^{-1})$ perform $p \xrightarrow{!}_{S_i} p'$ and $q \xrightarrow{!}_{S_i} q'$. If $p' \mathrel{\shortparallel\!\!\twoheadrightarrow}_{P \cup P^{-1}} q'$ does not hold, set $U := U \cup \{q \to p'\}$. If $U \neq \emptyset$ choose a non-empty $U' \subseteq U$ and continue with Step 2 with $R_{i+1} := R_i \cup U'$ and $i := i + 1$.

Step 4: Let $U := \emptyset$. For each $\langle p, q \rangle \in CP(S_i, S_i)$ perform $p \xrightarrow{!}_{S_i} p'$ and $q \xrightarrow{!}_{S_i} q'$. If $p' \mathrel{\leftarrow\!\!\shortparallel\!\!\rightarrow}_{P \cup P^{-1}} q'$ does not hold, set $U := U \cup \{p' \approx q'\}$. If $U = \emptyset$ return $Success$. Otherwise choose at least one rewrite rule of $U' \subseteq (U \cup U^{-1}) \cap \xleftrightarrow{*}_{P_i}$ and continue with Step 2 with $R_{i+1} := R_i \cup U'$ and $i := i + 1$.

Note: The originate TRS R is confluent, if we obtain a $Success$ as Output.

Example 3.5. Let

$$R = \begin{cases} 1: \quad x * 1 \to x & 2: \quad x * y \to y * x \\ 3: \quad x * 0 \to 0 & 4: \quad (x * y) * z \to x * (y * z). \end{cases}$$

Step 1: Set $R_0 = R$ and $i = 0$.

Step 2: We have to split TRS R_0 into two TRSs S_0 and P_0, where S_0 has to be left-linear and terminating and P_0 has to be reversible.

We choose $S_0 = \{1, 3\}$ and $P_0 = \{2, 4\}$ and prove the conditions:

Left-linearity: S_0 is left-linear as the left-hand sides of the rules 1 and 3 contain the single variable x, which occurs in both rules at most once.

Termination: S_0 is terminating as the right-hand sides of the rules 1 and 3 are proper subterms of their left-hand sides.

Reversibility: To show that P_0 is reversible both rules in P_0 have to be reversible:

2: $y * x \xrightarrow{2}_{P_0} x * y$ ✓

4: $x * (y * z) \xrightarrow{2}_{P_0} (y * z) * x \xrightarrow{2}_{P_0} (z * y) * x \xrightarrow{4}_{P_0} z * (y * x) \xrightarrow{2}_{P_0}$ $(y * x) * z \xrightarrow{2}_{P_0} (x * y) * z$ ✓

The conditions are satisfied for S_0 and P_0. Furthermore, $CP_{in}(P_0 \cup P_0^{-1}, S_0) = \emptyset$, so we continue with Step 3.

Step 3: In the following we generate the set $CP(S_0, P_0 \cup P_0^{-1})$:

Note: The abbreviation x' denotes the step where rule x of the set P_0^{-1} was applied. Furthermore, $2 \equiv 2'$.

(a) $$x \xleftarrow{1} x * 1 \xrightarrow{2} 1 * x$$

(b) $$x * z \xleftarrow{1} (x * 1) * z \xrightarrow{4} x * (1 * z)$$

(c) $$x * y \xleftarrow{1} (x * y) * 1 \xrightarrow{4} x * (y * 1)$$

(d) $$x * y \xleftarrow{1} x * (y * 1) \xrightarrow{4'} (x * y) * 1$$

(e) $$0 \xleftarrow{3} x * 0 \xrightarrow{2} 0 * x$$

(f) $$0 * z \xleftarrow{3} (x * 0) * z \xrightarrow{4} x * (0 * z)$$

$$(g)\quad \begin{tikzcd} & (x*y)*0 \arrow[dl, "3"'] \arrow[dr, "4"] & \\ 0 & & x*(y*0) \end{tikzcd} \qquad (h)\quad \begin{tikzcd} & x*(y*0) \arrow[dl, "3"'] \arrow[dr, "4'"] & \\ x*0 & & (x*y)*0 \end{tikzcd}$$

For the eight critical pairs we calculate $\xrightarrow{!}_{S_0}$ of both sides and then the additional $\nrightarrow_{P_0 \cup P_0^{-1}}$-steps have to be performed:

(a) no step possible, so $U := U \cup \{1 * x \to x\}$

(b) no step possible, so $U := U \cup \{x * (1 * z) \to x * z\}$

(c) $x * y \xleftarrow{1}_{S_0} x * (y * 1)$ ✓

(d) $x * y \xleftarrow{1}_{S_0} (x * y) * 1$ ✓

(e) no step possible, so $U := U \cup \{0 * x \to 0\}$

(f) no step possible, so $U := U \cup \{x * (0 * z) \to 0 * z\}$

(g) $0 \xleftarrow{3}_{S_0} x * 0 \xleftarrow{3}_{S_0} (x * (y * 0))$ ✓

(h) $x * 0 \xrightarrow{3}_{S_0} 0 \xleftarrow{3}_{S_0} ((x * y) * 0)$ ✓

Because of the non-joinable critical pairs $(a), (b), (e)$ and (f), confluence is not shown yet. To make these four critical pairs joinable we have to add further rules. We choose $U' \subseteq U$ to consist of the following two rules:

$5 : \; 1 * x \to x$

$6 : \; 0 * x \to 0$

We only take these two rules from the set U, because the left-hand side of 5 and 6 are proper subterms of the left-hand sides of the other two rules in U and with them we are able to make the critical pairs $(a), (b), (e)$ and (f) joinable.
We set $R_1 = R_0 \cup \{5, 6\}$ and $i = 1$ and continue with Step 2.

Step 2: We choose

$$S_1 = \{1, 3, 5, 6\}.$$

We have to check all conditions again, except the reversibility of P_1, as P_1 is equal to P_0. S_1 is left-linear as the left-hand sides of the rules 1, 3, 5 and 6 contain the single variable x, which occurs in all four rules at most once. S_1 is terminating as the right-hand sides of the rules 1, 3, 5 and 6 are proper subterms of their left-hand sides. The conditions for S_1 and P_1 are satisfied and $CP_{in}(P_1 \cup P_1^{-1}, S_1) = \emptyset$.

Step 3: It is sufficient to calculate $CP(S_1 \backslash \{1,3\}, P_1 \cup P_1^{-1})$, as $CP(\{1,3\}, P_1 \cup P_1^{-1})$ is joinable with the rules 5 and 6. In the following we generate the set $CP(\{5,6\}, P_1 \cup P_1^{-1})$:

$$(i)\quad \begin{tikzcd} & 1*x \arrow[dl, "5"'] \arrow[dr, "2"] & \\ x & & x*1 \end{tikzcd} \qquad (j)\quad \begin{tikzcd} & (1*y)*z \arrow[dl, "5"'] \arrow[dr, "4"] & \\ y*z & & 1*(y*z) \end{tikzcd} \qquad (k)\quad \begin{tikzcd} & 1*(y*z) \arrow[dl, "5"'] \arrow[dr, "4'"] & \\ y*z & & (1*y)*z \end{tikzcd}$$

$$(l)\quad \begin{tikzcd} & x*(1*z) \arrow[dl, "5"'] \arrow[dr, "4'"] & \\ x*z & & (x*1)*z \end{tikzcd} \qquad (m)\quad \begin{tikzcd} & 0*x \arrow[dl, "6"'] \arrow[dr, "2"] & \\ 0 & & x*0 \end{tikzcd} \qquad (n)\quad \begin{tikzcd} & (0*y)*z \arrow[dl, "6"'] \arrow[dr, "4"] & \\ 0*z & & 0*(y*z) \end{tikzcd}$$

$$(o)\quad \begin{tikzcd} & 0*(y*z) \arrow[dl, "6"'] \arrow[dr, "4'"] & \\ 0 & & (0*y)*z \end{tikzcd} \qquad (p)\quad \begin{tikzcd} & x*(0*z) \arrow[dl, "6"'] \arrow[dr, "4'"] & \\ x*0 & & (x*0)*z \end{tikzcd}$$

(i) $x \xleftarrow{1}_{S_1} x*1$ ✓

(j) $y*z \xleftarrow{5}_{S_1} 1*(y*z)$ ✓

(k) $y*z \xleftarrow{5}_{S_1} (1*y)*z$ ✓

(l) $x*z \xleftarrow{1}_{S_1} (x*1)*z$ ✓

(m) $0 \xleftarrow{3}_{S_1} x*0$ ✓

(n) $0*z \xrightarrow{6}_{S_1} 0 \xleftarrow{6}_{S_1} 0*(y*z)$ ✓

(o) $0 \xleftarrow{6}_{S_1} 0*z \xleftarrow{6}_{S_1} (0*y)*z$ ✓

(p) $x*0 \xrightarrow{3}_{S_1} 0 \xleftarrow{6}_{S_1} 0*z \xleftarrow{3}_{S_1} (x*0)*z$ ✓

We observe that all critical pairs are joinable and continue with Step 4.

Step 4: In the following we generate the set $CP(S_1, S_1)$:

$$
(a')\begin{tikzcd} & 0*0 \arrow[dl, "6"'] \arrow[dr, "1"] & \\ 0 & & 0 \end{tikzcd}
\quad
(b')\begin{tikzcd} & 0*1 \arrow[dl, "6"'] \arrow[dr, "3"] & \\ 0 & & 0 \end{tikzcd}
\quad
(c')\begin{tikzcd} & 1*0 \arrow[dl, "5"'] \arrow[dr, "1"] & \\ 0 & & 0 \end{tikzcd}
\quad
(d')\begin{tikzcd} & 1*1 \arrow[dl, "5"'] \arrow[dr, "3"] & \\ 1 & & 1 \end{tikzcd}
$$

All critical pairs are trivial, so criterion (i) of Theorem 3.2 holds. We have shown, that

$$S_1 \cup P_1 \text{ is confluent}$$

and *Success* is returned. So our starting TRS R is confluent.

In the following we give an example, where the procedure needs further improvements.

Example 3.6. Let

$$R = \begin{cases} 1: & 1 * y \to y & 2: & x * y \to y * x \\ 3: & x * f(y) \to f(x * y) & 4: & (x * y) * z \to x * (y * z). \end{cases}$$

The only possible combination is to set $S_0 = \{1, 3\}$ and $P_0 = \{2, 4\}$. S_0 is left-linear and terminating and P_0 is reversible. We have $CP_{in}(P_0 \cup P_0^{-1}, S_0) = \emptyset$. So the problem occurs in $CP(S_0, P_0 \cup P_0^{-1})$. We get eight critical pairs, where four of them are not joinable. So we have to add the following three rules:

$5: \; y * 1 \to y$

$6: \; f(y) * x \to f(x * y)$

$7: \; x * (f(y) * z) \to f(x * y) * z$

We need all three rules, because with them all eight critical pairs are joinable. The only possible combination is to set $S_1 = \{1, 3, 5, 6, 7\}$ and $P_1 = \{2, 4\}$. So S_1 is left-linear and terminating and P_1 is reversible. But now we get some critical pairs in $CP_{in}(P_1 \cup P_1^{-1}, S_1)$. We will not come to a satisfying solution. Definition 3.4 without improvements gives us no confluence for this example.

Definition 3.7. We improve the step by step procedure by additional steps. The first two improvents can be obtained during Step 2.

Step 2a: If $CP_{in}(P_i \cup P_i^{-1}, S_i) \neq \emptyset$ and $\exists\, l \to r \in S_i$ with $CP_{in}(P_i \cup P_i^{-1}, \{l \to r\}) \neq \emptyset$ and $\exists\, r'$ with $r \leftrightarrow_{P_i} r'$, we set $R_{i+1} := (R_i \backslash \{l \to r\}) \cup \{l \to r'\}$ and $i := i + 1$.
Step 2b: Let $\langle p, q \rangle \in CP_{in}(P_i \cup P_i^{-1}, S_i)$ and perform $q \xrightarrow{!}_{S_i} q'$. We set $R_{i+1} := R_i \cup \{p \to q'\}$ and $i := i + 1$.

The next improvement can be obtained when continuing with Step 2 after having performed Step 3.

Step 3a: We set $S_i := S_{i-1}$ and $P_i := P_{i-1}$. If $\exists\ l \to r \in S_i$ and $\exists\ r'$ with $r \leftrightarrow_{P_i} r'$ and a critical pair $\langle p, q\rangle \in CP(\{l \to r\}, P_i \cup P_i^{-1})$ we perform $p \xrightarrow{!}_{S_i} p'$ and $q \xrightarrow{!}_{S_i} q'$. If $p' \nleftrightarrow_{P \cup P^{-1}} q'$ does not hold, we set $R_{i+1} := (R_i \backslash \{l \to r\}) \cup \{l \to r'\}$ and $i := i + 1$.

Finally, the last improvement can be obtained when continuing with Step 2 after having performed Step 4.

Step 4a: We set $S_i := S_{i-1}$ and $P_i := P_{i-1}$. If $\exists\ l \to r \in S_i$ and $\exists\ r'$ with $r \leftrightarrow_{P_i} r'$ and a critical pair $\langle p, q\rangle \in CP(\{l \to r\}, S_i) \cup CP(S_i, \{l \to r\})$ we perform $p \xrightarrow{!}_{S_i} p'$ and $q \xrightarrow{!}_{S_i} q'$. If $p' \nleftrightarrow_{P \cup P^{-1}} q'$ does not hold, we set $R_{i+1} := (R_i \backslash \{l \to r\} \cup \{l \to r'\}$ and $i := i + 1$.

Example 3.8. Let

$$R = \begin{cases} 1: & 1 * y \to y \qquad\qquad 2: \quad x * y \to y * x \\ 3: & x * f(y) \to f(x * y) \qquad 4: \quad (x * y) * z \to x * (y * z). \end{cases}$$

The only possible combination is to set $S_0 = \{1, 3\}$ and $P_0 = \{2, 4\}$. S_0 is left-linear and terminating and P_0 is reversible. We have $CP_{in}(P_0 \cup P_0^{-1}, S_0) = \emptyset$. If we calculate $CP(S_0, P_0 \cup P_0^{-1})$, we get eight critical pairs, where four of them are not joinable. So we add the following two rules:

$5:\ y * 1 \to y$

$6:\ f(y) * x \to f(x * y)$

We need only these two rules, because with the improvement of 3a, we change rule 6 to rule 7 and then all computed critical pairs are joinable. Rule 7 looks like this:

$7:\ f(y) * x \to f(y * x)$

We can do this, because we set $P_1 = P_0$, $S_1 = S_0$ and we have a rule in P_1 which allows $f(x * y) \to_{P_1} f(y * x)$. So we have the following sets:

$$S_2 = \{1, 3, 5, 7\},\ P_2 = \{2, 4\}$$

Now $CP_{in}(P_2 \cup P_2^{-1}, S_2) = \emptyset$. $CP(S_2, P_2 \cup P_2^{-1})$ has sixteen (eight new) joinable elements. $CP(S_2, S_2)$ has four joinable elements. So with the improvements this example can be shown to be confluent.

4 Confluence Criterion using Parallel Critical Pairs

In the next example Theorem 3.2 fails at showing confluence.

Example 4.1. Let

$$R = \begin{cases} 1: & f(a,a,a) \to f(c,c,c) \qquad\qquad 2: \quad a \to d \\ 3: & d \to a \end{cases}$$

The only possible combination is to set $S_0 = \{1\}$ and $P_0 = \{2,3\}$. S_0 is left-linear and terminating and P_0 is reversible. But $CP_{in}(P_0 \cup P_0^{-1}, S_0)$ is not empty. We get as one of the problematic critical pairs: $\langle f(d,a,a), f(c,c,c)\rangle$. We can neither apply improvement $2a$ nor improvement $2b$. So we will not come to a satisfying solution. Theorem 3.2 gives us no confluence for this example.

Definition 4.2. Let $s_1, \ldots, s_n, t$ be $n+1$ terms without common variables and $p_1, \ldots, p_n$ be n pairwise parallel non-variable positions in t. The terms $s_1, \ldots, s_n$ *parallel overlap* on term t at positions $p_1, \ldots, p_n$, if $\{s_1 \approx t|_{p_1}, \ldots, s_n \approx t_n|_{p_n}\}$ is unifiable.

Definition 4.3. A parallel critical pair is generated when there exists a parallel overlap between left-hand sides of $n+1$ rewrite rules. Let $l_1 \to r_1, \ldots, l_n \to r_n$ (from a TRS S) and $l \to r$ (from a TRS T) be $n+1$ rewrite rules without any common variables and suppose $l_1, \ldots, l_n$ has a parallel overlap on l at parallel positions $p_1, \ldots, p_n$. The most general unifier for the unification problem $\{l_i \approx l|p_i \mid 1 \leq i \leq n\}$ is called σ. The *parallel critical pair* is the following:

$$\langle l[r_1, \ldots, r_n]_{p_1,\ldots,p_n}\sigma, r\sigma\rangle$$

$PCP(S,T)$ is the set of all parallel critical pairs originating from overlaps with $l_1 \to r_1, \ldots, l_n \to r_n$ in S on $l \to r$ in T. If no rule in S overlaps on a rule in T the set of parallel critical pairs is empty. We write $\langle l'[r_1, \ldots, r_n]_{p_1,\ldots,p_n}\sigma, r'\sigma\rangle_X$ if $X = \bigcup_{1\leq i\leq n} \mathcal{V}ar(l'\sigma|_{p_i})$.

Note: $CP(S,T) \subseteq PCP(S,T)$ always holds.

Definition 4.4. The term outer (inner) parallel critical pair denotes a parallel critical pairs with $p_1 = \epsilon$ ($p_i > \epsilon$ for all i). The abbreviation PCP_{out} (PCP_{in}) is used.

Note: For two TRSs S and T we have: $PCP_{out}(S,T) = CP_{out}(S,T)$.

Example 4.5. Let

$$R = \Big\{ 1: \quad f(a,a,a) \to c \qquad\qquad 2: \quad a \to b$$

The first step is to calculate the critical pairs $CP(R,R)$:

(a) $\langle f(b,a,a), c\rangle$

(b) $\langle f(a,b,a), c\rangle$

(c) $\langle f(a,a,b), c\rangle$

Then, we calculate the parallel critical pairs with an increasing number of parallel-overlaps, starting with two parallel positions:

(d) $\langle f(b, b, a), c \rangle$

(e) $\langle f(b, a, b), c \rangle$

(f) $\langle f(a, b, b), c \rangle$

Finally, there is one parallel critical pair with three parallel-positions:

(g) $\langle f(b, b, b), c \rangle$

The union of the seven generated parallel critical pairs is:

$$PCP(R, R) = \{(a), (b), (c), (d), (e), (f), (g)\}$$

Theorem 4.6. *[1, Theorem 3.9] Let P, S be TRSs such that S is left-linear and terminating and P is reversible. Suppose*

(i) $CP(S, S) \subseteq \xrightarrow{*}_S \circ \mathrel{\leftarrow\!\!\!\shortparallel\!\!\!\rightarrow}_{P \cup P^{-1}} \circ \xleftarrow{*}_S$

(ii) for all $\langle u, v \rangle_X \in PCP_{in}(P \cup P^{-1}, S)$, $u \xrightarrow{*}_S u' \mathrel{\leftarrow\!\!\!\shortparallel\!\!\!\rightarrow}_{V, P \cup P^{-1}} v' \xleftarrow{*}_S v$ *for some* u',v' *and* V *satisfying* $\bigcup_{q \in V} \mathcal{V}ar(v'|_q) \subseteq X$

(iii) $CP(S, P \cup P^{-1}) \subseteq \xrightarrow{*}_S \circ \mathrel{\shortparallel\!\!\!\twoheadrightarrow}_{P \cup P^{-1}} \circ \xleftarrow{*}_S$

then $S \cup P$ is confluent.

Note: Theorem 4.6 subsumes Theorem 3.2.

The step by step procedure is a straightforward modification of Definition 3.4. So let's look at an example where Theorem 3.2 does not apply but Theorem 4.6 can show confluence.

Example 4.7. Consider the TRS R of Example 4.1

Note: In this example there are no variables, so we omitted to check the variable condition in (ii) of Theorem 4.6.

Step 1: Set $R_0 = R$ and $i = 0$.

Step 2: We have to split TRS R_0 into two TRSs S_0 and P_0, where S_0 has to be left-linear and terminating and P_0 has to be reversible.

We choose $S_0 = \{1\}$ and $P_0 = \{2, 3\}$ and prove the conditions:

Left-linearity: S_0 is left-linear as the left-hand side of the rule 1 has no variables.

Termination: S_0 is terminating using LPO with precedence $a > c$.

Reversibility: To show that P_0 is reversible all rules in P_0 have to be reversible:

2: $d \xrightarrow{3}_{P_0} a$ ✓

3: $a \xrightarrow{2}_{P_0} d$ ✓

The conditions are satisfied for S_0 and P_0. So we continue with Step 3.

Step 3: In the following we generate the set $PCP_{in}(P_0 \cup P_0^{-1}, S_0)$:

(a)
$$\begin{tikzcd} & f(a,a,a) \arrow[dl, "3"'] \arrow[dr, "1"] & \\ f(d,a,a) & & f(c,c,c) \end{tikzcd}$$

(b)
$$\begin{tikzcd} & f(a,a,a) \arrow[dl, "3"'] \arrow[dr, "1"] & \\ f(a,d,a) & & f(c,c,c) \end{tikzcd}$$

(c)
$$\begin{tikzcd} & f(a,a,a) \arrow[dl, "3"'] \arrow[dr, "1"] & \\ f(a,a,d) & & f(c,c,c) \end{tikzcd}$$

(d)
$$\begin{tikzcd} & f(a,a,a) \arrow[dl, "{3,3}"'] \arrow[dr, "1"] & \\ f(d,d,a) & & f(c,c,c) \end{tikzcd}$$

(e)
$$\begin{tikzcd} & f(a,a,a) \arrow[dl, "{3,3}"'] \arrow[dr, "1"] & \\ f(a,d,d) & & f(c,c,c) \end{tikzcd}$$

(f)
$$\begin{tikzcd} & f(a,a,a) \arrow[dl, "{3,3}"'] \arrow[dr, "1"] & \\ f(d,a,d) & & f(c,c,c) \end{tikzcd}$$

(g)
$$\begin{tikzcd} & f(a,a,a) \arrow[dl, "{3,3,3}"'] \arrow[dr, "1"] & \\ f(d,d,d) & & f(c,c,c) \end{tikzcd}$$

For the seven critical pairs we calculate $\xrightarrow{!}_{S_0}$ of both sides and then the additional $\mathrel{\leftarrow\!\!\!\!\!\parallel\!\!\!\!\!\rightarrow}_{P_0 \cup P_0^{-1}}$-steps have to be performed:

(a) no step possible, so $U := U \cup \{f(d,a,a) \to f(c,c,c)\}$
(b) no step possible, so $U := U \cup \{f(a,d,a) \to f(c,c,c)\}$
(c) no step possible, so $U := U \cup \{f(a,a,d) \to f(c,c,c)\}$
(d) no step possible, so $U := U \cup \{f(d,d,a) \to f(c,c,c)\}$
(e) no step possible, so $U := U \cup \{f(a,d,d) \to f(c,c,c)\}$
(f) no step possible, so $U := U \cup \{f(d,a,d) \to f(c,c,c)\}$
(g) no step possible, so $U := U \cup \{f(d,d,d) \to f(c,c,c)\}$

The critical pairs are not joinable, so confluence is not shown yet. To make these critical pairs joinable we have to add further rules. The only choice for $U' \subseteq U$ is $U' = U$:

$$\begin{aligned}
4 &: \; f(d,a,a) \to f(c,c,c) \\
5 &: \; f(a,d,a) \to f(c,c,c) \\
6 &: \; f(a,a,d) \to f(c,c,c) \\
7 &: \; f(d,d,a) \to f(c,c,c) \\
8 &: \; f(a,d,d) \to f(c,c,c) \\
9 &: \; f(d,a,d) \to f(c,c,c) \\
10 &: \; f(d,d,d) \to f(c,c,c)
\end{aligned}$$

We set $R_1 = R_0 \cup \{4,5,6,7,8,9,10\}$ and $i = 1$ and continue with Step 2.

Step 2: We choose

$$S_1 = \{1,4,5,6,7,8,9,10\}.$$

We have to check all conditions again, except the reversibility of P_1, as P_1 is equal to P_0. S_1 is left-linear as the left-hand sides of the rules $1,4,5,6,7,8,9$ and 10 have no variables. S_1 is terminating using LPO with precedence $a > d > c$. The conditions for S_1 and P_1 are satisfied. So we continue with Step 3.

Step 3: It is sufficient to calculate $PCP_{in}(P_1 \cup P_1^{-1}, S_1 \backslash \{1\})_X$, as $PCP_{in}(P_1 \cup P_1^{-1}, \{1\})$ is joinable with the rules $4,5,6,7,8,9$ and 10. The set $PCP_{in}(P_1 \cup P_1^{-1}, \{4,5,6,7,8,9,10\})$ has 49 joinable critical pairs. So we continue with Step 4.

Step 4: The set $CP(S_1, P_1 \cup P_1^{-1}) = \emptyset$. So we continue with Step 5.

Step 5: The set $CP(S_1, S_1) = \emptyset$.
All properties are fullfilled, so we have shown that

$$S_1 \cup P_1 \text{ is confluent}$$

and *Success* is returned. So our starting TRS R is confluent.

5 Confluence Criterion for Linear TRSs

In the next example Theorem 4.6 fails to show confluence.

Example 5.1. Let

$$R = \begin{cases} 1: \quad f(g(x), y) \to g(f(x,y)) & 2: \quad g(x) \to g(g(x)) \\ 3: \quad f(x, g(y)) \to g(f(x,y)) & 4: \quad g(g(x)) \to g(x) \\ 5: \quad f(x,y) \to f(y,x) & \end{cases}$$

The only possible combination is to set $S_0 = \{1,3\}$ and $P_0 = \{2,4,5\}$. S_0 is left-linear and terminating and P_0 is reversible. But in $PCP_{in}(P_0 \cup P_0^{-1}, S_0)$ we get one of the problematic parallel critical pairs: $\langle f(g(g(x)), y), g(f(x,y))\rangle_X$ with $X = \{x\}$. We can still go from $f(g(g(x)), y) \xrightarrow{1}_{S_0} g(f(g(x), y)) \xrightarrow{1}_{S_0} g(g(f(x,y)))$ and there is a rule in P_0 that makes $g(g(f(x,y)))$ and $g(f(x,y))$ joinable. But that harms the variable condition ($\{x,y\} \nsubseteq \{x\}$). The only thing we can do is to add a rule from $f(g(g(x)), y) \to g(f(x,y))$. In the following steps we get for every $i \in \mathbb{N}$ the same problem in $PCP_{in}(P_i \cup P_i^{-1}, S_i)$ with more $g's$ on the left-hand side of the parallel critical pair. So we will not reach a satisfying solution. Theorem 4.6 gives us no confluence for this example.

Theorem 5.2. *[1, Theorem 3.13] Let P, S be TRSs such that S is linear and terminating and P is reversible. Suppose*

(i) $CP(S,S) \subseteq \xrightarrow{*}_S \circ \xleftarrow{=}_{P \cup P^{-1}} \circ \xleftarrow{*}_S$

(ii) $CP(P \cup P^{-1}, S) \subseteq \xrightarrow{*}_S \circ \xleftarrow{=}_{P \cup P^{-1}} \circ \xleftarrow{*}_S$

(iii) $CP(S, P \cup P^{-1}) \subseteq \xrightarrow{*}_S \circ \xrightarrow{=}_{P \cup P^{-1}} \circ \xleftarrow{*}_S$

then $S \cup P$ is confluent.

The difference between Theorem 5.2 and Theorem 4.6 is listed in the following:

- Theorem 5.2 has more restrictions:
  - It works only for linear TRSs.
  - We can not perform a parallel $P \cup P^{-1}$-step in (i), (ii) or in (iii), we can only do single $P \cup P^{-1}$-steps.
- Theorem 5.2 has fewer restrictions:
  - In (ii) we have to check only the set of critical pairs $CP(P \cup P^{-1}, S)$ and not the set of parallel critical pairs.
  - We have no set of parallel critical pairs, so we do not need a variable condition as in Theorem 4.6.

The step by step procedure is a straightforward modification of Definition 3.4. So let's look at an example where Theorem 4.6 does not apply but Theorem 5.2 can show confluence.

Example 5.3. Consider the TRS R of Example 5.1

Step 1: Set $R_0 = R$ and $i = 0$.

Step 2: We have to split TRS R_0 into two TRSs S_0 and P_0, where S_0 has to be linear and terminating and P_0 has to be reversible.

We choose $S_0 = \{1, 3\}$ and $P_0 = \{2, 4, 5\}$ and prove the conditions:

Linearity: S_0 is linear as the left and the right-hand sides of the rules 1 and 3 contain the variables x and y at most once.

Termination: S_0 is terminating using LPO with precedence $f > g$.

Reversibility: To show that P_0 is reversible all rules in P_0 have to be reversible:

$2: \ g(g(x)) \xrightarrow{4}_{P_0} g(x)$ ✓

$4: \ g(x) \xrightarrow{2}_{P_0} g(g(x))$ ✓

$5: \ f(y * x) \xrightarrow{5}_{P_0} f(x * y)$ ✓

The conditions are satisfied for S_0 and P_0. So we continue with Step 3.

Step 3: In the following we generate the set $CP(P_0 \cup P_0^{-1}, S_0)$:

(a)
$$\begin{tikzcd} & f(g(x),y) \arrow[dl, "2"'] \arrow[dr, "1"] & \\ f(g(g(x)),y) & & g(f(x,y)) \end{tikzcd}$$

(b)
$$\begin{tikzcd} & f(x,g(y)) \arrow[dl, "2"'] \arrow[dr, "3"] & \\ f(x,g(g(y))) & & g(f(x,y)) \end{tikzcd}$$

(c)
$$\begin{tikzcd} & f(g(x),y) \arrow[dl, "5"'] \arrow[dr, "1"] & \\ f(y,g(x)) & & g(f(x,y)) \end{tikzcd}$$

(d)
$$\begin{tikzcd} & f(x,g(y)) \arrow[dl, "5"'] \arrow[dr, "3"] & \\ f(g(y),x) & & g(f(x,y)) \end{tikzcd}$$

For the four critical pairs we calculate $\xrightarrow{!}_{S_0}$ of both sides and then the additional $\xleftarrow{=}_{P_0 \cup P_0^{-1}}$-steps have to be performed:

(a) $f(g(g(x)), y) \xrightarrow{1}_{S_0} g(f(g(x), y)) \xrightarrow{1}_{S_0} g(g(f(x, y))) \xleftarrow{2}_{P_0} g(f(x, y))$ ✓

(b) $f(x, g(g(y))) \xrightarrow{3}_{S_0} g(f(x, g(y))) \xrightarrow{3}_{S_0} g(g(f(x, y))) \xleftarrow{2}_{P_0} g(f(x, y))$ ✓

(c) $f(y * g(x)) \xrightarrow{3}_{S_0} g(f(y, x)) \xleftarrow{5}_{P_0} g(f(x, y))$ ✓

(d) $f(g(y) * x) \xrightarrow{1}_{S_0} g(f(y, x)) \xleftarrow{5}_{P_0} g(f(x, y))$ ✓

All critical pairs are joinable. So continue with Step 4.

Step 4: In the following we generate the set $CP(S_0, P_0 \cup P_0^{-1})$:

$$\begin{tikzcd} & f(g(x), y) \arrow[dl, "1"'] \arrow[dr, "5"] & \\ g(f(x, y)) & & f(y, g(x)) \end{tikzcd}$$

(a')

$$\begin{tikzcd} & f(x, g(y)) \arrow[dl, "3"'] \arrow[dr, "5"] & \\ g(f(x, y)) & & f(g(y), x) \end{tikzcd}$$

(b')

For the two critical pairs we calculate $\xrightarrow{!}_{S_0}$ of both sides and then the additional $\xrightarrow{=}_{P_0 \cup P_0^{-1}}$-steps have to be performed:

(a') $g(f(x, y)) \xrightarrow{5}_{P_0} g(f(y, x)) \xleftarrow{3}_{S_0} f(y * g(x))$ ✓

(b') $g(f(x, y)) \xrightarrow{5}_{P_0} g(f(y, x)) \xleftarrow{1}_{S_0} f(g(y) * x)$ ✓

All critical pairs are joinable. So continue with Step 5.

Step 5: In the following we generate the set $CP(S_1, S_1)$.

$$\begin{tikzcd} & f(g(x), g(y)) \arrow[dl, "3"'] \arrow[dr, "1"] & \\ g(f(x, g(y))) & & g(f(g(x), y)) \end{tikzcd}$$

(a'')

For the critical pair we calculate $\xrightarrow{!}_{S_0}$ of both sides and then the additional $\xleftarrow{=}_{P_0 \cup P_0^{-1}}$-steps have to be performed:

(a'') $g(f(x, g(y))) \xrightarrow{3}_{s_0} g(g(f(x, y))) \xleftarrow{1}_{S_0} g(f(g(x), y))$ ✓

All properties are fullfilled, so we have shown that

$$S_0 \cup P_0 \text{ is confluent}$$

and $Success$ is returned. So our starting TRS R is confluent.

6 Conclusion

We presented three methods to show confluence of TRSs with non-terminating rewrite rules. The key point of the methods is to split the TRS into two parts, a terminating part and a potentially non-terminating part. These methods are able to deal with associative and commutative rules, where other methods like Newman's Lemma fail. For implementing these rules we identified two main challenges. The first main challenge is the efficient splitting of the input term rewriting system. In [1] and in [2] a possible solution for this challenge is presented. Instead of trying all possible combinations they suggest to try combinations based on a certain heuristic. The second key challenge in programming these methods is to choose a subset U' efficiently. In [1] and [2] this choice is not described. A first approach would be to check wether the left-hand side of a certain rule s in U is a proper subterm of the left-hand side of another rule t in U. If that is the case U' should not contain t. If there are no such rules s and t in U we set $U' = U$.

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