Confluence of Root-E-overlapping Term Rewrite Systems

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Contents

Literature

Gomi, H., Oyamaguchi, M., Ohta, Y.:
On the Church-Rosser property of root-E-overlapping and strongly depth-preserving term rewriting systems.
Trans. IPSJ 39(4), 992–1005 (1998)

Motivation

Example

The following TRS (due to Barendregt) is not confluent:

$$\mathcal{R}_B = \{\mathsf{f}(x,x) \to \mathsf{a}, \mathsf{g}(x) \to \mathsf{f}(x,\mathsf{g}(x)), \mathsf{c} \to \mathsf{g}(\mathsf{c})\}$$

But the following TRS is:

$$\mathcal{R}_M = \{\mathsf{f}(x,x) \to \mathsf{a}, \mathsf{h}(x) \to \mathsf{f}(x,\mathsf{g}(x)), \mathsf{c} \to \mathsf{g}(\mathsf{c})\}$$

- confluence for non-terminating, non-left-linear systems is hard
- E-overlaps yield direct confluence criterion for such systems
- decidable approximation → can be implemented (e.g., ACP)

Progress

Main theorem

Theorem

Every strongly depth-preserving, root-E-closed TRS is confluent.

Strongly depth-preserving TRSs

Definition

A rule $\ell \to r$ is strongly depth-preserving iff for all $v \in \mathcal{V}\mathrm{ar}(\ell)$, there is some $p \in \mathcal{P}\mathrm{os}_{\{v\}}(\ell)$ such that $|p| \geq |q|$ for all $q \in \mathcal{P}\mathrm{os}_{\{v\}}(r)$.
A TRS is strongly depth-preserving iff all its rules are.

Example

The following TRS is strongly depth-preserving:

$$\mathcal{R}_2 = \{\mathsf{f}(\mathsf{g}(x), \mathsf{g}(\mathsf{g}(x))) \to \mathsf{h}(\mathsf{g}(x), \mathsf{g}(\mathsf{c})), \mathsf{c} \to \mathsf{g}(\mathsf{c})\}$$

But $\mathcal{R}_B$ is not:

$$\mathcal{R}_B = \{\mathsf{f}(x, x) \to \mathsf{a}, \mathsf{g}(x) \to \mathsf{f}(x, \mathsf{g}(x)), \mathsf{c} \to \mathsf{g}(\mathsf{c})\}$$

Parallel reduction sequences (Proofs)

- $\Leftrightarrow\!\!\!\!\!\mid\!\mid$ is defined by $\leftrightarrow^{=} \subseteq$ $\Leftrightarrow\!\!\!\!\!\mid\!\mid$ and $f(\Leftrightarrow\!\!\!\!\!\mid\!\mid, \ldots, \Leftrightarrow\!\!\!\!\!\mid\!\mid) \subseteq$ $\Leftrightarrow\!\!\!\!\!\mid\!\mid$ for $f \in \mathcal{F}$.
- $\alpha : t_0 \Leftrightarrow\!\!\!\!\!\mid\!\mid t_1 \ldots t_{n-1} \Leftrightarrow\!\!\!\!\!\mid\!\mid t_n$ is a proof of length n:
- $\text{height}(\alpha) = \max\{\text{height}(t_0), \ldots, \text{height}(t_n)\}$
- cut proof: If $\alpha = f(\alpha_1, \ldots, \alpha_m)$, then $\alpha|_i = \alpha_i$. (also $\alpha|_p$)

E-overlaps

Definition

Rules $\ell \to r$ and $\ell' \to r'$ have an E-overlap at $p \in \mathcal{P}\mathrm{os}_{\mathcal{F}}(\ell')$ iff there are substitutions σ, σ' such that

$$\ell\sigma \mathrel{\Leftrightarrow\!\!\!\!\!\!|\!|}^{*}_{>\epsilon} (\ell'|_p)\sigma'.$$

If $p = \epsilon$ (a root E-overlap) we require $\ell \to r \neq \ell' \to r'$.

Example

$$\mathcal{R}_3 = \{\mathsf{f}(x,x) \to \mathsf{h}(x,x), \mathsf{f}(\mathsf{g}(x),x) \to \mathsf{a}, \mathsf{c} \to \mathsf{g}(\mathsf{c}), \mathsf{h}(\mathsf{g}(x),x) \to \mathsf{a}\}$$

The first two rules are root-E-overlapping:

$$h(\mathsf{c},\mathsf{c}) \leftarrow \mathsf{f}(\mathsf{c},\mathsf{c}) \to_{>\epsilon} \mathsf{f}(\mathsf{g}(\mathsf{c}),\mathsf{c}) \to a$$

Root-E-closed TRSs

Definition

A TRS $\mathcal{R}$ is called root-E-closed if

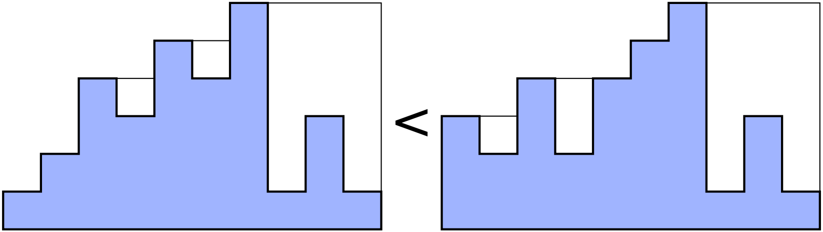
1. every E-overlap is a root E-overlap
2. for $\gamma : r\sigma \leftarrow l\sigma \twoheadleftarrow\!\!\!\twoheadrightarrow^*_P l'\sigma' \rightarrow r'\sigma'$ with $P \cap \mathcal{P}\mathrm{os}_{\mathcal{F}}(l) \cap \mathcal{P}\mathrm{os}_{\mathcal{F}}(l') = \varnothing$, there is $\delta : r\sigma \twoheadleftarrow\!\!\!\twoheadrightarrow^*_P r'\sigma'$ with $\delta \preceq \gamma$ and
   - one of the comparisons 2 or 3 of $\preceq$ (see below) is strict, or
   - $\delta \neq \leftarrow_\epsilon \cdot \twoheadleftarrow\!\!\!\twoheadrightarrow^*$, or
   - $\delta = \leftarrow_\epsilon \cdot \twoheadleftarrow\!\!\!\twoheadrightarrow^*_{>\epsilon}$

where $\delta : t_0 \twoheadleftarrow\!\!\!\twoheadrightarrow^* t_n \preceq \gamma : s_0 \twoheadleftarrow\!\!\!\twoheadrightarrow^* s_m$ iff with $\mathrm{mh}(T) = \max(\mathrm{height}(T))$,

1. $n = m$,
2. $t_i = t_{i+1}$ no more often than $s_i = s_{i+1}$,
3. $\{\mathrm{mh}\{t_0 \ldots t_i\} \mid 0 \leq i \leq n\} \leq_{\mathrm{mul}} \{\mathrm{mh}\{s_0 \ldots s_j\} \mid 0 \leq j \leq n\}$, and
4. $\{\mathrm{mh}\{t_i \ldots t_n\} \mid 0 \leq i \leq n\} \leq_{\mathrm{mul}} \{\mathrm{mh}\{s_i \ldots s_n\} \mid 0 \leq j \leq n\}$.

Mountains

$$\{\mathsf{mh}\{t_0 \dots t_i\} \mid 0 \leq i \leq n\} \leq_{\mathrm{mul}} \{\mathsf{mh}\{s_0 \dots s_i\} \mid 0 \leq j \leq n\}$$



Main theorem

Theorem

Every strongly depth-preserving, root-E-closed TRS is confluent.

Example

$$\mathcal{R}_3 = \{\mathsf{f}(x,x) \to \mathsf{h}(x,x), \mathsf{f}(\mathsf{g}(x),x) \to \mathsf{a}, \mathsf{c} \to \mathsf{g}(\mathsf{c}), \mathsf{h}(\mathsf{g}(x),x) \to \mathsf{a}\}$$

Consider a root E-overlap

$$\mathsf{h}(\sigma(x),\sigma(x)) \leftarrow \mathsf{f}(\sigma(x),\sigma(x)) \mathrel{\Leftrightarrow\!\!\!\!\!\!|\!|}^*_{>\epsilon} \mathsf{f}(\mathsf{g}(\sigma'(x)),\sigma'(x)) \to \mathsf{a}.$$

It can be closed by

$$\mathsf{h}(\sigma(x),\sigma(x)) \mathrel{\Leftrightarrow\!\!\!\!\!\!|\!|} \mathsf{h}(\sigma(x),\sigma(x)) \mathrel{\Leftrightarrow\!\!\!\!\!\!|\!|}^*_{>\epsilon} \mathsf{h}(\mathsf{g}(\sigma'(x)),\sigma'(x)) \to \mathsf{a}.$$

So $\mathcal{R}_3$ is root-E-closed and strongly depth-preserving, hence confluent.

Progress

E-peaks

Definition

An E-peak is a proof of the form $\leftarrow_\epsilon \cdot \Leftrightarrow\!\!\!\!\!\Vert\;^*_{>\epsilon} \cdot \rightarrow_\epsilon$.

Lemma (E-peak elimination)

Every proof $\gamma : s \Leftrightarrow\!\!\!\!\!\Vert\;^* t$ *has an equivalent proof* $\delta : s \Leftrightarrow\!\!\!\!\!\Vert\;^* t$ *without E-peaks (even nested), such that* $\delta \preceq \gamma$*.*

E-Peak elimination

Proof sketch.
Use induction on length(γ) and γ ordered by $\prec$.
Key case: $\delta = \delta_\wedge \cdot \delta'$, where $\delta_\wedge$ is an E-peak. So let

$$\delta_\wedge : s = r\sigma \leftarrow l\sigma \,\#\!\!\!\!\leftrightarrow^*_{>\epsilon}\, l'\sigma' \to r'\sigma',$$

where $\#\!\!\!\!\leftrightarrow^*_{>\epsilon}$ has no peaks by IH.

Only root-E-overlaps: All $\#\!\!\!\!\leftrightarrow_{>\epsilon}$ are below $\mathcal{P}\mathrm{os}_{\mathcal{F}}(l') \cap \mathcal{P}\mathrm{os}_{\mathcal{F}}(l)$.

Strong depth decrease: If $l \to r = l' \to r'$, then
$\delta_* : s = r\sigma \,\#\!\!\!\!\leftrightarrow\, r\sigma \,\#\!\!\!\!\leftrightarrow^*\, r\sigma' \,\#\!\!\!\!\leftrightarrow\, r\sigma'$ has $\delta_* \prec \delta_\wedge$.

Root-E-closedness: If $l \to r \neq l' \to r'$, we obtain $\delta_* \preceq \delta_\wedge$ by assumption.

In both cases, we continue the proof with $\delta_* \cdot \delta'$.

...

Progress

Approximating E-overlaps

Definition

The rules $l \to r$, $l' \to r'$ strongly overlap at $p \in \mathcal{P}os_{\mathcal{F}}(l')$ if l and $l'|_p$ unify after linearization (i.e., making all variables distinct).
If $p = \epsilon$, we require the rules to be distinct.

Example

$$\mathcal{R}_3 = \{\mathsf{f}(x,x) \to \mathsf{h}(x,x), \mathsf{f}(\mathsf{g}(x),x) \to \mathsf{a}, \mathsf{c} \to \mathsf{g}(\mathsf{c}), \mathsf{h}(\mathsf{g}(x),x) \to \mathsf{a}\}$$

has a strong root-overlap between the first two rules.

Lemma

The Main Theorem remains valid if we require that all strong overlaps are root-overlaps rather than using E-overlaps in condition 1 of root-E-closedness.

Approximating root-E-closedness

Lemma

Let $\mathcal{R}$ be a strongly root-overlapping (overlay) TRS. If for any strongly overlapping, variable disjoint rules $l \to r$, $l' \to r'$, we have either

$$r \mathrel{\Vert\!\!\!\!\!\!\leftrightarrow} t \Rightarrow r' \quad \text{or} \quad r \Rightarrow t \mathrel{\Vert\!\!\!\!\!\!\leftrightarrow} r',$$

then $\mathcal{R}$ is root-E-closed, where $\Rightarrow$ replaces (some) subterms that equal $l|_p$ by $l'|_p$ for all $p \in \min(\mathcal{P}\mathrm{os}_{\mathcal{V}}(l) \cup \mathcal{P}\mathrm{os}_{\mathcal{V}}(r))$.

Note that if we have an root E-overlap

$$r\sigma \leftarrow l\sigma \mathrel{\Vert\!\!\!\!\!\!\leftrightarrow}^*_{>\epsilon} l'\sigma' \to r'\sigma'$$

then $\Rightarrow$ can be replaced by some proofs from the $\mathrel{\Vert\!\!\!\!\!\!\leftrightarrow}^*_{>\epsilon}$ part.

Final example

Example

The strong root-overlap between the first two rules in

$$\mathcal{R}_3 = \{\mathsf{f}(x_1, x_1) \to \mathsf{h}(x_1, x_1), \mathsf{f}(\mathsf{g}(x_2), x_2) \to \mathsf{a}, \mathsf{c} \to \mathsf{g}(\mathsf{c}), \mathsf{h}(\mathsf{g}(x_3), x_3) \to \mathsf{a}\}$$

can be joined by $\mathsf{h}(x_1, x_1) \Rightarrow \mathsf{h}(\mathsf{g}(x_2), x_2) \to a$.
Since $\mathcal{R}_3$ is also strongly depth-preserving, it is confluent.

Progress

- Introduction
- E-overlaps and Main Result
- E-Peak Elimination
- A Decidable Criterion
- Conclusion

Summary

- strongly depth-preserving, root-E-closed TRSs are confluent
- key ingredient: parallel reductions $\twoheadrightarrow\!\!\!\!\!\!\parallel$
- decidable criterion based on strong overlaps
- in particular: strongly depth-preserving, strongly non-overlapping TRSs are confluent

Outlook

- extension: strongly weight-preserving TRSs. This covers

$$\mathcal{R}_M = \{\mathsf{f}(x,x) \to \mathsf{a}, \mathsf{h}(x) \to \mathsf{f}(x,\mathsf{g}(x)), \mathsf{c} \to \mathsf{g}(\mathsf{c})\}$$

  (weight generalizes height)
- technical proofs—can they be generalized or simplified?
- can the root-E-closedness condition be made symmetric?

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Thank you!