Proving Confluence of TRSs with AC-Rules

Seminar Report

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June 20, 2012

Overview

- Literature
- Motivation
- Confluence including AC-Rules
  - Confluence Criterion
  - Improvements
  - Confluence Criterion using Parallel Critical Pairs
  - Confluence Criterion on linear TRS
- Implementation challenges
- Summary

Literature

Takahito Aoto and Yoshihito Toyama
A reduction-preserving completion for proving confluence of non-terminating Term Rewriting Systems, 22nd International Conference on Rewriting Techniques and Applications (RTA'11), pp. 91–106, 2011

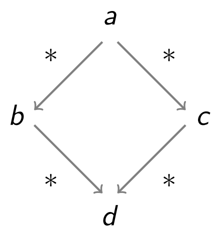
Takahito Aoto and Yoshihito Toyama
A reduction-preserving completion for proving confluence of non-terminating Term Rewriting Systems, Logical Methods in Computer Science, 2012

Overview

Motivation

Confluence:

$\forall a, b, c\ \exists d$



Horizontal notation: $\overset{*}{\leftarrow} \circ \overset{*}{\rightarrow} \subseteq \overset{*}{\rightarrow} \circ \overset{*}{\leftarrow}$

Motivation

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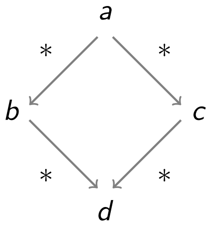

How can we prove confluence?

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Motivation

Confluence:

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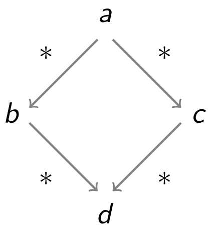

How can we prove confluence?

- Newman's Lemma is often used
- = termination + local confluence

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Motivation

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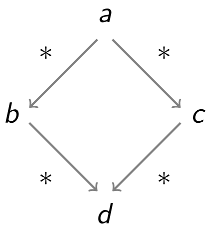

How can we prove confluence?

- Newman's Lemma is often used
- = termination + local confluence
- **But:** What about non-terminating TRSs?

Horizontal notation: $\overset{*}{\leftarrow} \circ \overset{*}{\rightarrow} \subseteq \overset{*}{\rightarrow} \circ \overset{*}{\leftarrow}$

Motivation

Example (TRS R)

1 :	$x * 1 \to x$	2 :	$x * y \to y * x$
3 :	$x * 0 \to 0$	4 :	$(x * y) * z \to x * (y * z)$

Motivation

Example (TRS R)

$$
\begin{array}{llll}
1: & x * 1 \to x & 2: & x * y \to y * x \\
3: & x * 0 \to 0 & 4: & (x * y) * z \to x * (y * z)
\end{array}
$$

Maybe someone can give me a termination proof?

Motivation

Example (TRS R)

$1:$	$x * 1 \to x$	$2:$	$x * y \to y * x$
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Maybe someone can give me a termination proof?

Newman's Lemma fails, because of AC-Rules. **But:** R is confluent as we see later

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Preliminaries

Let R, S be TRSs and s, t two terms:

- $s \xrightarrow{!}_R t$ if $s \xrightarrow{*}_R t$ and $t \in NF(R)$
- $s \mathrel{+\!\!\!\!\!\twoheadrightarrow}_R t$ denotes a parallel R-step from s to t
- $s \mathrel{+\!\!\!\!\!\twoheadrightarrow}_{R \cup S} t$ if $s \mathrel{+\!\!\!\!\!\twoheadrightarrow}_R t$ or $s \mathrel{+\!\!\!\!\!\twoheadrightarrow}_S t$
- R is reversible if for all rewrite rules $l \to r \in R$: $r \xrightarrow{*}_R l$ holds
- $CP_{out}(CP_{in})$ is the set of critical pairs on position $= \epsilon$ $(> \epsilon)$

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Theorem 1

Theorem (Confluence Criterion)

Let S, P be TRSs such that S is left-linear and terminating and P is reversible. Suppose

(i) $CP(S,S) \subseteq \xrightarrow{*}_S \circ \mathrel{\leftarrow\!\!\!+\!\!\!+}_{P \cup P^{-1}} \circ \xleftarrow{*}_S$

(ii) $CP_{in}(P \cup P^{-1}, S) = \emptyset$

(iii) $CP(S, P \cup P^{-1}) \subseteq \xrightarrow{*}_S \circ \mathrel{+\!\!\!+\!\!\!\rightarrow}_{P \cup P^{-1}} \circ \xleftarrow{*}_S$

then $S \cup P$ is confluent.

Step by step procedure

Input: TRS R
Output: *Success* or *Failure*

Step 1: Set $R_0 := R$ and $i := 0$.
Step 2: Choose $S_i \cup P_i = R_i$ such that S_i is left-linear and terminating, P_i is reversible and $CP_{in}(P_i \cup P_i^{-1}, S_i) = \emptyset$. If S_i and P_i do not exist return *Failure*.
Step 3: Let $U := \emptyset$. For each $\langle p, q \rangle \in CP(S_i, P_i \cup P_i^{-1})$ perform $p \xrightarrow{!}_{S_i} p'$ and $q \xrightarrow{!}_{S_i} q'$. If $p' \twoheadrightarrow\!\!\!\!\!\!\!\!\mid\;_{P \cup P^{-1}} q'$ does not hold, set $U := U \cup \{q \to p'\}$. If $U \neq \emptyset$ choose non-empty $U' \subseteq U$ and continue with Step 2 with $R_{i+1} := R_i \cup U'$ and $i := i + 1$.
Step 4: Let $U := \emptyset$. For each $\langle p, q \rangle \in CP(S_i, S_i)$ perform $p \xrightarrow{!}_{S_i} p'$ and $q \xrightarrow{!}_{S_i} q'$. If $p' \twoheadleftarrow\!\!\!\!\!\!\!\!\mid\;_{P \cup P^{-1}} q'$ does not hold, set $U := U \cup \{p' \approx q'\}$. If $U = \emptyset$ return *Success*. Otherwise choose at least one rewrite rule of $U' \subseteq (U \cup U^{-1}) \cap \xleftrightarrow{*}_{P_i}$ and continue with Step 2 with $R_{i+1} := R_i \cup U'$ and $i := i + 1$.

Example for Theorem 1

Example (TRS R)

1 :	$x * 1 \to x$	2 :	$x * y \to y * x$
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Step 1: Set $R_0 = R$ and $i = 0$.

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Step 2: $S_0 = \{1, 3\}$, $P_0 = \{2, 4\}$ and $CP_{in}(P_0 \cup P_0^{-1}, S_0) = \emptyset$.

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Step 2: $S_0 = \{1, 3\}$, $P_0 = \{2, 4\}$ and $CP_{in}(P_0 \cup P_0^{-1}, S_0) = \emptyset$.

Step 3: Four critical pairs of $CP(S_0, P_0 \cup P_0^{-1}) = \{$
$\langle x, 1 * x\rangle, \langle 0, 0 * x\rangle, \langle x * z, x * (1 * z)\rangle$, $\langle x * y, x * (y * 1)\rangle$,
$\langle x * y, (x * y) * 1\rangle$, $\langle 0 * z, x * (0 * z)\rangle$, $\langle 0, x * (y * 0)\rangle$,
$\langle x * 0, (x * y) * 10\rangle$ $\}$ are joinable and four are not joinable $\to$
add two new rules. Set $R_1 = R_0 \cup \{5, 6\}$ and $i = 1$.

Example for Theorem 1

Example (TRS R)

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Step 3: All critical pairs of $CP(S_1, P_1 \cup P_1^{-1})$ are joinable.

Step 4: All critical pairs of $CP(S_1, S_1)$ are joinable. So starting TRS R is confluent.

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Improvements of step by step procedure

Step 2a: If $CP_{in}(P_i \cup P_i^{-1}, S_i) \neq \emptyset$ and $\exists\ l \to r \in S_i$ with $CP_{in}(P_i \cup P_i^{-1}, \{l \to r\}) \neq \emptyset$ and $\exists\ r'$ with $r \leftrightarrow_{P_i} r'$, we set $R_{i+1} := (R_i \backslash \{l \to r\}) \cup \{l \to r'\}$ and $i := i + 1$.

Improvements of step by step procedure

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Step 2b: Let $\langle p, q \rangle \in CP_{in}(P_i \cup P_i^{-1}, S_i)$ and perform $q \xrightarrow{!}_{S_i} q'$. We set $R_{i+1} := R_i \cup \{p \to q'\}$ and $i := i + 1$.

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Step 3a: We set $S_i := S_{i-1}$ and $P_i := P_{i-1}$. If $\exists\ r'$ with $r \leftrightarrow_{P_i} r'$ and a critical pair $\langle p, q \rangle \in CP(\{l \to r\}, P_i \cup P_i^{-1})$ we perform $p \xrightarrow{!}_{S_i} p'$ and $q \xrightarrow{!}_{S_i} q'$. If $p' \not\leftrightarrow^*_{P \cup P^{-1}} q'$ does not hold, we set $R_{i+1} := (R_i \backslash \{l \to r\} \cup \{l \to r'\}$ and $i := i + 1$.

Improvements of step by step procedure

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Step 4a: We set $S_i := S_{i-1}$ and $P_i := P_{i-1}$. If $\exists\ r'$ with $r \leftrightarrow_{P_i} r'$ and a critical pair $\langle p, q \rangle \in CP(\{l \to r\}, S_i) \cup CP(S_i, \{l \to r\})$ we perform $p \xrightarrow{!}_{S_i} p'$ and $q \xrightarrow{!}_{S_i} q'$. If $p' \twoheadleftarrow\!\!\!\!\!\!\not\,\not\;\;_{P \cup P^{-1}} q'$ does not hold, we set $R_{i+1} := (R_i \backslash \{l \to r\} \cup \{l \to r'\}$ and $i := i + 1$.

Example without Improvements

Example (TRS R)

1 :	$1 * y \to y$	2:	$x * y \to y * x$
3 :	$x * f(y) \to f(x * y)$	4:	$(x * y) * z \to x * (y * z)$

Step 2: $S_0 = \{1, 3\}$ and $P_0 = \{2, 4\}$

$CP_{in}(P_0 \cup P_0^{-1}, S_0) = \emptyset$

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Add rules: $5 : y * 1 \to y$, $6 : f(y) * x \to f(x * y)$,

$7 : x * (f(y) * z) \to f(x * y) * z$

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Step 2: $S_1 = \{1, 3, 5, 6, 7\}$ and $P_1 = \{2, 4\}$

$CP_{in}(P_1 \cup P_1^{-1}, S_1) \neq \emptyset$, so we get no confluence for starting TRS R.

Example with Improvements

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Improvement 3a: change 6 to $7 : (f(y) * x \to f(y * x))$

Possible because rule in P_1: $f(x * y) \to_{P_1} f(y * x)$

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Step 2: $S_2 = \{1, 3, 5, 7\}$ and $P_2 = \{2, 4\}$

- $CP_{in}(P_2 \cup P_2^{-1}, S_2) = \emptyset$, all critical pairs of $CP(S_2, P_2 \cup P_2^{-1})$ and $CP(S_2, S_2)$ are joinable. So starting TRS R is confluent.

Overview

Limitation of Theorem 1

Example (TRS R)

1 :	$f(a, a, a) \to f(c, c, c)$	2:	$a \to d$
3 :	$d \to a$		

- Only possible combination $S_0 = \{1\}$ and $P_0 = \{2, 3\}$

Limitation of Theorem 1

Example (TRS R)

1 :	$f(a, a, a) \to f(c, c, c)$	2:	$a \to d$
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- Only possible combination $S_0 = \{1\}$ and $P_0 = \{2, 3\}$
- But $CP_{in}(P_0 \cup P_0^{-1}, S_0)$ not empty
- Theorem 1 does not help us to show confluence of R.

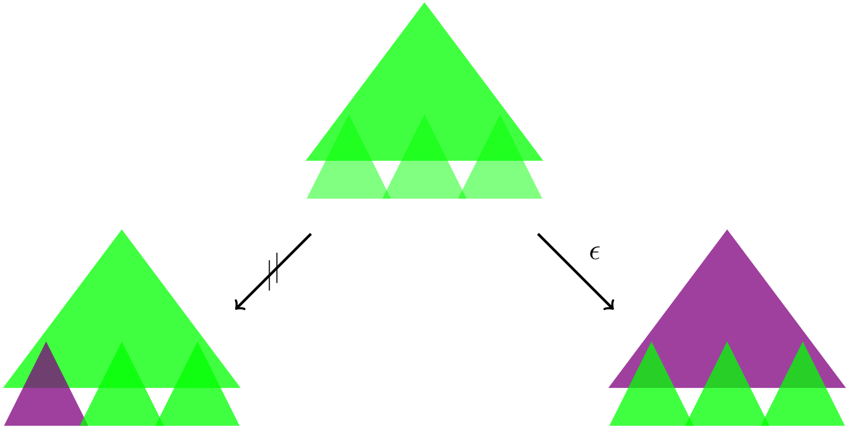
Parallel critical pairs

Definition

Parallel critical pairs originate from parallel overlaps between left-hand sides of $n+1$ rewrite rules. Let $l_1 \to r_1, \ldots, l_n \to r_n$ (from a TRS S) and $l' \to r'$ (from a TRS T) be $n+1$ rewrite rules without any common variables and suppose $l_1, \ldots, l_n$ has a parallel overlap on l' at parallel positions $p_1, \ldots, p_n$. The mgu for $l_1, \ldots, l_n$ and $l'|_{p_1}, \ldots, l'|_{p_n}$ is called σ. The parallel critical pair is the following:

$$\langle l'[r_1, \ldots, r_n]_{p_1,\ldots,p_n}\sigma, r'\sigma\rangle$$

Parallel critical pairs



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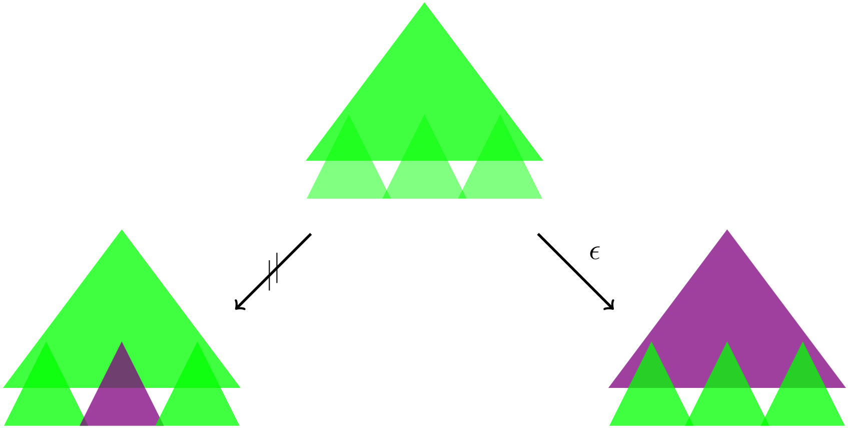

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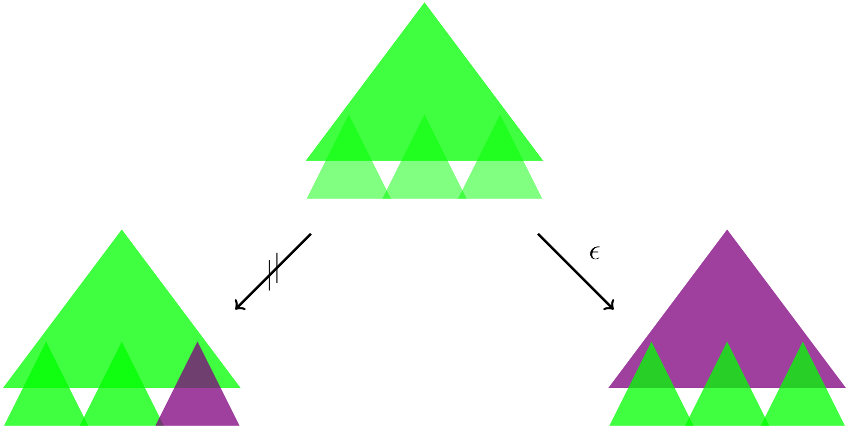

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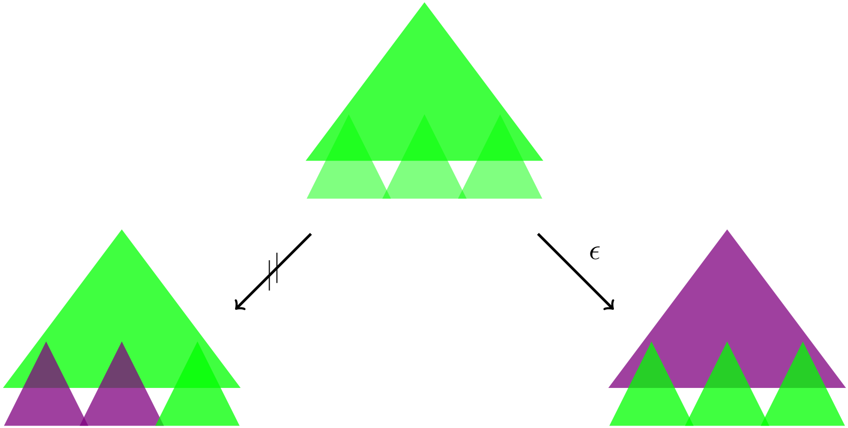

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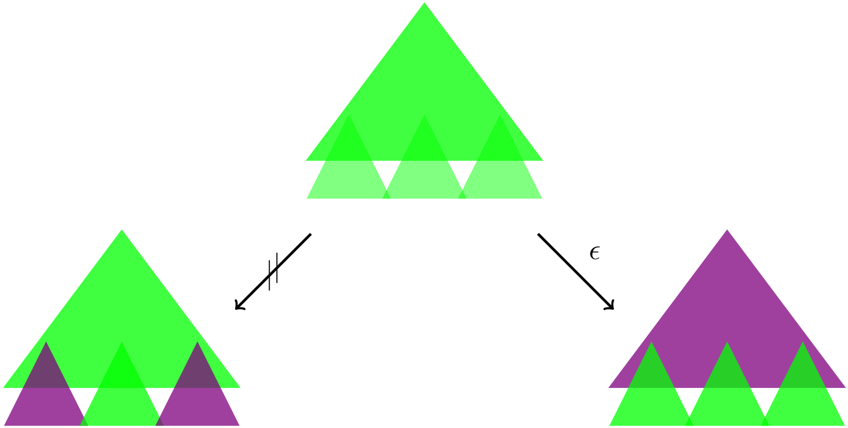

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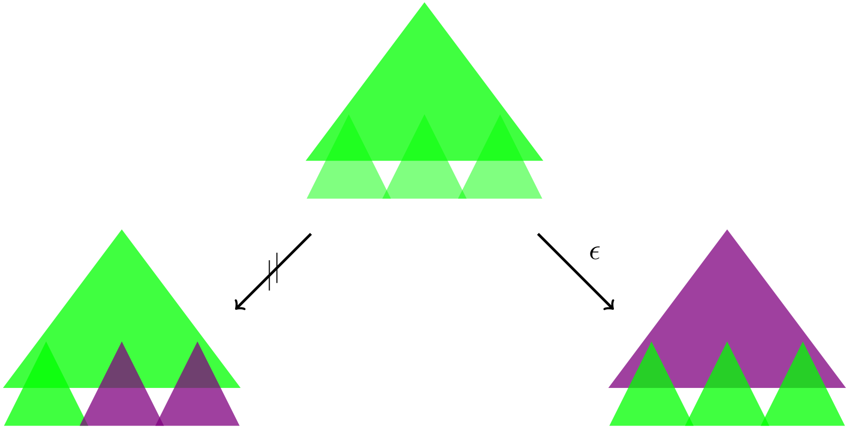
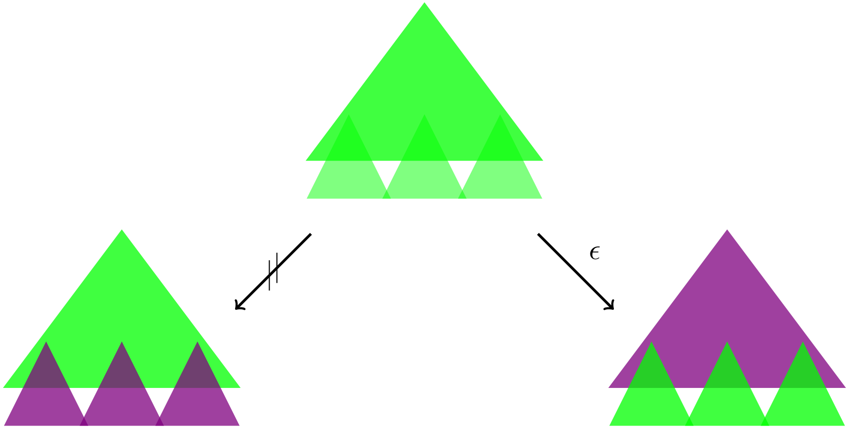


Parallel critical pairs



- The set of all parallel critical pairs is written as PCP
- $\langle l'[r_1, \ldots, r_n]_{p_1 \ldots, p_n}\sigma, r'\sigma\rangle_X$ if $X = \bigcup_{1 \leq i \leq n} \mathcal{V}ar(l'\sigma|_{p_i})$

Theorem 2

Theorem (Confluence Criterion using Parallel Critical Pairs)

Let P, S be TRSs such that S is left-linear and terminating and P is reversible. Suppose

(i) $CP(S,S) \subseteq \xrightarrow{*}_S \circ \mathrel{\leftarrow\!\!\!\!\!\!\!\!+\!\!\!\!+}_{P \cup P^{-1}} \circ \xleftarrow{*}_S$

(ii) *for all* $\langle u, v\rangle_X \in PCP_{in}(P \cup P^{-1}, S)$, $u \xrightarrow{*}_S u' \mathrel{\leftarrow\!\!\!\!\!\!\!\!+\!\!\!\!+}_{V, P \cup P^{-1}} v' \xleftarrow{*}_S v$ *for some* u',v' *and* V *satisfying* $\bigcup_{q \in V} \mathcal{V}ar(v'|_q) \subseteq X$

(iii) $CP(S, P \cup P^{-1}) \subseteq \xrightarrow{*}_S \circ \mathrel{+\!\!\!\!+\!\!\!\!\!\!\!\!\rightarrow}_{P \cup P^{-1}} \circ \xleftarrow{*}_S$

then $S \cup P$ is confluent.

Example for Theorem 2

Example (TRS R)

1 :	$f(a,a,a) \to f(c,c,c)$	2 :	$a \to d$
3 :	$d \to a$		

Step 1: Set $R_0 = R$ and $i = 0$.

Step 2: $S_0 = \{1\}$ and $P_0 = \{2,3\}$

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Step 2: $S_0 = \{1\}$ and $P_0 = \{2, 3\}$

Step 3: The seven elements of $PCP_{in}(P_0 \cup P_0^{-1}, S_0)$ are not joinable. Add all seven new rules and set $R_1 = R_0 \cup \{4, 5, 6, 7, 8, 9, 10\}$ and $i = 1$.

Example for Theorem 2

Example (TRS R)

1 :	$f(a,a,a) \to f(c,c,c)$	2 :	$a \to d$
4 :	$f(a,a,d) \to f(c,c,c)$	3 :	$d \to a$
5 :	$f(a,d,a) \to f(c,c,c)$	6 :	$f(d,a,d) \to f(c,c,c)$
7 :	$f(d,a,a) \to f(c,c,c)$	8 :	$f(a,d,d) \to f(c,c,c)$
9 :	$f(d,d,a) \to f(c,c,c)$	10 :	$f(d,d,d) \to f(c,c,c)$

Step 2: $S_1 = \{1,4,5,6,7,8,9,10\}$ and $P_1 = \{2,3\}$

Example for Theorem 2

Example (TRS R)

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Step 2: $S_1 = \{1, 4, 5, 6, 7, 8, 9, 10\}$ and $P_1 = \{2, 3\}$

Step 3: All pairs in $PCP_{in}(P_1 \cup P_1^{-1}, S_1)$ are joinable.

Example for Theorem 2

Example (TRS R)

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Step 3: All pairs in $PCP_{in}(P_1 \cup P_1^{-1}, S_1)$ are joinable.

Step 4: $CP(S_1, P_1 \cup P_1^{-1}) = \emptyset$

Step 5: $CP(S_1, S_1) = \emptyset$ so starting TRS R is confluent.

Overview

Limitation of Theorem 2

Example (TRS R)

1 :	$f(g(x), y) \to g(f(x, y))$	2 :	$g(x) \to g(g(x))$
3 :	$f(x, g(y)) \to g(f(x, y))$	4 :	$g(g(x)) \to g(x)$
5 :	$f(x, y) \to f(y, x)$		

- Only possible combination $S_0 = \{1, 3\}$ and $P_0 = \{2, 4, 5\}$

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- Only possible combination $S_0 = \{1, 3\}$ and $P_0 = \{2, 4, 5\}$
- But: problem occurs in $PCP_{in}(P_0 \cup P_0^{-1}, S_0)$
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- Still joinable but harms variable condition $\{x, y\} \not\subseteq \{x\}$

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- Only possible combination $S_0 = \{1, 3\}$ and $P_0 = \{2, 4, 5\}$
- But: problem occurs in $PCP_{in}(P_0 \cup P_0^{-1}, S_0)$
- One problematic critical pair $\langle f(g(g(x)), y), g(f(x, y))\rangle_X$ with $X = \{x\}$
- Still joinable but harms variable condition $\{x, y\} \nsubseteq \{x\}$
- Adding rule $f(g(g(x)), y) \to g(f(x, y))$ not helpful $\to$ looping problem with incremental $g's$

Theorem 3

Theorem (Confluence Criterion on linear TRSs)

Let P, S be TRSs such that S is linear and terminating and P is reversible. Suppose

(i) $CP(S, S) \subseteq \xrightarrow{*}_S \circ \xleftarrow{=}_{P \cup P^{-1}} \circ \xleftarrow{*}_S$

(ii) $CP(P \cup P^{-1}, S) \subseteq \xrightarrow{*}_S \circ \xleftarrow{=}_{P \cup P^{-1}} \circ \xleftarrow{*}_S$

(iii) $CP(S, P \cup P^{-1}) \subseteq \xrightarrow{*}_S \circ \xrightarrow{=}_{P \cup P^{-1}} \circ \xleftarrow{*}_S$

then $S \cup P$ is confluent.

Overview

- Literature
- Motivation
- Confluence including AC-Rules
- Implementation challenges
- Summary

Two main challenges

1. How to find $S_i \cup P_i$ efficiently?

Two main challenges

1. How to find $S_i \cup P_i$ efficiently?
2. How to find the 'best' subset of U?

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Summary

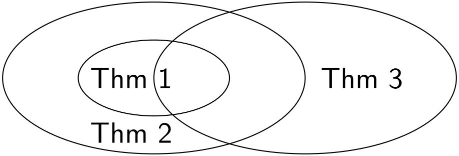

Figure: Relationship between theorems

- Theorems are useful for proving confluence of TRSs with AC-Rules

Summary

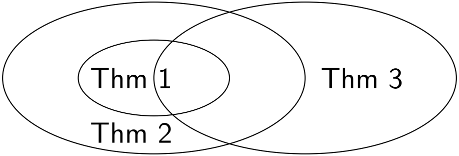

Figure: Relationship between theorems

- Theorems are useful for proving confluence of TRSs with AC-Rules
- Theorems do not cover all confluent TRSs with AC-Rules