Higher-
Order
Termination

Cynthia Kop

June 20, 2012

Overview

1. What is Higher-Order Termination?
2. What is so difficult about this?
3. What do we do about that?
4. How do we do that?
   - Polynomial Interpretations
   - Path Orderings
   - Dependency Pairs
5. Where does that bring us?

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$$\begin{array}{rcl} \mathtt{rec}(0, y, F) & \Rightarrow & y \\ \mathtt{rec}(\mathtt{s}(x), y, F) & \Rightarrow & F \cdot x \cdot \mathtt{rec}(x, y, F) \end{array}$$

$$\begin{array}{rcl} \mathtt{and}(P, \mathtt{forall}(\lambda x.Q(x))) & \Rightarrow & \mathtt{forall}(\lambda x.\mathtt{and}(P, Q(x))) \\ \mathtt{or}(P, \mathtt{forall}(\lambda x.Q(x))) & \Rightarrow & \mathtt{forall}(\lambda x.\mathtt{or}(P, Q(x))) \\ \mathtt{not}(\mathtt{forall}(\lambda x.Q(x))) & \Rightarrow & \mathtt{exists}(\lambda x.\mathtt{not}(Q(x))) \end{array}$$

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What is a Higher-Order Term Rewriting System?

Not one clear definition!

In practice: first-order rewriting plus application and/or abstraction

- CS [Aczel, 1978]
- CRS [Klop, 1980]
- ERS [Khasidashvili, 1990]
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- applicative untyped TRSs

- Higher-Order Logic (Isabelle)
- Calculus Of Constructions (Coq)
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What systems do I consider?

Systems with λ-abstraction and β-reduction.
(or meta-variables, which can easily encode β-reduction).

Otherwise typically convertible to first-order systems!

Also needed: type system.
(or possibility to derive types)

Otherwise typically non-terminating!

Still many possibilities:

- function symbols with or without arity
- matching with variables or meta-variables
- application present or absent
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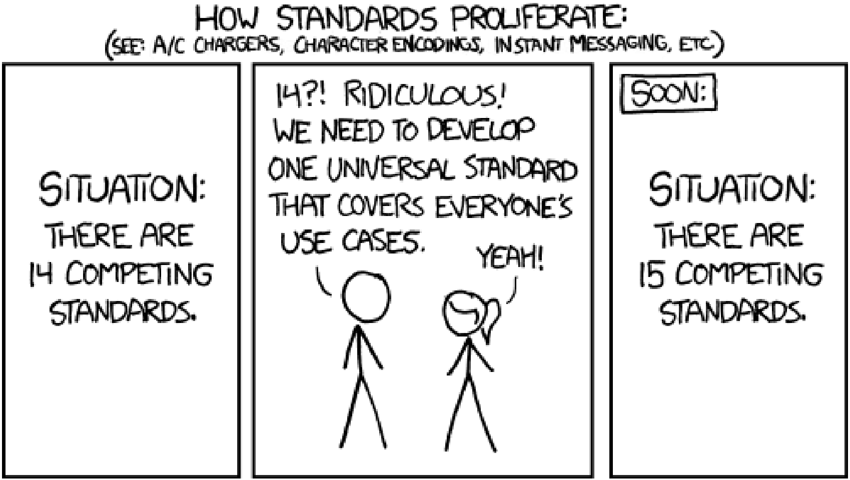

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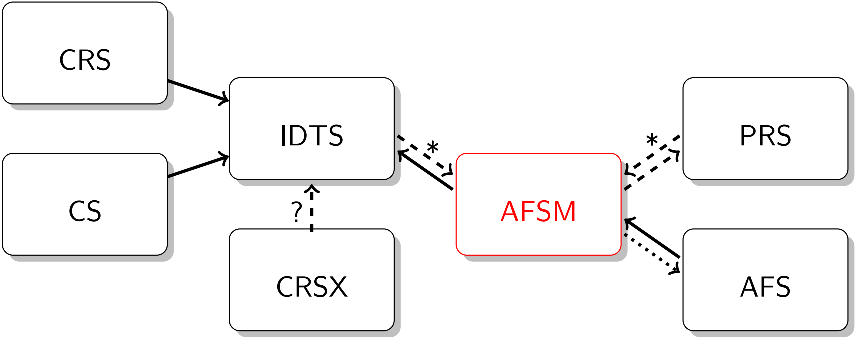

What is Termination?

Goal: study full termination

- no reduction strategy
- arbitrary start terms
  (in particular: start terms may contain λ-abstractions)

Outline

$$\mathtt{A}(\mathtt{L}(x)) \Rightarrow x$$

Non-terminating if $\mathtt{A} : [o] \to o \to o$ and $\mathtt{L} : [o \to o] \to o$

Define:

$$\omega := \mathtt{L}(\lambda x.(\mathtt{A}(x) \cdot x))$$

Then:

$$\begin{aligned} \mathtt{A}(\omega) \cdot \omega &= \mathtt{A}(\mathtt{L}(\lambda x.(\mathtt{A}(x) \cdot x))) \cdot \omega \\ &\Rightarrow (\lambda x.(\mathtt{A}(x) \cdot x)) \cdot \omega \\ &\Rightarrow_\beta \mathtt{A}(\omega) \cdot \omega \end{aligned}$$

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Outline

Solutions

- don't copy first-order methods, generalise the idea
- methods should be type-aware
- use techniques like computability

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$$\begin{array}{rcl} \mathtt{minus}(x, \mathtt{0}) & \Rightarrow & x \\ \mathtt{minus}(\mathtt{0}, x) & \Rightarrow & \mathtt{0} \\ \mathtt{minus}(\mathtt{s}(x), \mathtt{s}(y)) & \Rightarrow & \mathtt{minus}(x, y) \end{array}$$

Use $[\![\_]\!]$ with:

- $[\![\mathtt{0}]\!] = 0$
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- $[\![\mathtt{minus}]\!] = \lambda nm.n + m + 1$

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x+1 & = & [\![\texttt{minus}(x,\texttt{0})]\!] & > & [\![x]\!] & = & x \\
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Required:

$$[\![(\lambda x.s) \cdot t]\!] \geq [\![s[x := t]]\!]$$

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- $[\![\mathtt{nil}]\!] = 1$
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- $[\![\mathtt{map}]\!] = \lambda fn.f(0) + 2 \cdot n + n \cdot f(n)$
- $[\![\cdot^{\mathtt{nat} \to \mathtt{nat}}]\!] = \lambda fn.f(n) + n$

Constraints:

- $F(0) + 2 \cdot 1 + 1 \cdot F(1) > 1$
- $F(h+t+1) + h \cdot f(h+t+1) + t \cdot F(h+t+1) + 2 \cdot h + 1 > F(h) + t \cdot F(t) + 0$

Interpretations in Weakly Monotonic Functionals (Pol '96).

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Higher-Order Polynomials

Definition

- variables are polynomials
- if p, q polynomials, then $p + q$ is a polynomial
- if p, q polynomials, then $p \cdot q$ is a polynomial
- if F is a variable of type $\tau_1 \rightarrow \ldots \rightarrow \tau_m \rightarrow \iota$ with ι a base type, and $p_1 \in Pol^{\tau_1}, \ldots, p_m \in Pol^{\tau_m}$, then $F(p_1, \ldots, p_m)$ is a higher-order polynomial
  - here, $Pol^{\sigma_1 \rightarrow \ldots \rightarrow \sigma_n}$ contains elements $\boldsymbol{\lambda} y_1 \in WM_{\sigma_1} \ldots y_n \in WM_{\sigma_n}.q$, with q a higher-order polynomial.

In particular, if $p_1, \ldots, p_n$ are higher-order polynomials, and $F : \iota_1 \rightarrow \ldots \rightarrow \iota_n \rightarrow \kappa$ a second-order variable, then $F(p_1, \ldots, p_n)$ is a higher-order polynomial.

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Ingredients:

- well-founded precedence $>$
- status function mapping each symbol f to *lex* or *mul*

Definition:

1. $f(s_1, \ldots, s_n) \succ t$ if $s_i \succeq t$ (some i);
2. $f(s_1, \ldots, s_n) \succ g(t_1, \ldots, t_m)$ if $f > g$, $\forall i. f(\vec{s}) \succ t_i$;
3. $f(s_1, \ldots, s_n) \succ f(t_1, \ldots, t_m)$ if $stat(f) = lex$, $[s_1, \ldots, s_n] \succ_{\mathrm{lex}} [t_1, \ldots, t_m]$, $\forall i. f(\vec{s}) \succ t_i$;
4. $f(s_1, \ldots, s_n) \succ f(t_1, \ldots, t_m)$ if $stat(f) = mul$, $\{\{s_1, \ldots, s_n\}\} \succ_{\mathrm{mul}} \{\{t_1, \ldots, t_m\}\}$;
5. $x \succeq x$;
6. $f(s_1, \ldots, s_n) \succeq f(t_1, \ldots, t_n)$ if $stat(f) = lex$ and each $s_i \succeq t_i$;
7. $f(s_1, \ldots, s_n) \succeq f(t_1, \ldots, t_n)$ if $stat(f) = mul$ and $\{\{s_1, \ldots, s_n\}\} \succeq_{\mathrm{mul}} \{\{t_1, \ldots, t_n\}\}$.
8. $f(s_1, \ldots, s_n) \succeq t$ if $f(s_1, \ldots, s_n) \succ t$.

Ingredients:

- well-founded precedence $>$
- status function mapping each symbol f to *lex* or *mul*

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5. $x \succeq x$;
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8. $f(s_1,\ldots,s_n) \succeq t$ if $f(s_1,\ldots,s_n) \succ t$.

Ingredients:

- well-founded precedence $>$
- status function mapping each symbol f to *lex* or *mul*

Definition:

1. $f(s_1, \ldots, s_n) \succ t$ if $f^\star(s_1, \ldots, s_n) \succeq t$;
2. $f^\star(s_1, \ldots, s_n) \succeq t$ if $s_i \succeq t$ (some i);
3. $f^\star(s_1, \ldots, s_n) \succeq g(t_1, \ldots, t_m)$ if $f > g$, $\forall i. f^\star(\vec{s}) \succeq t_i$;
4. $f^\star(s_1, \ldots, s_n) \succeq f(t_1, \ldots, t_m)$ if $stat(f) = lex$, $[s_1, \ldots, s_n] \succ_{\mathrm{lex}} [t_1, \ldots, t_m]$, $\forall i. f^\star(\vec{s}) \succeq t_i$;
5. $f^\star(s_1, \ldots, s_n) \succeq f(t_1, \ldots, t_m)$ if $stat(f) = mul$, $\{\{s_1, \ldots, s_n\}\} \succ_{\mathrm{mul}} \{\{t_1, \ldots, t_m\}\}$;
6. $x \succeq x$;
7. $f(s_1, \ldots, s_n) \succeq f(t_1, \ldots, t_n)$ if $stat(f) = lex$ and each $s_i \succeq t_i$;
8. $f(s_1, \ldots, s_n) \succeq f(t_1, \ldots, t_n)$ if $stat(f) = mul$ and $\{\{s_1, \ldots, s_n\}\} \succeq_{\mathrm{mul}} \{\{t_1, \ldots, t_n\}\}$.
9. $f(s_1, \ldots, s_n) \succeq t$ if $f^\star(s_1, \ldots, s_n) \succeq t$.

Ingredients:

- application becomes a function symbol
- well-founded precedence $>$
- status function mapping each symbol f to *lex* or *mul*

So

$$\mathtt{map}(F, \mathtt{cons}(h, t)) \Rightarrow \mathtt{cons}(F \cdot h, \mathtt{map}(F, t))$$

becomes

$$\mathtt{map}(F, \mathtt{cons}(h, t)) \Rightarrow \mathtt{cons}(@^{\mathtt{nat} \to \mathtt{nat}}(F, h), \mathtt{map}(F, t))$$

Ingredients:

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1. $\lambda\vec{x}.f(s_1,\ldots,s_n) \succ t$ if $\lambda\vec{x}.f^\star(s_1,\ldots,s_n) \succeq t$;
2. $f^\star(s_1,\ldots,s_n) \succeq t$ if $s_i\langle f^\star(\vec{s}),\ldots,f^\star(\vec{s})\rangle \succeq t$ (some i);
3. $f^\star(s_1,\ldots,s_n) \succeq g(t_1,\ldots,t_m)$ if $f > g$, $\forall i$: $f^\star_{\tau_i}(\vec{s}) \succeq t_1,\ldots,t_n$;
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6. $f^\star(s_1,\ldots,s_n) \succeq \lambda x.t$ if $f^\star(s_1,\ldots,s_n,x) \succeq t$;
7. $x \succeq x$;
8. $\lambda x.s \succeq \lambda x.t$ if $s \succeq t$;
9. $F(s_1,\ldots,s_n) \succeq F(t_1,\ldots,t_n)$ if $\forall i.s_i \succeq t_i$;
10. $f(s_1,\ldots,s_n) \succeq f(t_1,\ldots,t_n)$ if $stat(f) = lex$ and each $s_i \succeq t_i$;
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12. $f(s_1,\ldots,s_n) \succeq t$ if $f^\star(s_1,\ldots,s_n) \succeq t$.

$$\begin{array}{rcl} \texttt{nil} & : & \mathsf{o} \\ \texttt{cons} & : & [\mathsf{o} \times \mathsf{o}] \to \mathsf{o} \\ \texttt{map} & : & [(\mathsf{o} \to \mathsf{o}) \times \mathsf{o}] \to \mathsf{o} \\ @^{\mathsf{o}\to\mathsf{o}} & : & [(\mathsf{o} \to \mathsf{o}) \times \mathsf{o}] \to \mathsf{o} \end{array}$$

$$\begin{array}{rcl} \texttt{map}(F, \texttt{nil}) & \Rightarrow & \texttt{nil} \\ \texttt{map}(F, \texttt{cons}(h, t)) & \Rightarrow & \texttt{cons}(@^{\mathsf{o}\to\mathsf{o}}(F, h), \texttt{map}(F, t)) \end{array}$$

Choose $\texttt{map} > \texttt{cons}, @^{\mathsf{o}\to\mathsf{o}}$, all statuses *mul*.

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Choose $\mathtt{map} > \mathtt{cons}, @^{\mathsf{o} \to \mathsf{o}}$, all statuses *mul*.

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Choose $\mathtt{map} > \mathtt{cons}, \mathtt{@}^{o \rightarrow o}$, all statuses *mul*.

Goal: $\mathtt{map}(F, \mathtt{cons}(h, t)) \succ \mathtt{cons}(\mathtt{@}^{o \rightarrow o}(F, h), \mathtt{map}(F, t))$

$$\begin{array}{rcl} \texttt{nil} & : & \mathsf{o} \\ \texttt{cons} & : & [\mathsf{o} \times \mathsf{o}] \to \mathsf{o} \\ \texttt{map} & : & [(\mathsf{o} \to \mathsf{o}) \times \mathsf{o}] \to \mathsf{o} \\ \texttt{@}^{\mathsf{o}\to\mathsf{o}} & : & [(\mathsf{o} \to \mathsf{o}) \times \mathsf{o}] \to \mathsf{o} \end{array}$$

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Choose $\texttt{map} > \texttt{cons}, \texttt{@}^{\mathsf{o}\to\mathsf{o}}$, all statuses *mul*.

Goal: $\texttt{map}(F, \texttt{cons}(h, t)) \succ \texttt{cons}(\texttt{@}^{\mathsf{o}\to\mathsf{o}}(F, h), \texttt{map}(F, t))$
Because (1)

- $\texttt{map}^{\star}(F, \texttt{cons}(h, t)) \succeq \texttt{cons}(\texttt{@}^{\mathsf{o}\to\mathsf{o}}(F, h), \texttt{map}(F, t))$

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Choose $\mathtt{map} > \mathtt{cons}, \mathtt{@}^{o \to o}$, all statuses *mul*.

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Goal: $\texttt{map}^{\star}(F, \texttt{cons}(h, t)) \succeq \texttt{cons}(@^{o \rightarrow o}(F, h), \texttt{map}(F, t))$
Because (3)

- $\texttt{map} > \texttt{cons}$
- $\texttt{map}^{\star}(F, \texttt{cons}(h, t)) \succeq @^{o \rightarrow o}(F, h)$
- $\texttt{map}^{\star}(F, \texttt{cons}(h, t)) \succeq \texttt{map}(F, t)$

$$\begin{array}{rcl} \mathtt{nil} & : & o \\ \mathtt{cons} & : & [o \times o] \to o \\ \mathtt{map} & : & [(o \to o) \times o] \to o \\ @^{o\to o} & : & [(o \to o) \times o] \to o \end{array}$$

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Choose $\mathtt{map} > \mathtt{cons}, @^{o\to o}$, all statuses *mul*.

Goal 1: $\mathtt{map}^\star(F, \mathtt{cons}(h, t)) \succeq @^{o\to o}(F, h)$
Goal 2: $\mathtt{map}^\star(F, \mathtt{cons}(h, t)) \succeq \mathtt{map}(F, t)$

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Choose $\mathtt{map} > \mathtt{cons}, @^{o \to o}$, all statuses *mul*.

Goal 1: $\mathtt{map}^\star(F, \mathtt{cons}(h, t)) \succeq @^{o \to o}(F, h)$
Goal 2: $\mathtt{map}^\star(F, \mathtt{cons}(h, t)) \succeq \mathtt{map}(F, t)$
Because (5)

- $F \succeq F$ (by 9)
- $\mathtt{cons}(h, t) \succ t$

$$\begin{array}{rcl} \texttt{nil} & : & o \\ \texttt{cons} & : & [o \times o] \to o \\ \texttt{map} & : & [(o \to o) \times o] \to o \\ @^{o \to o} & : & [(o \to o) \times o] \to o \end{array}$$

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Choose $\texttt{map} > \texttt{cons}, @^{o \to o}$, all statuses *mul*.

Goal 1: $\texttt{map}^{\star}(F, \texttt{cons}(h, t)) \succeq @^{o \to o}(F, h)$
Goal 2: $\texttt{cons}(h, t) \succ t$

$$\begin{array}{rcl} \texttt{nil} & : & o \\ \texttt{cons} & : & [o \times o] \to o \\ \texttt{map} & : & [(o \to o) \times o] \to o \\ @^{o \to o} & : & [(o \to o) \times o] \to o \end{array}$$

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Goal 1: $\texttt{map}^{\star}(F, \texttt{cons}(h, t)) \succeq @^{o \to o}(F, h)$
Goal 2: $\texttt{cons}(h, t) \succ t$
Because (1)

- $\texttt{cons}^{\star}(h, t) \succeq t$

$$\begin{array}{rcl} \mathtt{nil} & : & \mathsf{o} \\ \mathtt{cons} & : & [\mathsf{o} \times \mathsf{o}] \to \mathsf{o} \\ \mathtt{map} & : & [(\mathsf{o} \to \mathsf{o}) \times \mathsf{o}] \to \mathsf{o} \\ @^{\mathsf{o} \to \mathsf{o}} & : & [(\mathsf{o} \to \mathsf{o}) \times \mathsf{o}] \to \mathsf{o} \end{array}$$

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Choose $\mathtt{map} > \mathtt{cons}, @^{\mathsf{o} \to \mathsf{o}}$, all statuses *mul*.

Goal 1: $\mathtt{map}^{\star}(F, \mathtt{cons}(h, t)) \succeq @^{\mathsf{o} \to \mathsf{o}}(F, h)$
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Choose $\mathtt{map} > \mathtt{cons}, @^{o \to o}$, all statuses *mul*.

Goal 1: $\mathtt{map}^{\star}(F, \mathtt{cons}(h, t)) \succeq @^{o \to o}(F, h)$
Goal 2: $\mathtt{cons}^{\star}(h, t) \succeq t$
Because (2)

- $t \succeq t$ (by 9)

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$$\begin{array}{rcl} \texttt{map}(F, \texttt{nil}) & \Rightarrow & \texttt{nil} \\ \texttt{map}(F, \texttt{cons}(h, t)) & \Rightarrow & \texttt{cons}(\texttt{@}^{o \to o}(F, h), \texttt{map}(F, t)) \end{array}$$

Choose $\texttt{map} > \texttt{cons}, \texttt{@}^{o \to o}$, all statuses *mul*.

Goal 1: $\texttt{map}^{\star}(F, \texttt{cons}(h, t)) \succeq \texttt{@}^{o \to o}(F, h)$

$$\begin{array}{rcl} \mathtt{nil} & : & o \\ \mathtt{cons} & : & [o \times o] \rightarrow o \\ \mathtt{map} & : & [(o \rightarrow o) \times o] \rightarrow o \\ @^{o \rightarrow o} & : & [(o \rightarrow o) \times o] \rightarrow o \end{array}$$

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Choose $\mathtt{map} > \mathtt{cons}, @^{o \rightarrow o}$, all statuses *mul*.

Goal 1: $\mathtt{map}^{\star}(F, \mathtt{cons}(h, t)) \succeq @^{o \rightarrow o}(F, h)$
Because (3)

- $\mathtt{map} > @^{o \rightarrow o}$
- $\mathtt{map}^{\star}_{o \rightarrow o}(F, \mathtt{cons}(h, t)) \succeq F$
- $\mathtt{map}^{\star}(F, \mathtt{cons}(h, t)) \succeq t$

$$\begin{array}{rcl} \mathtt{nil} & : & o \\ \mathtt{cons} & : & [o \times o] \to o \\ \mathtt{map} & : & [(o \to o) \times o] \to o \\ @^{o \to o} & : & [(o \to o) \times o] \to o \end{array}$$

$$\begin{array}{rcl} \mathtt{map}(F, \mathtt{nil}) & \Rightarrow & \mathtt{nil} \\ \mathtt{map}(F, \mathtt{cons}(h, t)) & \Rightarrow & \mathtt{cons}(@^{o \to o}(F, h), \mathtt{map}(F, t)) \end{array}$$

Choose $\mathtt{map} > \mathtt{cons}, @^{o \to o}$, all statuses *mul*.

Goal 1: $\mathtt{map}^{\star}_{o \to o}(F, \mathtt{cons}(h, t)) \succeq F$
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$$\begin{array}{rcl} \texttt{map}(F, \texttt{nil}) & \Rightarrow & \texttt{nil} \\ \texttt{map}(F, \texttt{cons}(h, t)) & \Rightarrow & \texttt{cons}(@^{\mathsf{o} \to \mathsf{o}}(F, h), \texttt{map}(F, t)) \end{array}$$

Choose $\texttt{map} > \texttt{cons}, @^{\mathsf{o} \to \mathsf{o}}$, all statuses *mul*.

Goal 1: $\texttt{map}^{\star}_{\mathsf{o} \to \mathsf{o}}(F, \texttt{cons}(h, t)) \succeq F$
Goal 2: $\texttt{map}^{\star}(F, \texttt{cons}(h, t)) \succeq h$
Because (2)

- $\texttt{cons}(h, t) \succeq h$

$$\begin{array}{rcl} \texttt{nil} & : & \mathsf{o} \\ \texttt{cons} & : & [\mathsf{o} \times \mathsf{o}] \to \mathsf{o} \\ \texttt{map} & : & [(\mathsf{o} \to \mathsf{o}) \times \mathsf{o}] \to \mathsf{o} \\ \texttt{@}^{\mathsf{o} \to \mathsf{o}} & : & [(\mathsf{o} \to \mathsf{o}) \times \mathsf{o}] \to \mathsf{o} \end{array}$$

$$\begin{array}{rcl} \texttt{map}(F, \texttt{nil}) & \Rightarrow & \texttt{nil} \\ \texttt{map}(F, \texttt{cons}(h, t)) & \Rightarrow & \texttt{cons}(\texttt{@}^{\mathsf{o} \to \mathsf{o}}(F, h), \texttt{map}(F, t)) \end{array}$$

Choose $\texttt{map} > \texttt{cons}, \texttt{@}^{\mathsf{o} \to \mathsf{o}}$, all statuses *mul*.

Goal 1: $\texttt{map}^{\star}_{\mathsf{o} \to \mathsf{o}}(F, \texttt{cons}(h, t)) \succeq F$
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Choose $\texttt{map} > \texttt{cons}, @^{\mathsf{o} \to \mathsf{o}}$, all statuses *mul*.

Goal 1: $\texttt{map}^{\star}_{\mathsf{o} \to \mathsf{o}}(F, \texttt{cons}(h, t)) \succeq F$
Goal 2: $\texttt{cons}(h, t) \succeq h$
Because (12)

- $\texttt{cons}^{\star}(h, t) \succeq h$

$$\begin{array}{rcl} \mathtt{nil} & : & \mathsf{o} \\ \mathtt{cons} & : & [\mathsf{o} \times \mathsf{o}] \to \mathsf{o} \\ \mathtt{map} & : & [(\mathsf{o} \to \mathsf{o}) \times \mathsf{o}] \to \mathsf{o} \\ @^{\mathsf{o} \to \mathsf{o}} & : & [(\mathsf{o} \to \mathsf{o}) \times \mathsf{o}] \to \mathsf{o} \end{array}$$

$$\begin{array}{rcl} \mathtt{map}(F, \mathtt{nil}) & \Rightarrow & \mathtt{nil} \\ \mathtt{map}(F, \mathtt{cons}(h, t)) & \Rightarrow & \mathtt{cons}(@^{\mathsf{o} \to \mathsf{o}}(F, h), \mathtt{map}(F, t)) \end{array}$$

Choose $\mathtt{map} > \mathtt{cons}, @^{\mathsf{o} \to \mathsf{o}}$, all statuses *mul*.

Goal 1: $\mathtt{map}^{\star}_{\mathsf{o} \to \mathsf{o}}(F, \mathtt{cons}(h, t)) \succeq F$
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Because (2)

- $h \succeq h$ (by 9)

$$\begin{array}{rcl} \mathtt{nil} & : & o \\ \mathtt{cons} & : & [o \times o] \to o \\ \mathtt{map} & : & [(o \to o) \times o] \to o \\ @^{o\to o} & : & [(o \to o) \times o] \to o \end{array}$$

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Termination proved!

$$\begin{array}{lcl} \mathtt{A}^{\sigma,\tau} & : & [(\sigma \to \tau) \times \sigma] \to \tau \\ \mathtt{L}^{\sigma,\tau} & : & [\sigma \to \tau] \to \sigma \to \tau \end{array}$$

$$\mathtt{A}^{\sigma,\tau}(\mathtt{L}^{\sigma,\tau}(\lambda x.F(x)), Y) \Rightarrow F(Y)$$

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Goal: $\mathtt{A}(\mathtt{L}(\lambda x.F(x)), Y) \succ F(Y)$

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Because: (1)

- $\mathtt{A}^{\star}(\mathtt{L}(\lambda x.F(x)), Y) \succeq F(Y)$

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Goal: $\mathtt{A}^{\star}(\mathtt{L}(\lambda x.F(x)), Y) \succeq F(Y)$
Because: (2)

- $\mathtt{L}(\lambda x.F(x)) : \sigma \to \tau$ and $F(Y) : \tau$
- $\mathtt{L}^{\star}(\lambda x.F(x), \mathtt{A}^{\star}_{\sigma}(\mathtt{L}(\lambda x.F(x)), Y)) \succeq F(Y)$

$$\begin{array}{lcl} \mathtt{A}^{\sigma,\tau} & : & [(\sigma \to \tau) \times \sigma] \to \tau \\ \mathtt{L}^{\sigma,\tau} & : & [\sigma \to \tau] \to \sigma \to \tau \end{array}$$

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Because: (2)

- $\lambda x.F(x) : \sigma \to \tau$ and $F(Y) : \tau$
- $F(\mathtt{L}^{\star}_{\sigma}(\lambda x.F(x), \mathtt{A}^{\star}_{\sigma}(\mathtt{L}(\lambda x.F(x)), Y))) \succeq F(Y)$

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$$\mathtt{A}^{\sigma,\tau}(\mathtt{L}^{\sigma,\tau}(\lambda x.F(x)), Y) \Rightarrow F(Y)$$

Termination proved!

First-order Dependency Pairs

[Arts and Giesl 1997]
rewrite rules:

$$\begin{array}{rcl} \mathtt{A}(\mathtt{0}, n) & \Rightarrow & \mathtt{s}(n) \\ \mathtt{A}(\mathtt{s}(m), \mathtt{0}) & \Rightarrow & \mathtt{A}(m, \mathtt{s}(\mathtt{0})) \\ \mathtt{A}(\mathtt{s}(m), \mathtt{s}(n)) & \Rightarrow & \mathtt{A}(m, \mathtt{A}(\mathtt{s}(m), n)) \end{array}$$

dependency pairs:

$$\begin{array}{lrcl} 1. & \mathtt{A}^\sharp(\mathtt{s}(m), \mathtt{0}) & \Rightarrow & \mathtt{A}^\sharp(m, \mathtt{s}(\mathtt{0})) \\ 2. & \mathtt{A}^\sharp(\mathtt{s}(m), \mathtt{s}(n)) & \Rightarrow & \mathtt{A}^\sharp(m, \mathtt{A}(\mathtt{s}(m), n)) \\ 3. & \mathtt{A}^\sharp(\mathtt{s}(m), \mathtt{s}(n)) & \Rightarrow & \mathtt{A}^\sharp(\mathtt{s}(m), n) \end{array}$$

infinite reduction $\Leftrightarrow$ infinite dependency chain

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Higher-order Dependency Pairs

$$
\begin{array}{rcl}
\mathtt{I}(\mathtt{0}) & \Rightarrow & \mathtt{0} \\
\mathtt{I}(\mathtt{s}(n)) & \Rightarrow & \mathtt{s}(\mathtt{twice}(\lambda x.\mathtt{I}(x), n)) \\
\mathtt{twice}(F, n) & \Rightarrow & F \cdot (F \cdot n)
\end{array}
$$

Two styles of Dependency Pairs!

Dynamic Dependency Pairs:

$$
\begin{array}{rcl}
\mathtt{I}^\sharp(\mathtt{s}(n)) & \Rightarrow & \mathtt{twice}^\sharp(\lambda x.\mathtt{I}(x), n) \\
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Higher-order Dependency Pairs

argument filterings remain powerful (assuming some restrictions)

dependency graph relatively weak in the presence of collapsing DPs (but can still be used, and extended with bound-variable occurrence checks)

usable rules relatively weak in the presence of collapsing rules (but can be useful in particular with static dependency pairs, and extended with type information)

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Outline

Conclusions

- first-order termination techniques can often be generalised
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- specific methods for higher-order rewriting can also be designed
  in particular: transformations
- automatability: similar to the first-order case
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- automatability: similar to the first-order case
  although higher risk of exponential blow-up

Conclusions

- first-order termination techniques can often be generalised
  often even in more than one way!
- specific methods for higher-order rewriting can also be designed
  in particular: transformations
- automatability: similar to the first-order case
  although higher risk of exponential blow-up

Future Work

- further extension of first-order techniques
  - narrowing (non-termination)
  - usable and formative rules wrt an argument filtering
  - max-interpretations
- extension to more complicated type systems
  - polymorphism
  - dependent types
- using WANDA for real applications

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