Ordinals, Subrecursive Hierarchies and All That

Georg Moser

Institute of Computer Science
University of Innsbruck

Master Seminar, November 9, 2011

Overview

- Ordinals
- Demystification of Ordinals
- Subrecursive Hierarchies
- Application ①
- More on the Slow-Growing Hierarchy
- Application ②

Ordinals

Definition

an ordinal is a set α such that

1. α is totally ordered with respect to membership

$$\forall \beta, \gamma \in \alpha \colon \beta \in \gamma \vee \gamma \in \beta \vee \beta = \gamma$$

2. every element of α is a subset of α (aka α is transitive)

$$\forall \beta \colon \beta \in \alpha \text{ implies } \beta \subseteq \alpha$$

Ordinals

Definition

an ordinal is a set α such that

1. α is totally ordered with respect to membership
$$\forall \beta, \gamma \in \alpha \colon \beta \in \gamma \vee \gamma \in \beta \vee \beta = \gamma$$
2. every element of α is a subset of α (aka α is transitive)
$$\forall \beta \colon \beta \in \alpha \text{ implies } \beta \subseteq \alpha$$

Lemma

ordinal α is either

- 0
- $\beta \cup \{\beta\} = \beta + 1$ *successor ordinal*
- $\bigcup \alpha$ *limit ordinal*

Further Properties or Ordinals

Lemma

let α be an ordinal, then

- α *is well-ordered by* $\in$
  1. α *is totally ordered*
  2. *there are no infinite descending sequences*

$$\neg \exists \; \alpha_1 \ni \alpha_2 \ni \alpha_3 \cdots \qquad\qquad (\alpha_i \in \alpha)$$

- *if* $\beta \in \alpha$*, then* β *is an ordinal*
- $\alpha = \bigcup_{\beta \in \alpha} \beta$

Lemma

let X be a non-empty set of ordinals, then

- $\bigcap X$ *is an ordinal and* $\bigcap X = \inf X$
- $\bigcup X$ *is an ordinal and* $\bigcup X = \sup X$

Arithmetic of Ordinals

Definition (Addition)

$$\alpha + \beta = \begin{cases} \alpha & \text{if } \beta = 0 \\ (\alpha + \beta') + 1 & \text{if } \beta = \beta' + 1 \\ \bigcup_{\beta' < \beta}(\alpha + \beta') & \text{if } \beta \text{ is a limit ordinal} \end{cases}$$

Arithmetic of Ordinals

Definition (Addition)

$$\alpha + \beta = \begin{cases} \alpha & \text{if } \beta = 0 \\ (\alpha + \beta') + 1 & \text{if } \beta = \beta' + 1 \\ \bigcup_{\beta' < \beta} (\alpha + \beta') & \text{if } \beta \text{ is a limit ordinal} \end{cases}$$

Definition (Multiplication)

$$\alpha \cdot \beta = \begin{cases} 0 & \text{if } \beta = 0 \\ \alpha \cdot \beta' + \alpha & \text{if } \beta = \beta' + 1 \\ \bigcup_{\beta' < \beta} (\alpha \cdot \beta') & \text{if } \beta \text{ is a limit ordinal} \end{cases}$$

Arithmetic of Ordinals

Definition (Addition)

$$\alpha + \beta = \begin{cases} \alpha & \text{if } \beta = 0 \\ (\alpha + \beta') + 1 & \text{if } \beta = \beta' + 1 \\ \bigcup_{\beta' < \beta} (\alpha + \beta') & \text{if } \beta \text{ is a limit ordinal} \end{cases}$$

Definition (Multiplication)

$$\alpha \cdot \beta = \begin{cases} 0 & \text{if } \beta = 0 \\ \alpha \cdot \beta' + \alpha & \text{if } \beta = \beta' + 1 \\ \bigcup_{\beta' < \beta} (\alpha \cdot \beta') & \text{if } \beta \text{ is a limit ordinal} \end{cases}$$

Definition (Exponentiation)

. . .

Lemma

for all ordinals α, β, γ

- $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$
- $\alpha \cdot (\beta \cdot \gamma) = (\alpha \cdot \beta) \cdot \gamma$

Lemma

for all ordinals α, β, γ

- $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$
- $\alpha \cdot (\beta \cdot \gamma) = (\alpha \cdot \beta) \cdot \gamma$

Example

neither $+$ nor $\cdot$ are commutative

$$1 + \omega = \omega \neq \omega + 1 \qquad 2 \cdot \omega = \omega \neq \omega \cdot 2$$

Lemma

for all ordinals α, β, γ

- $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$
- $\alpha \cdot (\beta \cdot \gamma) = (\alpha \cdot \beta) \cdot \gamma$

Example

neither $+$ nor $\cdot$ are commutative

$$1 + \omega = \omega \neq \omega + 1 \qquad 2 \cdot \omega = \omega \neq \omega \cdot 2$$

Theorem

ordinal $\alpha \neq 0$ *is representable in Cantor Normal Form (CNF)*

$$\alpha = \omega^{\alpha_1} + \cdots + \omega^{\alpha_n}$$

where $\alpha_1 \geqslant \cdots \geqslant \alpha_n$ *and* α_i *in CNF*

Demystification of Ordinals

Definitions

let $(A, <)$, $(B, \prec)$ be partially ordered sets

- a mapping $f\colon A \to B$ is order preserving if

$$x < y \text{ implies } f(x) \prec f(y)$$

- if f is a bijection and f and f^{-1} are order preserving then $(A, <)$ is isomorphic to $(B, \prec)$

Demystification of Ordinals

Definitions

let $(A, <)$, $(B, \prec)$ be partially ordered sets

- a mapping $f \colon A \to B$ is order preserving if
$$x < y \text{ implies } f(x) \prec f(y)$$
- if f is a bijection and f and f^{-1} are order preserving then $(A, <)$ is isomorphic to $(B, \prec)$

Lemma

if two well-ordered sets W_1 and W_2 are isomorphic, then the isomorphism of W_1 onto W_2 is unique

Demystification of Ordinals

Definitions

let $(A, <)$, $(B, \prec)$ be partially ordered sets

- a mapping $f\colon A \to B$ is order preserving if

$$x < y \text{ implies } f(x) \prec f(y)$$

- if f is a bijection and f and f^{-1} are order preserving then $(A, <)$ is isomorphic to $(B, \prec)$

Lemma

if two well-ordered sets W_1 and W_2 are isomorphic, then the isomorphism of W_1 onto W_2 is unique

Theorem

every well-ordered set is isomorphic to a unique ordinal

Ordinals as Order Types

Alternative "Definition"

an ordinal is an equivalence class of well-orders with respect to isomorphism

Ordinals as Order Types

Alternative “Definition”

an ordinal is an equivalence class of well-orders with respect to isomorphism

Definition

an ordinal term α is an element of the inductively defined set E;
definition makes use of auxiliary set P

Ordinals as Order Types

Alternative "Definition"

an ordinal is an equivalence class of well-orders with respect to isomorphism

Definition

an ordinal term α is an element of the inductively defined set E; definition makes use of auxiliary set P

1. $0 \in E$

Ordinals as Order Types

Alternative "Definition"

an ordinal is an equivalence class of well-orders with respect to isomorphism

Definition

an ordinal term α is an element of the inductively defined set E; definition makes use of auxiliary set P

1. $0 \in E$
2. if $\alpha_1, \ldots, \alpha_m \in P$, then $\alpha_1 + \cdots + \alpha_m \in E$

Ordinals as Order Types

Alternative "Definition"

an ordinal is an equivalence class of well-orders with respect to isomorphism

Definition

an ordinal term α is an element of the inductively defined set E; definition makes use of auxiliary set P

1. $0 \in E$
2. if $\alpha_1, \ldots, \alpha_m \in P$, then $\alpha_1 + \cdots + \alpha_m \in E$
3. if $\alpha \in E$, then $\omega^\alpha \in P$, and $\omega^\alpha \in E$

Ordinals as Order Types

Alternative "Definition"

an ordinal is an equivalence class of well-orders with respect to isomorphism

Definition

an ordinal term α is an element of the inductively defined set E; definition makes use of auxiliary set P

1. $0 \in E$
2. if $\alpha_1, \dots, \alpha_m \in P$, then $\alpha_1 + \cdots + \alpha_m \in E$
3. if $\alpha \in E$, then $\omega^\alpha \in P$, and $\omega^\alpha \in E$

we write 1 instead of ω^0

Ordinals as Order Types

Alternative "Definition"

an ordinal is an equivalence class of well-orders with respect to isomorphism

Definition

an ordinal term α is an element of the inductively defined set E; definition makes use of auxiliary set P

1. $0 \in E$
2. if $\alpha_1, \ldots, \alpha_m \in P$, then $\alpha_1 + \cdots + \alpha_m \in E$
3. if $\alpha \in E$, then $\omega^\alpha \in P$, and $\omega^\alpha \in E$

we write 1 instead of ω^0

Lemma

any $\alpha \neq 0 \in E$ can be uniquely represented as $\alpha = \omega^{\alpha_1} + \cdots + \omega^{\alpha_m}$, where for each $\alpha_i \neq 0$ the same holds

Definition

simultaneous definition of $\approx$ and $\prec$ on E

Definition

simultaneous definition of $\approx$ and $\prec$ on E

1. 0 is the minimal element of $\prec$

Definition

simultaneous definition of $\approx$ and $\prec$ on E

1. 0 is the minimal element of $\prec$
2. $\alpha \prec \beta$ iff $\omega^\alpha \prec \omega^\beta$

Definition

simultaneous definition of $\approx$ and $\prec$ on E

1. 0 is the minimal element of $\prec$
2. $\alpha \prec \beta$ iff $\omega^\alpha \prec \omega^\beta$
3. let α, β be of form

$$\alpha = \gamma + \omega^{\alpha_i} + \omega^{\alpha_{i+1}} + \delta \qquad \beta = \gamma + \omega^{\alpha_{i+1}} + \delta$$

where $\alpha_{i+1} \succ \alpha_i$, then $\alpha \approx \beta$

Definition

simultaneous definition of $\approx$ and $\prec$ on E

1. 0 is the minimal element of $\prec$
2. $\alpha \prec \beta$ iff $\omega^\alpha \prec \omega^\beta$
3. let α, β be of form
$$\alpha = \gamma + \omega^{\alpha_i} + \omega^{\alpha_{i+1}} + \delta \qquad \beta = \gamma + \omega^{\alpha_{i+1}} + \delta$$
where $\alpha_{i+1} \succ \alpha_i$, then $\alpha \approx \beta$
4. let $\alpha = \omega^{\alpha_1} + \cdots + \omega^{\alpha_m}$ and let $\beta = \omega^{\beta_1} + \cdots + \omega^{\beta_n}$ such that
$$\alpha_1 \succcurlyeq \alpha_2 \succcurlyeq \cdots \succcurlyeq \alpha_m \qquad \beta_1 \succcurlyeq \beta_2 \succcurlyeq \cdots \succcurlyeq \beta_n$$
then $\alpha \prec \beta$ iff $\exists i\colon \omega^{\alpha_i} \prec \omega^{\beta_i} \wedge \forall j < i\colon \omega^{\alpha_j} \approx \omega^{\beta_j}$

Lemma (Cantor Normal Form (again))

ordinal term $\alpha \neq 0$ is uniquely representable in Cantor Normal Form (CNF)

$$\alpha = \omega^{\alpha_1} + \cdots + \omega^{\alpha_n}$$

where $\alpha_1 \succcurlyeq \cdots \succcurlyeq \alpha_n$ and α_i in CNF

Lemma (Cantor Normal Form (again))

ordinal term $\alpha \neq 0$ is uniquely representable in Cantor Normal Form (CNF)

$$\alpha = \omega^{\alpha_1} + \cdots + \omega^{\alpha_n}$$

where $\alpha_1 \succcurlyeq \cdots \succcurlyeq \alpha_n$ and α_i in CNF

Lemma

the set $(E, \prec)$ is isomorphic to $(\epsilon_0, \in)$

Lemma (Cantor Normal Form (again))

ordinal term $\alpha \neq 0$ is uniquely representable in Cantor Normal Form (CNF)

$$\alpha = \omega^{\alpha_1} + \cdots + \omega^{\alpha_n}$$

where $\alpha_1 \succcurlyeq \cdots \succcurlyeq \alpha_n$ and α_i in CNF

Lemma

the set $(E, \prec)$ is isomorphic to $(\epsilon_0, \in)$

Notations

- ordinal terms are ordinals, collected in the set $\mathcal{O}$
- $\approx$ becomes $=$
- $\prec$ becomes $<$
- a limit ordinal is an ordinal which is neither 0 nor a successor ordinal

Subrecursive Hierarchies

Definition

we define the family of fundamental sequences $\lambda[x]_{x \in \mathbb{N}}$ as follows (λ limit ordinal):

$$\lambda[x] = \begin{cases} & \text{if } \lambda = \omega \\ & \text{if } \lambda = \beta + \omega^{\alpha+1} \\ & \text{if } \lambda = \beta + \omega^{\alpha}, \alpha \text{ limit} \end{cases}$$

Subrecursive Hierarchies

Definition

we define the family of fundamental sequences $\lambda[x]_{x \in \mathbb{N}}$ as follows (λ limit ordinal):

$$\lambda[x] = \begin{cases} x+1 & \text{if } \lambda = \omega \\ & \text{if } \lambda = \beta + \omega^{\alpha+1} \\ & \text{if } \lambda = \beta + \omega^{\alpha}, \ \alpha \text{ limit} \end{cases}$$

Subrecursive Hierarchies

Definition

we define the family of fundamental sequences $\lambda[x]_{x\in\mathbb{N}}$ as follows (λ limit ordinal):

$$\lambda[x] = \begin{cases} x+1 & \text{if } \lambda = \omega \\ \beta + \omega^{\alpha}\cdot(x+1) & \text{if } \lambda = \beta + \omega^{\alpha+1} \\ & \text{if } \lambda = \beta + \omega^{\alpha},\ \alpha \text{ limit} \end{cases}$$

Subrecursive Hierarchies

Definition

we define the family of fundamental sequences $\lambda[x]_{x\in\mathbb{N}}$ as follows (λ limit ordinal):

$$\lambda[x] = \begin{cases} x+1 & \text{if } \lambda = \omega \\ \beta + \omega^\alpha \cdot (x+1) & \text{if } \lambda = \beta + \omega^{\alpha+1} \\ \beta + \omega^{\alpha[x]} & \text{if } \lambda = \beta + \omega^\alpha\text{, } \alpha \text{ limit} \end{cases}$$

Subrecursive Hierarchies

Definition

we define the family of fundamental sequences $\lambda[x]_{x \in \mathbb{N}}$ as follows (λ limit ordinal):

$$\lambda[x] = \begin{cases} x+1 & \text{if } \lambda = \omega \\ \beta + \omega^{\alpha} \cdot (x+1) & \text{if } \lambda = \beta + \omega^{\alpha+1} \\ \beta + \omega^{\alpha[x]} & \text{if } \lambda = \beta + \omega^{\alpha},\ \alpha \text{ limit} \end{cases}$$

Definition

the family of slow-growing functions $(\mathsf{G}_\alpha)_{\alpha \in \mathcal{O}}$ is defined as follows:

$$\mathsf{G}_0(x) = 0 \qquad \mathsf{G}_{\alpha+1}(x) = \mathsf{G}_\alpha(x) + 1 \qquad \mathsf{G}_\lambda(x) = \mathsf{G}_{\lambda[x]}(x) \quad (\lambda \text{ limit})$$

Subrecursive Hierarchies

Definition

we define the family of fundamental sequences $\lambda[x]_{x \in \mathbb{N}}$ as follows (λ limit ordinal):

$$\lambda[x] = \begin{cases} x + 1 & \text{if } \lambda = \omega \\ \beta + \omega^{\alpha} \cdot (x + 1) & \text{if } \lambda = \beta + \omega^{\alpha+1} \\ \beta + \omega^{\alpha[x]} & \text{if } \lambda = \beta + \omega^{\alpha},\ \alpha \text{ limit} \end{cases}$$

Definition

the family of slow-growing functions $(\mathsf{G}_{\alpha})_{\alpha \in \mathcal{O}}$ is defined as follows:

$$\mathsf{G}_0(x) = 0 \qquad \mathsf{G}_{\alpha+1}(x) = \mathsf{G}_{\alpha}(x) + 1 \qquad \mathsf{G}_{\lambda}(x) = \mathsf{G}_{\lambda[x]}(x) \quad (\lambda \text{ limit})$$

Subrecursive Hierarchies

Definition

we define the family of fundamental sequences $\lambda[x]_{x \in \mathbb{N}}$ as follows (λ limit ordinal):

$$\lambda[x] = \begin{cases} x + 1 & \text{if } \lambda = \omega \\ \beta + \omega^{\alpha} \cdot (x+1) & \text{if } \lambda = \beta + \omega^{\alpha+1} \\ \beta + \omega^{\alpha[x]} & \text{if } \lambda = \beta + \omega^{\alpha},\ \alpha \text{ limit} \end{cases}$$

Definition

the family of slow-growing functions $(\mathsf{G}_\alpha)_{\alpha \in \mathcal{O}}$ is defined as follows:

$$\mathsf{G}_0(x) = 0 \qquad \mathsf{G}_{\alpha+1}(x) = \mathsf{G}_\alpha(x) + 1 \qquad \mathsf{G}_\lambda(x) = \mathsf{G}_{\lambda[x]}(x) \quad (\lambda \text{ limit})$$

Subrecursive Hierarchies

Definition

we define the family of fundamental sequences $\lambda[x]_{x \in \mathbb{N}}$ as follows (λ limit ordinal):

$$\lambda[x] = \begin{cases} x+1 & \text{if } \lambda = \omega \\ \beta + \omega^\alpha \cdot (x+1) & \text{if } \lambda = \beta + \omega^{\alpha+1} \\ \beta + \omega^{\alpha[x]} & \text{if } \lambda = \beta + \omega^\alpha,\ \alpha \text{ limit} \end{cases}$$

Definition

the family of slow-growing functions $(\mathsf{G}_\alpha)_{\alpha \in \mathcal{O}}$ is defined as follows:

$$\mathsf{G}_0(x) = 0 \qquad \mathsf{G}_{\alpha+1}(x) = \mathsf{G}_\alpha(x) + 1 \qquad \mathsf{G}_\lambda(x) = \mathsf{G}_{\lambda[x]}(x) \quad (\lambda \text{ limit})$$

Subrecursive Hierarchies

Definition

we define the family of fundamental sequences $\lambda[x]_{x\in\mathbb{N}}$ as follows (λ limit ordinal):

$$\lambda[x] = \begin{cases} x+1 & \text{if } \lambda = \omega \\ \beta + \omega^{\alpha}\cdot(x+1) & \text{if } \lambda = \beta + \omega^{\alpha+1} \\ \beta + \omega^{\alpha[x]} & \text{if } \lambda = \beta + \omega^{\alpha}, \ \alpha \text{ limit} \end{cases}$$

Definition

the family of slow-growing functions $(\mathsf{G}_\alpha)_{\alpha\in\mathcal{O}}$ is defined as follows:

$$\mathsf{G}_0(x) = 0 \qquad \mathsf{G}_{\alpha+1}(x) = \mathsf{G}_\alpha(x) + 1 \qquad \mathsf{G}_\lambda(x) = \mathsf{G}_{\lambda[x]}(x) \quad (\lambda \text{ limit})$$

Example

$$\mathsf{G}_\omega(x) = x+1 \quad \mathsf{G}_{\omega^{\omega^\omega}}(x) = (x+1)^{x+1^{x+1}} \quad \mathsf{G}_{\omega\cdot 2}(10) = (10+1)\cdot 2 = 22$$

Lemma

- $\mathsf{G}_{\alpha_1+\cdots+\alpha_n}(x) = \mathsf{G}_{\alpha_1}(x) + \cdots + \mathsf{G}_{\alpha_n}(x)$
- $\mathsf{G}_{\omega^\alpha}(x) = (x+1)^{\mathsf{G}_\alpha(x)}$

Lemma

- $G_{\alpha_1+\cdots+\alpha_n}(x) = G_{\alpha_1}(x) + \cdots + G_{\alpha_n}(x)$
- $G_{\omega^\alpha}(x) = (x+1)^{G_\alpha(x)}$

Definition

we define a function $P_x(\alpha)$ that allows to "subtract" 1 from α

$$P_x(0) = 0 \qquad P_x(\alpha+1) = \alpha \qquad P_x(\lambda) = P_x(\lambda[x]) \qquad (\lambda \text{ limit})$$

Lemma

- $G_{\alpha_1+\cdots+\alpha_n}(x) = G_{\alpha_1}(x) + \cdots + G_{\alpha_n}(x)$
- $G_{\omega^\alpha}(x) = (x+1)^{G_\alpha(x)}$

Definition

we define a function $P_x(\alpha)$ that allows to "subtract" 1 from α

$$P_x(0) = 0 \qquad P_x(\alpha+1) = \alpha \qquad P_x(\lambda) = P_x(\lambda[x]) \qquad (\lambda \text{ limit})$$

Lemma

- $G_{\alpha_1+\cdots+\alpha_n}(x) = G_{\alpha_1}(x) + \cdots + G_{\alpha_n}(x)$
- $G_{\omega^\alpha}(x) = (x+1)^{G_\alpha(x)}$

Definition

we define a function $P_x(\alpha)$ that allows to "subtract" 1 from α

$$P_x(0) = 0 \qquad P_x(\alpha+1) = \alpha \qquad P_x(\lambda) = P_x(\lambda[x]) \qquad (\lambda \text{ limit})$$

Lemma

- $\mathsf{G}_{\alpha_1+\cdots+\alpha_n}(x) = \mathsf{G}_{\alpha_1}(x) + \cdots + \mathsf{G}_{\alpha_n}(x)$
- $\mathsf{G}_{\omega^\alpha}(x) = (x+1)^{\mathsf{G}_\alpha(x)}$

Definition

we define a function $\mathsf{P}_x(\alpha)$ that allows to “subtract” 1 from α

$$\mathsf{P}_x(0) = 0 \qquad \mathsf{P}_x(\alpha+1) = \alpha \qquad \mathsf{P}_x(\lambda) = \mathsf{P}_x(\lambda[x]) \qquad (\lambda \text{ limit})$$

Lemma

- $G_{\alpha_1+\cdots+\alpha_n}(x) = G_{\alpha_1}(x) + \cdots + G_{\alpha_n}(x)$
- $G_{\omega^\alpha}(x) = (x+1)^{G_\alpha(x)}$

Definition

we define a function $P_x(\alpha)$ that allows to "subtract" 1 from α

$$P_x(0) = 0 \qquad P_x(\alpha+1) = \alpha \qquad P_x(\lambda) = P_x(\lambda[x]) \qquad (\lambda \text{ limit})$$

Lemma

for $\alpha \in \mathcal{O}$, $x \in \mathbb{N}$, *we have* $G_{P_x(\alpha)}(x) = P_x(G_\alpha(x)) = G_\alpha(x) - 1$

Proof.

by induction on α, e.g.

$$G_{P_x(\beta+1)}(x) = G_\beta(x) = P_x(G_\beta(x)+1) = P_x(G_{\beta+1}(x))$$

∎

Application ①

Definition (Goodstein Sequence)

1. start with arbitrary number N in hereditary base x representation
2. replace base x by base $x + 1$, then subtract 1
3. continue

Application ①

Definition (Goodstein Sequence)

1. start with arbitrary number N in hereditary base x representation
2. replace base x by base $x + 1$, then subtract 1
3. continue

Theorem (Kirby and Paris)

this process terminates, but this is not provable in Peano Arithmetic

Application ①

Definition (Goodstein Sequence)

1. start with arbitrary number N in hereditary base x representation
2. replace base x by base $x + 1$, then subtract 1
3. continue

Theorem (Kirby and Paris)

this process terminates, but this is not provable in Peano Arithmetic

Definition

the family of Hardy functions $(\mathsf{H}_\alpha)_{\alpha \in \mathcal{O}}$ is defined as follows:

$$\mathsf{H}_0(x) = x \qquad \mathsf{H}_{\alpha+1}(x) = \mathsf{H}_\alpha(x+1) \qquad \mathsf{H}_\lambda(x) = \mathsf{H}_{\lambda[x]}(x) \quad (\lambda \text{ limit})$$

Application ①

Definition (Goodstein Sequence)

1. start with arbitrary number N in hereditary base x representation
2. replace base x by base $x+1$, then subtract 1
3. continue

Theorem (Kirby and Paris)

this process terminates, but this is not provable in Peano Arithmetic

Definition

the family of Hardy functions $(\mathsf{H}_\alpha)_{\alpha\in\mathcal{O}}$ is defined as follows:

$$\mathsf{H}_0(x) = x \qquad \mathsf{H}_{\alpha+1}(x) = \mathsf{H}_\alpha(x+1) \qquad \mathsf{H}_\lambda(x) = \mathsf{H}_{\lambda[x]}(x) \quad (\lambda \text{ limit})$$

Application ①

Definition (Goodstein Sequence)

1. start with arbitrary number N in hereditary base x representation
2. replace base x by base $x+1$, then subtract 1
3. continue

Theorem (Kirby and Paris)

this process terminates, but this is not provable in Peano Arithmetic

Definition

the family of Hardy functions $(\mathsf{H}_\alpha)_{\alpha \in \mathcal{O}}$ is defined as follows:

$$\mathsf{H}_0(x) = x \qquad \mathsf{H}_{\alpha+1}(x) = \mathsf{H}_\alpha(x+1) \qquad \mathsf{H}_\lambda(x) = \mathsf{H}_{\lambda[x]}(x) \quad (\lambda \text{ limit})$$

Application ①

Definition (Goodstein Sequence)

1. start with arbitrary number N in hereditary base x representation
2. replace base x by base $x + 1$, then subtract 1
3. continue

Theorem (Kirby and Paris)

this process terminates, but this is not provable in Peano Arithmetic

Definition

the family of Hardy functions $(\mathsf{H}_\alpha)_{\alpha \in \mathcal{O}}$ is defined as follows:

$$\mathsf{H}_0(x) = x \qquad \mathsf{H}_{\alpha+1}(x) = \mathsf{H}_\alpha(x+1) \qquad \mathsf{H}_\lambda(x) = \mathsf{H}_{\lambda[x]}(x) \quad (\lambda \text{ limit})$$

Application ①

Definition (Goodstein Sequence)

1. start with arbitrary number N in hereditary base x representation
2. replace base x by base $x+1$, then subtract 1
3. continue

Theorem (Kirby and Paris)

this process terminates, but this is not provable in Peano Arithmetic

Definition

the family of Hardy functions $(\mathsf{H}_\alpha)_{\alpha\in\mathcal{O}}$ is defined as follows:

$$\mathsf{H}_0(x) = x \qquad \mathsf{H}_{\alpha+1}(x) = \mathsf{H}_\alpha(x+1) \qquad \mathsf{H}_\lambda(x) = \mathsf{H}_{\lambda[x]}(x) \quad (\lambda \text{ limit})$$

Remark

the Hardy functions are fast-growing

A Closer Look

Definition

A Closer Look

Definition

1. start with arbitrary number N in hereditary base x representation

A Closer Look

Definition

1. start with arbitrary number N in hereditary base x representation
2. replace x by ω to obtain α; note $\mathsf{G}_\alpha(x-1) = N$

A Closer Look

Definition

1. start with arbitrary number N in hereditary base x representation
2. replace x by ω to obtain α; note $\mathsf{G}_\alpha(x-1) = N$
3. change base to $x+1$ obtaining $\mathsf{G}_\alpha(x)$

A Closer Look

Definition

1. start with arbitrary number N in hereditary base x representation
2. replace x by ω to obtain α; note $\mathsf{G}_\alpha(x-1) = N$
3. change base to $x+1$ obtaining $\mathsf{G}_\alpha(x)$
4. subtract 1: $\mathsf{G}_\alpha(x) - 1 = \mathsf{P}_x(\mathsf{G}_\alpha(x)) = \mathsf{G}_{\mathsf{P}_x(\alpha)}(x)$

A Closer Look

Definition

1. start with arbitrary number N in hereditary base x representation
2. replace x by ω to obtain α; note $\mathsf{G}_\alpha(x-1) = N$
3. change base to $x+1$ obtaining $\mathsf{G}_\alpha(x)$
4. subtract 1: $\mathsf{G}_\alpha(x) - 1 = \mathsf{P}_x(\mathsf{G}_\alpha(x)) = \mathsf{G}_{\mathsf{P}_x(\alpha)}(x)$
5. one more iteration yields $\mathsf{G}_{\mathsf{P}_{x+1} \circ \mathsf{P}_x(\alpha)}(x+1)$

A Closer Look

Definition

1. start with arbitrary number N in hereditary base x representation
2. replace x by ω to obtain α; note $\mathsf{G}_\alpha(x-1) = N$
3. change base to $x+1$ obtaining $\mathsf{G}_\alpha(x)$
4. subtract 1: $\mathsf{G}_\alpha(x) - 1 = \mathsf{P}_x(\mathsf{G}_\alpha(x)) = \mathsf{G}_{\mathsf{P}_x(\alpha)}(x)$
5. one more iteration yields $\mathsf{G}_{\mathsf{P}_{x+1} \circ \mathsf{P}_x(\alpha)}(x+1)$
6. one more iteration yields $\mathsf{G}_{\mathsf{P}_{x+2} \circ \mathsf{P}_{x+1} \circ \mathsf{P}_x(\alpha))}(x+2)$

A Closer Look

Definition

1. start with arbitrary number N in hereditary base x representation
2. replace x by ω to obtain α; note $\mathsf{G}_\alpha(x-1) = N$
3. change base to $x+1$ obtaining $\mathsf{G}_\alpha(x)$
4. subtract 1: $\mathsf{G}_\alpha(x) - 1 = \mathsf{P}_x(\mathsf{G}_\alpha(x)) = \mathsf{G}_{\mathsf{P}_x(\alpha)}(x)$
5. one more iteration yields $\mathsf{G}_{\mathsf{P}_{x+1} \circ \mathsf{P}_x(\alpha)}(x+1)$
6. one more iteration yields $\mathsf{G}_{\mathsf{P}_{x+2} \circ \mathsf{P}_{x+1} \circ \mathsf{P}_x(\alpha))}(x+2)$
7. ...

A Closer Look

Definition

1. start with arbitrary number N in hereditary base x representation
2. replace x by ω to obtain α; note $\mathsf{G}_\alpha(x-1) = N$
3. change base to $x+1$ obtaining $\mathsf{G}_\alpha(x)$
4. subtract 1: $\mathsf{G}_\alpha(x) - 1 = \mathsf{P}_x(\mathsf{G}_\alpha(x)) = \mathsf{G}_{\mathsf{P}_x(\alpha)}(x)$
5. one more iteration yields $\mathsf{G}_{\mathsf{P}_{x+1} \circ \mathsf{P}_x(\alpha)}(x+1)$
6. one more iteration yields $\mathsf{G}_{\mathsf{P}_{x+2} \circ \mathsf{P}_{x+1} \circ \mathsf{P}_x(\alpha))}(x+2)$
7. ...

Theorem (Goodstein)

the process terminates

Proof (due to Cichon).

Proof (due to Cichon).

- $\mathsf{G}_\alpha(x) = 0$ iff $\alpha = 0$

Proof (due to Cichon).

- $G_\alpha(x) = 0$ iff $\alpha = 0$
- process terminates iff $\forall\ \alpha \neq 0 \in \mathcal{O}$, $\exists\ y > x$ such that

$$P_y \circ P_{y-1} \circ \cdots \circ P_{x+2} \circ P_x(\alpha) = 0$$

Proof (due to Cichon).

- $\mathsf{G}_\alpha(x) = 0$ iff $\alpha = 0$
- process terminates iff $\forall\ \alpha \neq 0 \in \mathcal{O}$, $\exists\ y > x$ such that

$$\mathsf{P}_y \circ \mathsf{P}_{y-1} \circ \cdots \circ \mathsf{P}_{x+2} \circ \mathsf{P}_x(\alpha) = 0$$

- let $f(\alpha, x)$ be defined as follows

$$f(\alpha, x) = \text{least } y \;\; (\mathsf{P}_{y-1} \circ \mathsf{P}_{y-1} \circ \cdots \circ \mathsf{P}_{x+1} \circ \mathsf{P}_x(\alpha) = 0)$$

Proof (due to Cichon).

- $\mathsf{G}_\alpha(x) = 0$ iff $\alpha = 0$
- process terminates iff $\forall\ \alpha \neq 0 \in \mathcal{O}$, $\exists\ y > x$ such that
$$\mathsf{P}_y \circ \mathsf{P}_{y-1} \circ \cdots \circ \mathsf{P}_{x+2} \circ \mathsf{P}_x(\alpha) = 0$$
- let $f(\alpha, x)$ be defined as follows
$$f(\alpha, x) = \text{least } y\ \ (\mathsf{P}_{y-1} \circ \mathsf{P}_{y-1} \circ \cdots \circ \mathsf{P}_{x+1} \circ \mathsf{P}_x(\alpha) = 0)$$
- $\forall \alpha \neq 0$, $\forall x$: $\mathsf{H}_\alpha(x) = f(\alpha, x)$

Proof (due to Cichon).

- $\mathsf{G}_\alpha(x) = 0$ iff $\alpha = 0$
- process terminates iff $\forall\ \alpha \neq 0 \in \mathcal{O}$, $\exists\ y > x$ such that
$$\mathsf{P}_y \circ \mathsf{P}_{y-1} \circ \cdots \circ \mathsf{P}_{x+2} \circ \mathsf{P}_x(\alpha) = 0$$
- let $f(\alpha, x)$ be defined as follows
$$f(\alpha, x) = \text{least } y\ \ (\mathsf{P}_{y-1} \circ \mathsf{P}_{y-1} \circ \cdots \circ \mathsf{P}_{x+1} \circ \mathsf{P}_x(\alpha) = 0)$$
- $\forall \alpha \neq 0$, $\forall x$: $\mathsf{H}_\alpha(x) = f(\alpha, x)$

Proof (due to Cichon).

- $G_\alpha(x) = 0$ iff $\alpha = 0$
- process terminates iff $\forall\ \alpha \neq 0 \in \mathcal{O}$, $\exists\ y > x$ such that

$$P_y \circ P_{y-1} \circ \cdots \circ P_{x+2} \circ P_x(\alpha) = 0$$

- let $f(\alpha, x)$ be defined as follows

$$f(\alpha, x) = \text{least } y \ \ (P_{y-1} \circ P_{y-1} \circ \cdots \circ P_{x+1} \circ P_x(\alpha) = 0)$$

- $\forall \alpha \neq 0$, $\forall x$: $H_\alpha(x) = f(\alpha, x)$

Corollary (Kirby and Paris)

termination is not provable in Peano Arithmetic

Proof.

$\forall$ functions f, provable recursive in Peano Arithmetic
$\exists\ \alpha \in \mathcal{O}$ such that f is majorised by H_α

More on the Slow-Growing Hierarchy

Lemma

the family $(\mathsf{G}_\alpha)_{\alpha \in \mathcal{O}}$ *forms a hierarchy: for* $\alpha > \beta$*:*

$$\exists\ c \text{ such that } \forall\ x \geqslant c\text{: } \mathsf{G}_\alpha(x) > \mathsf{G}_\beta(x)$$

More on the Slow-Growing Hierarchy

Lemma

the family $(\mathsf{G}_\alpha)_{\alpha \in \mathcal{O}}$ *forms a hierarchy: for* $\alpha > \beta$*:*

$$\exists\, c \text{ such that } \forall\, x \geqslant c\colon \mathsf{G}_\alpha(x) > \mathsf{G}_\beta(x)$$

Example

$$\forall x \geqslant 1 \qquad \mathsf{G}_{\omega^\omega}(x) = (x+1)^{x+1} > (x+1) = \mathsf{G}_\omega(x)$$

$$\forall x \geqslant 1 \qquad \mathsf{G}_\omega(x) = x + 1 \not> y = \mathsf{G}_y(x) \qquad \text{whenever } y > x$$

More on the Slow-Growing Hierarchy

Lemma

the family $(\mathsf{G}_\alpha)_{\alpha \in \mathcal{O}}$ *forms a hierarchy: for* $\alpha > \beta$*:*

$$\exists\, c \text{ such that } \forall\, x \geqslant c\colon \mathsf{G}_\alpha(x) > \mathsf{G}_\beta(x)$$

Example

$$\forall x \geqslant 1 \qquad \mathsf{G}_{\omega^\omega}(x) = (x+1)^{x+1} > (x+1) = \mathsf{G}_\omega(x)$$

$$\forall x \geqslant 1 \qquad \mathsf{G}_\omega(x) = x + 1 \not> y = \mathsf{G}_y(x) \qquad \text{whenever } y > x$$

Definition

let $>_{(x)}$ denote the transitive closure of the fundamental sequence $\cdot[\cdot]$

$$\alpha >_{(x)} \beta \qquad \text{if } \beta = \alpha[x] \text{ or } \alpha[x] \geqslant_{(x)} \beta$$

More on the Slow-Growing Hierarchy

Lemma

the family $(\mathsf{G}_\alpha)_{\alpha \in \mathcal{O}}$ *forms a hierarchy: for* $\alpha > \beta$*:*

$$\exists\ c \text{ such that } \forall\ x \geqslant c\colon\ \mathsf{G}_\alpha(x) > \mathsf{G}_\beta(x)$$

Example

$$\forall x \geqslant 1 \qquad \mathsf{G}_{\omega^\omega}(x) = (x+1)^{x+1} > (x+1) = \mathsf{G}_\omega(x)$$

$$\forall x \geqslant 1 \qquad \mathsf{G}_\omega(x) = x + 1 \not> y = \mathsf{G}_y(x) \qquad \text{whenever } y > x$$

Definition

let $>_{(x)}$ denote the transitive closure of the fundamental sequence $\cdot[\cdot]$

$$\alpha >_{(x)} \beta \qquad \text{if } \beta = \alpha[x] \text{ or } \alpha[x] \geqslant_{(x)} \beta$$

where we set $0[x] = 0$, $(\alpha + 1)[x] = \alpha$

Lemma

1. G_α *is increasing (strictly if* α *is infinite)*
2. *if* $\alpha >_{(n)} \beta$, *then* $\mathsf{G}_\alpha(x) > \mathsf{G}_\beta(x)$ *for all* $x \geqslant n$

Lemma

1. G_α *is increasing (strictly if* α *is infinite)*
2. *if* $\alpha >_{(n)} \beta$, *then* $\mathsf{G}_\alpha(x) > \mathsf{G}_\beta(x)$ *for all* $x \geqslant n$

Definition

we define the norm $\mathsf{N}(\alpha)$ of α as follows

$$\mathsf{N}(0) = 0$$
$$\mathsf{N}(\omega^{\alpha_1} + \cdots + \omega^{\alpha_n}) = \mathsf{N}(\alpha_1) + \cdots + \mathsf{N}(\alpha_n) + n$$

Lemma

1. G_α *is increasing (strictly if* α *is infinite)*
2. *if* $\alpha >_{(n)} \beta$, *then* $\mathsf{G}_\alpha(x) > \mathsf{G}_\beta(x)$ *for all* $x \geqslant n$

Definition

we define the norm $\mathsf{N}(\alpha)$ of α as follows

$$\mathsf{N}(0) = 0$$
$$\mathsf{N}(\omega^{\alpha_1} + \cdots + \omega^{\alpha_n}) = \mathsf{N}(\alpha_1) + \cdots + \mathsf{N}(\alpha_n) + n$$

Lemma

if $\alpha > \beta$ *and* $n \geqslant \mathsf{N}(\beta)$, *then*

1. $\alpha >_{(n)} \beta$
2. $\forall x \geqslant n\colon \mathsf{G}_\alpha(x) > \mathsf{G}_\beta(x)$

Application: Transfinite Knuth-Bendix Orders

Definition (TKBOs)

...

Application: Transfinite Knuth-Bendix Orders

Definition (TKBOs)

...

Theorem (Ludwig and Waldmann)

TKBOs form a simplification order

Application: Transfinite Knuth-Bendix Orders

Definition (TKBOs)

...

Theorem (Ludwig and Waldmann)

TKBOs form a simplification order

Theorem (Kovacs, M., and Voronkov)

let signature F *be finite; any TKBO is equivalent to a TKBO restricting ordinal weights to ordinals* $< \omega^{\omega^{\omega}}$

Application: Transfinite Knuth-Bendix Orders

Definition (TKBOs)

. . .

Theorem (Ludwig and Waldmann)

TKBOs form a simplification order

Theorem (Kovacs, M., and Voronkov)

let signature F *be finite; any TKBO is equivalent to a TKBO restricting ordinal weights to ordinals* $< \omega^{\omega^{\omega}}$

Theorem (Winkler, Zankl, and Middeldorp)

if a finite TRS $\mathcal{R}$ *is compatible with a TKBO, then* $\mathcal{R}$ *is compatible with a finite TKBO*

Example

let $\mathsf{F} = \{\mathsf{f}, \mathsf{g}, \mathsf{h}, \mathsf{k}\}$ and consider the following rules:

$$\begin{aligned} \mathsf{f}(x) &\to \mathsf{g}(x) \\ \mathsf{h}(x) &\to \mathsf{f}(\mathsf{f}(x)) \\ \mathsf{k}(x,y) &\to \mathsf{h}(\mathsf{f}(x),\mathsf{f}(y)) \end{aligned}$$

$$\mathsf{w}(\mathsf{f}) = 5 \quad \mathsf{w}(\mathsf{g}) = 0 \quad \mathsf{w}(\mathsf{h}) = \omega \quad \mathsf{w}(\mathsf{k}) = \omega \cdot 2$$

subterm coefficents are $= 1$

Example

let $F = \{f, g, h, k\}$ and consider the following rules:

$$\begin{aligned} f(x) &\to g(x) \\ h(x,x) &\to f(f(x)) \\ k(x,y) &\to h(f(x), f(y)) \end{aligned}$$

$$\begin{aligned} w(f) &= 5 \\ w(g) &= 0 \\ w(h) &= \omega \\ w(k) &= \omega \cdot 2 \end{aligned}$$

subterm coefficents are $= 1$

Example

let $F = \{f, g, h, k\}$ and consider the following rules:

$$f(x) \succ_{tkbo} g(x) \qquad w(f) = 5$$
$$h(x, x) \succ_{tkbo} f(f(x)) \qquad w(g) = 0$$
$$k(x, y) \succ_{tkbo} h(f(x), f(y)) \qquad w(h) = \omega$$
$$w(k) = \omega \cdot 2$$

subterm coefficents are $= 1$

Example

let $F = \{f, g, h, k\}$ and consider the following rules:

$$\begin{aligned} f(x) &\succ_{tkbo} g(x) \\ h(x,x) &\succ_{tkbo} f(f(x)) \\ k(x,y) &\succ_{tkbo} h(f(x), f(y)) \end{aligned}$$

$$\begin{aligned} w(f) &= 5 \\ w(g) &= 0 \\ w(h) &= 11 \\ w(k) &= 11 \cdot 2 = 22 \end{aligned}$$

subterm coefficents are $= 1$

Example

let $F = \{f, g, h, k\}$ and consider the following rules:

$$f(x) \succ_{tkbo} g(x) \qquad w(f) = 5$$
$$h(x,x) \succ_{tkbo} f(f(x)) \qquad w(g) = 0$$
$$k(x,y) \succ_{tkbo} h(f(x), f(y)) \qquad w(h) = 11$$
$$w(k) = 11 \cdot 2 = 22$$

subterm coefficents are $= 1$

Proof of Theorem on Finite TKBOs.

- it suffices to show that $\exists\, x\ \forall\ l \to r \in \mathcal{R}$: $\mathsf{weight}(l) \geqslant_{(x)} \mathsf{weight}(r)$ as then $\forall\ l \to r \in \mathcal{R}$: $\mathsf{G}_{\mathsf{weight}(l)}(k) \geqslant \mathsf{G}_{\mathsf{weight}(r)}(k)$

Example

let $\mathsf{F} = \{\mathsf{f}, \mathsf{g}, \mathsf{h}, \mathsf{k}\}$ and consider the following rules:

$$\begin{aligned} \mathsf{f}(x) &\succ_{\mathsf{tkbo}} \mathsf{g}(x) & \mathsf{w}(\mathsf{f}) &= 5 \\ \mathsf{h}(x,x) &\succ_{\mathsf{tkbo}} \mathsf{f}(\mathsf{f}(x)) & \mathsf{w}(\mathsf{g}) &= 0 \\ \mathsf{k}(x,y) &\succ_{\mathsf{tkbo}} \mathsf{h}(\mathsf{f}(x),\mathsf{f}(y)) & \mathsf{w}(\mathsf{h}) &= 11 \\ & & \mathsf{w}(\mathsf{k}) &= 11 \cdot 2 = 22 \end{aligned}$$

subterm coefficents are $= 1$

Proof of Theorem on Finite TKBOs.

- it suffices to show that $\exists\ x\ \forall\ l \to r \in \mathcal{R}$: $\mathsf{weight}(l) \geqslant_{(x)} \mathsf{weight}(r)$ as then $\forall\ l \to r \in \mathcal{R}$: $\mathsf{G}_{\mathsf{weight}(l)}(k) \geqslant \mathsf{G}_{\mathsf{weight}(r)}(k)$
- set $k = \max\{\mathsf{N}(\mathsf{weight}(r)) \mid l \to r \in \mathcal{R}\}$

Example

let $\mathsf{F} = \{\mathsf{f}, \mathsf{g}, \mathsf{h}, \mathsf{k}\}$ and consider the following rules:

$$\begin{aligned} \mathsf{f}(x) &\succ_{\mathsf{tkbo}} \mathsf{g}(x) & \mathsf{w}(\mathsf{f}) &= 5 \\ \mathsf{h}(x, x) &\succ_{\mathsf{tkbo}} \mathsf{f}(\mathsf{f}(x)) & \mathsf{w}(\mathsf{g}) &= 0 \\ \mathsf{k}(x, y) &\succ_{\mathsf{tkbo}} \mathsf{h}(\mathsf{f}(x), \mathsf{f}(y)) & \mathsf{w}(\mathsf{h}) &= 11 \\ & & \mathsf{w}(\mathsf{k}) &= 11 \cdot 2 = 22 \end{aligned}$$

subterm coefficents are $= 1$

Proof of Theorem on Finite TKBOs.

- it suffices to show that $\exists\, x\ \forall\, l \to r \in \mathcal{R}$: $\mathsf{weight}(l) \geqslant_{(x)} \mathsf{weight}(r)$ as then $\forall\, l \to r \in \mathcal{R}$: $\mathsf{G}_{\mathsf{weight}(l)}(k) \geqslant \mathsf{G}_{\mathsf{weight}(r)}(k)$
- set $k = \max\{\mathsf{N}(\mathsf{weight}(r)) \mid l \to r \in \mathcal{R}\}$
- then $\mathsf{weight}(l) \geqslant \mathsf{weight}(r)$ implies $\mathsf{weight}(l) \geqslant_{(k)} \mathsf{weight}(r)$

Example

let $\mathsf{F} = \{\mathsf{f}, \mathsf{g}, \mathsf{h}, \mathsf{k}\}$ and consider the following rules:

$$\mathsf{f}(x) \succ_{\mathsf{tkbo}} \mathsf{g}(x) \qquad \mathsf{w}(\mathsf{f}) = 5$$

$$\mathsf{h}(x,x) \succ_{\mathsf{tkbo}} \mathsf{f}(\mathsf{f}(x)) \qquad \mathsf{w}(\mathsf{g}) = 0$$

$$\mathsf{k}(x,y) \succ_{\mathsf{tkbo}} \mathsf{h}(\mathsf{f}(x),\mathsf{f}(y)) \qquad \mathsf{w}(\mathsf{h}) = 11$$

$$\mathsf{w}(\mathsf{k}) = 11 \cdot 2 = 22$$

subterm coefficents are $= 1$

Proof of Theorem on Finite TKBOs.

- it suffices to show that $\exists\ x\ \forall\ l \to r \in \mathcal{R}$: $\mathsf{weight}(l) \geqslant_{(x)} \mathsf{weight}(r)$ as then $\forall\ l \to r \in \mathcal{R}$: $\mathsf{G}_{\mathsf{weight}(l)}(k) \geqslant \mathsf{G}_{\mathsf{weight}(r)}(k)$
- set $k = \max\{\mathsf{N}(\mathsf{weight}(r)) \mid l \to r \in \mathcal{R}\}$
- then $\mathsf{weight}(l) \geqslant \mathsf{weight}(r)$ implies $\mathsf{weight}(l) \geqslant_{(k)} \mathsf{weight}(r)$ ∎

Definition

we define the derivational complexity with respect to $\mathcal{R}$:

$$\mathrm{dheight}(t) = \max\{n \mid \exists u\ t \to^n u\}$$
$$\mathrm{dc}(n) = \max\{\mathrm{dheight}(t) \mid |t| \leqslant n\}$$

Definition

we define the derivational complexity with respect to $\mathcal{R}$:

$$\mathsf{dheight}(t) = \max\{n \mid \exists u\ t \to^n u\}$$

$$\mathsf{dc}(n) = \max\{\mathsf{dheight}(t) \mid |t| \leqslant n\}$$

Theorem (Lepper)

for any TRS $\mathcal{R}$, compatible with a KBO, the derivational complexity is bounded by a 2-recursive function, that is, $\mathsf{dc}(n) \in \mathsf{Ack}(O(n), 0)$

Definition

we define the derivational complexity with respect to $\mathcal{R}$:

$$\mathrm{dheight}(t) = \max\{n \mid \exists u\ t \to^n u\}$$

$$\mathrm{dc}(n) = \max\{\mathrm{dheight}(t) \mid |t| \leqslant n\}$$

Theorem

for any TRS $\mathcal{R}$, compatible with a TKBO, the derivational complexity is bounded by a 2-recursive function, that is, $\mathrm{dc}(n) \in \mathrm{Ack}(O(n), 0)$

Definition

we define the derivational complexity with respect to $\mathcal{R}$:

$$\mathsf{dheight}(t) = \max\{n \mid \exists u \; t \to^n u\}$$

$$\mathsf{dc}(n) = \max\{\mathsf{dheight}(t) \mid |t| \leqslant n\}$$

Theorem

for any TRS $\mathcal{R}$, compatible with a TKBO, the derivational complexity is bounded by a 2-recursive function, that is, $\mathsf{dc}(n) \in \mathsf{Ack}(O(n), 0)$

Future Work

- generalised KBOs compute weights based on weakly monotone simple algebras $\mathcal{A}$
- clarify restrictions on $\mathcal{A}$ so that ordinal weights again collapse to numbers

Thank You for Your Attention!