A New Order-theoretic Characterisation of the Polytime Computable Functions

Martin Avanzini[1] *Naohi Eguchi*[2] *Georg Moser*[1]

[1]Institute of Computer Science
University of Innsbruck, Austria

[2]Mathematical Institute
Tohoku University, Japan

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Implicit Computational Complexity (ICC)

characterise complexity classes

- − external restriction of resources
- \+ restrict program structure

Motivation

- broaden understanding
- apply in programming language theory

Cobhams Definition of FP

Alan Cobham

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Bounded Recursion on Notation

$$f(\varepsilon, \vec{y}) = g(\vec{y})$$
$$f(zi, \vec{y}) = h_i(z, \vec{y}, f(z, \vec{y})) \quad \text{for } i = 0, 1$$

- provided $f(z, \vec{y}) \leqslant p(z, \vec{y})$ for previously defined p and all $z, \vec{y}$
- requires smash function $x\#y := 2^{|x|\cdot|y|}$

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Predicative Recursion on Notation

$$f(\varepsilon, \vec{x}; \vec{y}) = g(\vec{x}; \vec{y})$$
$$f(zi, \vec{x}; \vec{y}) = h_i(z, \vec{x}; \vec{y}, f(z, \vec{x}; \vec{y})) \quad \text{for } i = 0, 1$$

- uses separation of arguments

$$f(\underbrace{n_1, \ldots, n_l}_{\textit{normal}}; \underbrace{n_{l+1}, \ldots, n_{l+k}}_{\textit{safe}})$$

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- *"complexity depends only on normal arguments"*

Runtime Complexity Analysis of TRSs

Computation

- conceive TRS $\mathcal{R}$ as first order program over signature $\mathcal{D} \uplus \mathcal{C}$
- values $\mathcal{V}\mathrm{al}$ are terms built from constructors $\mathcal{C}$
- for each $\mathsf{f} \in \mathcal{D}$, $\mathcal{R}$ computes functions $f : \mathcal{V}\mathrm{al}^n \to \mathcal{V}\mathrm{al}$ such that

$$f(v_1, \ldots, v_n) := u \qquad \Longleftrightarrow \qquad \mathsf{f}(v_1, \ldots, v_n) \to^{!}_{\mathcal{R}} u$$

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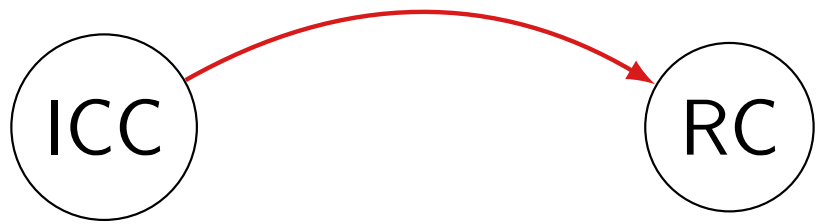
Runtime Complexity

- runtime complexity function

$$\mathrm{rc}_{\mathcal{R}}(n) := \max\{\mathrm{dh}(t, \to_{\mathcal{R}}) \mid t \text{ is basic term of size up to } n\}$$

$$\text{where } \mathrm{dh}(t, \to) := \max\{\ell \mid t \to t_1 \to t_2 \cdots \to t_\ell\}$$

- term $\mathsf{f}(v_1, \ldots, v_n)$ is basic if $f \in \mathcal{D}$ and all $v_i \in \mathcal{V}\mathrm{al}$


ICC
RC

Polynomial Path Order $>_{\mathsf{pop}*}$

M. Avanzini and G. Moser
Complexity Analysis by Rewriting.
Proc. of 9th International Symposium on Functional and Logic Programming, LNCS Vol. 4989, pages 130–146, 2008

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Example

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\begin{array}{ll}
1: & +(0,y) \to y \\
2: & +(\mathsf{s}(x),y) \to \mathsf{s}(+(x,y)) \\
5: & \mathsf{sq}(x\ ) \to \times(x,x\ )
\end{array}
\qquad
\begin{array}{ll}
3: & \times(0,y\ ) \to 0 \\
4: & \times(\mathsf{s}(x),y\ ) \to +(y,\times(x,y\ ))
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- TRS is compatible with $>_{\mathsf{mpo}}$ using precedence $\mathsf{sq} \succ \times \succ + \succ \mathsf{s}$

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Runtime Complexity Analysis

Let $\mathcal{R}$ denote a constructor TRS. There exists $k \in \mathbb{N}$ such that

$$\mathcal{R} \subseteq >_{\mathsf{pop}*} \quad \Longrightarrow \quad \mathsf{rc}^{\mathsf{i}}_{\mathcal{R}} \in \mathsf{O}(n^k)$$

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Implicit Computational Complexity

1. **Soundness**
   Let $\mathcal{R}$ be a constructor TRS that computes a function f.
   If $\mathcal{R} \subseteq >_{\mathsf{pop}*}$ then f is polytime computable.

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   Let f be a polytime computable function.
   There exists a constructor TRS $\mathcal{R}_f$ computing f with $\mathcal{R}_f \subseteq >_{\mathsf{pop}*}$.

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   There exists a constructor TRS $\mathcal{R}_f$ computing f with $\mathcal{R}_f \subseteq >_{\mathsf{pop}*}$.

Drawback

order $>_{\mathsf{pop}*}$ cannot precisely estimate degree of polynomial

Small Polynomial Path Order $>_{\mathsf{spop}*}$

- restriction of polynomial path order $>_{\mathsf{spop}*} \subseteq >_{\mathsf{pop}*}$
- estimate degree of complexity certificate in *"depth of recursion"*

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  1. restrict composition to safe arguments — weak safe composition

$$f(\vec{x};\vec{y}) = g(\vec{x};h_1(\vec{x};\vec{y}),\ldots,h_k(\vec{x};\vec{y}))$$

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  1. restrict composition to safe arguments — weak safe composition

$$f(\vec{x};\vec{y}) = g(\vec{x};h_1(\vec{x};\vec{y}),\ldots,h_k(\vec{x};\vec{y}))$$

  2. uses product-status

$$\begin{aligned}
\mathsf{f}_k(\mathsf{s}(x_1),x_2,x_3,\ldots,x_k;) &\to \mathsf{f}_k(x_1,x_2,x_3,\ldots,x_k;)\\
\mathsf{f}_k(0,\mathsf{s}(x_2),x_3,\ldots,x_k;) &\to \mathsf{f}_k(x_2,x_2,x_3,\ldots,x_k;)\\
&\vdots\\
\mathsf{f}_k(0,\ldots,0,\mathsf{s}(x_k);) &\to \mathsf{f}_k(x_k,\ldots,x_k,x_k;)
\end{aligned}$$

  - compatible with $>_{\mathsf{pop}*}$, admits runtime complexity in $\Theta(n^k)$

Small Polynomial Path Order $>_{\mathsf{spop}*}$

$f(s_1, \ldots, s_k; s_{k+1}, \ldots, s_{k+l}) >_{\mathsf{spop}*} t$ if

1. $s_i \geqslant_{\mathsf{spop}*} t$ for some argument s_i

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2. $t = g(t_1, \ldots, t_m; t_{m+1}, \ldots, t_{m+n})$ where $f \in \mathcal{D}$ and $f \succ g$
   - $f(s_1, \ldots, s_k; s_{k+1}, \ldots, s_{k+l}) \rhd_n t_j$ for all normal arguments t_j
   - $f(s_1, \ldots, s_k; s_{k+1}, \ldots, s_{k+l}) >_{\text{spop}*} t_j$ for all safe arguments t_j

$$f(\vec{x}; \vec{y}) = g(\vec{x}; h_1(\vec{x}; \vec{y}), \ldots, h_k(\vec{x}; \vec{y}))$$

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$$f(s_1, \ldots, s_k; \ldots) \rhd_{\mathsf{n}} t \quad :\Leftrightarrow \quad s_i \trianglerighteq \cdot \sim t$$

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3. $t = g(t_1, \ldots, t_k; t_{k+1}, \ldots, t_{k+l})$ where $f \in \mathcal{D}$ and $f \sim g$
   - $\langle s_1, \ldots, s_k \rangle >_{\text{spop}*} \langle t_{\pi(1)}, \ldots, t_{\pi(k)} \rangle$
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$$\mathsf{f}(\mathsf{s}(x);) \not\geqslant_{\mathsf{spop}*} \mathsf{c}(;\mathsf{f}(x;),\mathsf{f}(x;))$$

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3. $t = g(t_1, \ldots, t_k; t_{k+1}, \ldots, t_{k+l})$ where $f \in \mathcal{D}_{\text{rec}}$ and $f \sim g$
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assume designated set of recursive symbols $\mathcal{D}_{\text{rec}} \subseteq \mathcal{D}$

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Small Polynomial Path Order $>_{\mathsf{spop}*}$

Main Result

- the depth of recursion $\mathsf{rd}(f)$ is defined as follows:

$$\mathsf{rd}(f) := \begin{cases} 1 + \max\{\mathsf{rd}(g) \mid f \succ g\} & \text{if } f \in \mathcal{D}_{\mathsf{rec}} \\ \max\{\mathsf{rd}(g) \mid f \succ g\} & \text{otherwise} \end{cases}$$

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- constructor TRS $\mathcal{R}$ is predicative recursive of degree d if
  - $\mathcal{R}$ is compatible with $>_{\mathsf{spop}*}$
  - depth of recursion of symbol in $\mathcal{R}$ is at most d

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Theorem

1. if $\mathcal{R}$ is a predicative recursive TRS of degree d, then $\mathsf{rc}^{\mathsf{i}}_{\mathcal{R}} \in \mathsf{O}(n^d)$

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  - $\mathcal{R}$ is compatible with $>_{\mathsf{spop}*}$
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Theorem

1. if $\mathcal{R}$ is a predicative recursive TRS of degree d, then $\mathsf{rc}^{\mathsf{i}}_{\mathcal{R}} \in \mathsf{O}(n^d)$
2. for all d there exists a predicative recursive TRS of degree d such that $\mathsf{rc}^{\mathsf{i}}_{\mathcal{R}} \in \Theta(n^d)$

Small Polynomial Path Order $>_{\mathsf{spop}*}$

Example

$$
\begin{aligned}
1 &: \quad +(0; y) \to y & 3 &: \quad \times(0, y;) \to 0 \\
2 &: +(\mathsf{s}(x); y) \to \mathsf{s}(+(x; y)) & 4 &: \times(\mathsf{s}(x), y;) \to +(y; \times(x, y;)) \\
5 &: \quad \mathsf{sq}(x;) \to \times(x, x;)
\end{aligned}
$$

- TRS is compatible with $>_{\mathsf{spop}*}$ using precedence $\mathsf{sq} \succ \times \succ + \succ \mathsf{s}$
- only $\times$ and $+$ are recursive
- innermost runtime complexity is quadratic

Experimental Results

bound	MPO	POP*	sPOP*
yes	$76_{\backslash 0.09}$	$43_{\backslash 0.05}$	$39_{\backslash 0.07}$

Table: 757 constructor TRSs from TPDB 8.0

Experimental Results

bound	MPO	POP*	sPOP*
$O(1)$			$9_{\backslash 0.06}$
$O(n^1)$			$32_{\backslash 0.07}$
$O(n^2)$			$38_{\backslash 0.09}$
$O(n^3)$			$39_{\backslash 0.20}$
$O(n^k)$		$43_{\backslash 0.05}$	$39_{\backslash 0.20}$
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$O(n^k)$		$43_{\backslash 0.05}$	$39_{\backslash 0.20}$	$56_{\backslash 0.05}$	$54_{\backslash 0.22}$
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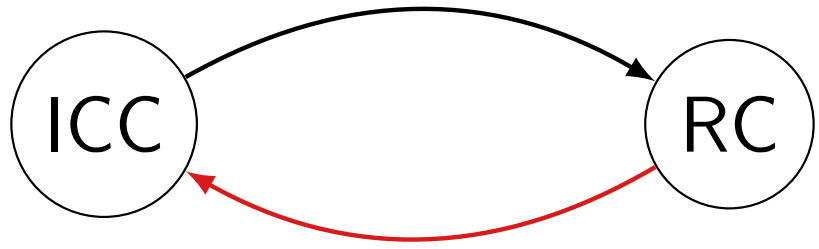
- Parameter Substitution

$$f(\varepsilon, \vec{x}; \vec{y}) = g(\vec{x}; \vec{y})$$

$$f(zi, \vec{x}; \vec{y}) = h_i(z, \vec{x}; \vec{y}, f(z, \vec{x}; p_1(z, \vec{x}; \vec{y}), \ldots, p_k(z, \vec{x}; \vec{y}))) \quad \text{for } i = 0, 1$$


ICC
RC

Applications in ICC

Main Results

Theorem

$>_{\mathsf{spop}*}$ is sound and complete for polytime functions

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- innermost runtime complexity is an invariant cost model

Applications in ICC

The Class $\mathcal{B}_{\mathsf{wsc}}$

class $\mathcal{B}_{\mathsf{wsc}}$ is smallest class of functions that

1. contains a small set of initial functions successors, conditional, ...
2. is closed under weak predicative composition

$$f(\vec{x};\vec{y}) = g(\vec{x}; h_1(\vec{x};\vec{y}), \ldots, h_k(\vec{x};\vec{y}))$$

3. is closed under safe recursion on notation

$$f(\varepsilon, \vec{x};\vec{y}) = g(\vec{x};\vec{y})$$
$$f(zi, \vec{x};\vec{y}) = h_i(z, \vec{x};\vec{y}, f(z, \vec{x};\vec{y})) \quad \text{for } i = 0, 1$$

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The Class $\mathcal{B}_{\mathsf{wsc}}$

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1. contains a small set of initial functions successors, conditional, ...
2. is closed under weak predicative composition

$$f(\vec{x};\vec{y}) = g(\vec{x}; h_1(\vec{x};\vec{y}), \ldots, h_k(\vec{x};\vec{y}))$$

3. is closed under safe recursion on notation

$$f(\varepsilon, \vec{x};\vec{y}) = g(\vec{x};\vec{y})$$
$$f(zi, \vec{x};\vec{y}) = h_i(z, \vec{x};\vec{y}, f(z, \vec{x};\vec{y})) \quad \text{for } i = 0, 1$$

Theorem

$\mathcal{B}_{\mathsf{wsc}}$ equals the polytime computable functions

Applications in ICC

Main Results

Theorem

$>_{\mathsf{spop}*}$ is sound and complete for polytime functions

- innermost runtime complexity is an invariant cost model
- every $f \in \mathcal{B}_{\mathsf{wsc}}$ can be formulated as predicative recursive TRS

Applications in ICC

More Precise...

Theorem *RM* $\Rightarrow$ *TRS*

Suppose R is register machine operating in time $O(n^d)$.
If f is computed by R, then f is computable by some predicative recursive TRS $\mathcal{R}_f$ of degree d.

Applications in ICC

More Precise...

Theorem *RM ⇒ TRS*

Suppose R is register machine operating in time $O(n^d)$.
If f is computed by R, then f is computable by some predicative recursive TRS $\mathcal{R}_f$ of degree d.

Theorem *TRS ⇒ RM*

Suppose predicative TRS $\mathcal{R}$ of degree d computes functions on strings.
If f is computed by $\mathcal{R}$, then f is computable on a register machine R_f operating in time $O(n^d)$.

Future Work

- can Theorem TRS $\Rightarrow$ RM be generalised?

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- can we replace weak safe composition by $f(\vec{x};\vec{y}) = g(h(\vec{x};);i(\vec{x};\vec{y}))$?

$$g \in \mathrm{O}(n^{d_g}), h \in \mathrm{O}(n^{d_h}), i \in \mathrm{O}(n^{d_i}) \quad \Rightarrow \quad f \in \mathrm{O}(n^{d})$$

where $d = \max\{\min\{1, d_g\} \cdot d_h, d_i\}$

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Conjecture

for any function f on strings, the following are equivalent

1. f is defined in $\mathcal{B}_{\mathrm{wsc}}$ via at most d-fold nested application of SRN
2. f is computed by predicative recursive TRS $\mathcal{R}_f$ of degree d
3. f is computed by register machine R_f operating in time $\mathrm{O}(n^d)$