# Signature Extensions for Termination

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# Overview

- Introduction
- Termination
- Relative Termination
- Innermost Termination
- Dependency Pairs
- Summary

# Motivation

- consider TRS

$$\mathsf{f}(\mathsf{s}(\mathsf{s}(x))) \to \mathsf{f}(\mathsf{s}(\mathsf{f}(x)))$$
$$+(\mathsf{0}, y) \to y$$
$$+(\mathsf{s}(x), y) \to \mathsf{s}(+(x, y))$$

- after applying LPO with precedence $+ > \mathsf{s} \equiv \mathsf{f}$, only

$$\mathsf{f}(\mathsf{s}(\mathsf{s}(x))) \to \mathsf{f}(\mathsf{s}(\mathsf{f}(x)))$$

  remains
- Question: does it suffice to prove termination on $\mathcal{T}(\{\mathsf{f}, \mathsf{s}\}, \mathcal{V})$?
- if so, then we can apply string reversal

$$\mathsf{s}(\mathsf{s}(\mathsf{f}(x))) \to \mathsf{f}(\mathsf{s}(\mathsf{f}(x)))$$

  and are done, as there are no dependency pairs

# Signature Extensions and Restrictions

- term $t$ is strongly normalizing w.r.t. relation $\to$ ($SN_{\to}(t)$) iff there is no infinite derivation

$$t \to t_1 \to t_2 \to t_3 \to \dots$$

- relation $\to$ is strongly normalizing ($SN(\to)$) iff $\forall t. SN_{\to}(t)$
- for any relation on terms $\to$ and signature $\mathcal{F}$ define

$$\xrightarrow{\mathcal{F}} \quad = \quad \to \cap \, \mathcal{T}(\mathcal{F}, \mathcal{V})^2$$

- for TRS $\mathcal{R}$, let $\to_{\mathcal{R}}$ be rewrite relation, $\mathcal{F}(\mathcal{R})$ be symbols in $\mathcal{R}$
- signature extension: $SN(\xrightarrow{\mathcal{F}(\mathcal{R})}_{\mathcal{R}})$ implies $SN(\to_{\mathcal{R}})$
- signature restriction: given infinite derivation w.r.t. $\to_{\mathcal{R}}$, construct infinite derivation w.r.t. $\xrightarrow{\mathcal{F}(\mathcal{R})}_{\mathcal{R}}$

# Aliens and cleaning

Let $\mathcal{F}$ be signature.

- aliens of $t$ are maximal subterms of $t$ where root is not in $\mathcal{F}$
  - $\mathsf{aliens}(x) = \{\}$
  - $\mathsf{aliens}(\mathsf{f}(t_1, \ldots, t_n)) = \bigcup_{i=1}^{n} \mathsf{aliens}(t_i)$, if $\mathsf{f} \in \mathcal{F}$
  - $\mathsf{aliens}(\mathsf{f}(t_1, \ldots, t_n)) = \{\mathsf{f}(t_1, \ldots, t_n)\}$, if $\mathsf{f} \notin \mathcal{F}$
- cleaning replaces alien subterms by terms over $\mathcal{F}$;
  major parameter: one-alien-clean-function
  $\mathsf{c} : \mathit{alien} \times \mathcal{T}(\mathcal{F}, \mathcal{V})\mathit{list} \Rightarrow \mathcal{T}(\mathcal{F}, \mathcal{V})$
  - $\mathsf{clean}(x) = x$
  - $\mathsf{clean}(\mathsf{f}(t_1, \ldots, t_n)) = \mathsf{f}(\mathsf{clean}(t_1), \ldots, \mathsf{clean}(t_n))$, if $\mathsf{f} \in \mathcal{F}$
  - $\mathsf{clean}(\mathsf{f}(t_1, \ldots, t_n)) = \mathsf{c}(\mathsf{f}(t_1, \ldots, t_n), [\mathsf{clean}(t_1), \ldots, \mathsf{clean}(t_n)])$,
    if $\mathsf{f} \notin \mathcal{F}$

# Examples

Let $\mathcal{F} = \{0, \mathsf{s}, +\}$, let $t = +(\mathsf{g}(\mathsf{s}(x)), \mathsf{h}(+(x, y), \mathsf{g}(y)))$

- cleaning by identical fresh variable $z$ ($\mathsf{clean}_z$): $\mathsf{c}(a, \mathit{list}) = z$

$$\mathsf{clean}_z(t) = +(z, z)$$

- cleaning by indexed fresh variable $z_a$ ($\mathsf{clean}_{z_a}$): $\mathsf{c}(a, \mathit{list}) = z_a$

$$\mathsf{clean}_{z_a}(t) = +(z_{\mathsf{g}(\mathsf{s}(\mathsf{x}))}, z_{\mathsf{h}(+(\mathsf{x},\mathsf{y}),\mathsf{g}(\mathsf{y}))})$$

- cleaning with collecting ($\mathsf{clean}_{coll}$):
  $\mathsf{c}(a, [t_1, \ldots, t_n]) = +(t_1, +(t_2, \ldots +(t_n, z)))$

$$\mathsf{clean}_{coll}(t) = +(+(\mathsf{s}(x), z), +(+(x, y), +(+(y, z), z)))$$

# Properties of cleaning ($\mathcal{F}(\mathcal{R}) \subseteq \mathcal{F}$)

| $c(a, [t_1, \ldots, t_n])$ | $z$ <br> $\mathsf{clean}_z$ | $z_a$ <br> $\mathsf{clean}_{z_a}$ | $f(t_1, f(t_2, \ldots), \ldots)$ <br> $\mathsf{clean}_{coll}$ |
|---|---|---|---|
| simulation of root step | 1 | 1 | 1 |
| simulation of arbitrary step | 0 – 1 | – | 1 |
| reverse simulation | *no* | *yes* | *no* |

- simulation of root step:
  $s \to_{\varepsilon,\mathcal{R}} t$ implies $\mathsf{clean}(s) \to^{?}_{\varepsilon,\mathcal{R}} \mathsf{clean}(t)$
- simulation of arbitrary step:
  $s \to_{\mathcal{R}} t$ implies $\mathsf{clean}(s) \to^{?}_{\mathcal{R}} \mathsf{clean}(t)$
- reverse simulation:
  $\mathsf{clean}(s) \to_{\mathcal{R}} u$ implies $\exists t. s \to_{\mathcal{R}} t \wedge u = \mathsf{clean}(t)$

# Problems

Let $\mathcal{F} = \{\mathsf{b}, \mathsf{f}, \mathsf{g}\}$

$$\mathsf{g}(\mathsf{f}(x, y)) \to \mathsf{b}$$
$$\mathsf{f}(x, x) \to \mathsf{b}$$

- $\mathsf{h}(\mathsf{f}(x, x)) \to \mathsf{h}(\mathsf{b})$, but
  $\mathsf{clean}_{z_a}(\mathsf{h}(\mathsf{f}(x, x))) = z_{\mathsf{h}(\mathsf{f}(\mathsf{x},\mathsf{x}))} \not\to^* z_{\mathsf{h}(\mathsf{b})} = \mathsf{clean}_{z_a}(\mathsf{h}(\mathsf{b}))$
- $\mathsf{clean}_z(\mathsf{f}(\mathsf{h}(x), \mathsf{h}(y))) = \mathsf{f}(z, z) \to \mathsf{b}$, but $\mathsf{f}(\mathsf{h}(x), \mathsf{h}(y)) \not\to$
- $\mathsf{clean}_{coll}(\mathsf{g}(\mathsf{h}(x))) = \mathsf{g}(\mathsf{f}(x, z)) \to \mathsf{b}$, but $\mathsf{g}(\mathsf{h}(x)) \not\to$

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# Goal: $\neg SN(\to_{\mathcal{R}})$ implies $\neg SN(\xrightarrow{\mathcal{F}(\mathcal{R})}_{\mathcal{R}})$

- goal: map infinite $\to_{\mathcal{R}}$-derivation to infinite $\xrightarrow{\mathcal{F}(\mathcal{R})}_{\mathcal{R}}$-derivation

| | $\mathsf{clean}_z$ | $\mathsf{clean}_{z_a}$ | $\mathsf{clean}_{coll}$ |
|---|---|---|---|
| simulation of root step | 1 | 1 | 1 |
| simulation of arbitrary step | $0-1$ | – | 1 |

- $\mathsf{clean}_{z_a}$ is obviously not applicable
- $\mathsf{clean}_{coll}$ directly achieves result but not applicable on unary signatures

$\Rightarrow$ for general result use $\mathsf{clean}_z$

# Goal: $\neg SN(\to_{\mathcal{R}})$ implies $\neg SN(\xrightarrow{\mathcal{F}(\mathcal{R})}_{\mathcal{R}})$

- from simulation of root steps obtain:
  $s \to_{\varepsilon,\mathcal{R}} t$ implies $\mathsf{clean}_z(s) \to_{\varepsilon,\mathcal{R}} \mathsf{clean}_z(t)$
- from simulation of arbitrary steps obtain:
  $s \to_{\mathcal{R}}^* t$ implies $\mathsf{clean}_z(s) \to_{\mathcal{R}}^* \mathsf{clean}_z(t)$
- to obtain infinitely many root steps use dependency pairs (minimal non-terminating terms)
- $\neg SN(\to_{\mathcal{R}})$ implies
  exists infinite $\to_{\varepsilon,DP(\mathcal{R})} \circ \to_{\mathcal{R}}^*$ derivation:
  $t \to_{\varepsilon,DP(\mathcal{R})} t_1 \to_{\mathcal{R}}^* t_2 \to_{\varepsilon,DP(\mathcal{R})} t_3 \to_{\mathcal{R}}^* \dots$ implies
  $\mathsf{clean}_z(t) \to_{\varepsilon,DP(\mathcal{R})} \mathsf{clean}_z(t_1) \to_{\mathcal{R}}^* \dots$ implies
  $\neg SN(\xrightarrow{\mathcal{F}(\mathcal{R})}_{\mathcal{R}})$

# Overview

# Goal: $\neg SN(\to_{\mathcal{R}} / \to_{\mathcal{S}})$ implies $\neg SN(\xrightarrow{\mathcal{F}}_{\mathcal{R}} / \xrightarrow{\mathcal{F}}_{\mathcal{S}})$

Let $\mathcal{F} = \mathcal{F}(\mathcal{R}) \cup \mathcal{F}(\mathcal{S})$

- relative rewriting: $(\to_{\mathcal{R}} / \to_{\mathcal{S}}) = (\to_{\mathcal{S}}^{*} \circ \to_{\mathcal{R}} \circ \to_{\mathcal{S}}^{*})$

| | $\mathsf{clean}_z$ | $\mathsf{clean}_{z_a}$ | $\mathsf{clean}_{coll}$ |
|---|---|---|---|
| simulation of root step | 1 | 1 | 1 |
| simulation of arbitrary step | $0-1$ | – | 1 |

- $\mathsf{clean}_{z_a}$ is obviously not applicable
- $\mathsf{clean}_{coll}$ directly achieves result but not applicable on unary signatures
- to complement $\mathsf{clean}_{coll}$ use $\mathsf{clean}_z$ for TRSs with unary signature
- dependency pairs are not applicable
- minimal non-terminating terms are again convenient

# Goal: $\neg SN(\to_{\mathcal{R}} / \to_{\mathcal{S}})$ implies $\neg SN(\xrightarrow{\mathcal{F}}_{\mathcal{R}} / \xrightarrow{\mathcal{F}}_{\mathcal{S}})$

- key lemma: $s \to t$ implies $\mathsf{clean}_z(s) \to \mathsf{clean}_z(t)$ or $\mathsf{clean}_z(s) = \mathsf{clean}_z(t) \wedge \mathsf{aliens}(s) \to^{mul}_{>\varepsilon} \mathsf{aliens}(t)$
- $s \to_{\mathcal{R}} t$ implies $\mathsf{clean}_z(s) \to_{\mathcal{R}} \mathsf{clean}_z(t)$ or $\mathsf{clean}_z(s) = \mathsf{clean}_z(t) \wedge \mathsf{aliens}(s) \to^{mul}_{>\varepsilon,\mathcal{R}} \mathsf{aliens}(t)$
- $s \to^*_{\mathcal{S}} t$ implies $\mathsf{clean}_z(s) \to^*_{\mathcal{S}} \mathsf{clean}_z(t)$
- simplifying idea: consider infinite reduction of minimal non-terminating term w.r.t. $\to_{\mathcal{R}} / \to_{\mathcal{S}}$
  - if infinitely many $\to_{\mathcal{R}}$-steps are preserved via $\mathsf{clean}_z$, we have proven $\neg SN(\xrightarrow{\mathcal{F}}_{\mathcal{R}} / \xrightarrow{\mathcal{F}}_{\mathcal{S}})$
  - otherwise, after a while each strict step becomes alien step $\mathsf{aliens}(s) \to^{mul}_{>\varepsilon,\mathcal{R}} \mathsf{aliens}(t)$
  - non-duplication of $\mathcal{S}$ ensures that $s \to_{\mathcal{S}} t$ implies $\mathsf{aliens}(s) \to^{mul,*}_{>\varepsilon,\mathcal{S}} \mathsf{aliens}(t)$
  - hence, there is some alien that is non-terminating w.r.t. $\to_{>\varepsilon,\mathcal{R}} / \to_{>\varepsilon,\mathcal{S}}$ in contradiction to minimality

# $SN(\xrightarrow{\mathcal{F}}_{\mathcal{R}} / \xrightarrow{\mathcal{F}}_{\mathcal{S}})$ implies $SN(\rightarrow_{\mathcal{R}} / \rightarrow_{\mathcal{S}})$

signature extensions for relative rewriting are sound if

- signature contains symbol with arity $\geq 2$ (use $\mathsf{clean}_{coll}$)
- signature contains at most unary symbols if (use $\mathsf{clean}_{z}$)
  - $\mathcal{R}$ and $\mathcal{S}$ satisfy variable condition of TRSs
  - $\mathcal{V}(\ell) \supseteq \mathcal{V}(r)$ for all $\ell \to r \in \mathcal{R} \cup \mathcal{S}$
    (previous proof does not require $\ell \notin \mathcal{V}$)
  - $\mathcal{V}(\ell) \supseteq \mathcal{V}(r)$ for all $\ell \to r \in \mathcal{S}$
    (if $\mathcal{V}(\ell) \not\supseteq \mathcal{V}(r)$ for $\ell \to r \in \mathcal{R}$, conclude $\neg SN(\xrightarrow{\mathcal{F}}_{\mathcal{R}} / \xrightarrow{\mathcal{F}}_{\mathcal{S}}))$

signature extension for relative rewriting is unsound for

$$\mathcal{R} = \{\mathsf{f}(\mathsf{a}) \to \mathsf{b}\}$$
$$\mathcal{S} = \{\mathsf{a} \to x\}$$

(where $\mathcal{F} = \{\mathsf{a}, \mathsf{b}, \mathsf{f}\}$)

# Overview

# Signature Extensions for Innermost Termination

| | $\mathsf{clean}_z$ | $\mathsf{clean}_{z_a}$ | $\mathsf{clean}_{coll}$ |
|---|---|---|---|
| simulation of root step | 1 | 1 | 1 |
| simulation of arbitrary step | $0-1$ | $1(\mathit{inn.})$ | 1 |
| reverse simulation | *no* | *yes* | *no* |

- without reverse simulation, we cannot guarantee that $NF(t)$ implies $NF(\mathsf{clean}(t))$

$\Rightarrow$ $\mathsf{clean}_z$ and $\mathsf{clean}_{coll}$ are not applicable for innermost rewriting

- however, $\mathsf{clean}_{z_a}$ becomes applicable, key lemma:
  $\mathsf{aliens}(s) \subseteq NF(\mathcal{R})$ and $s \xrightarrow{i}_{\mathcal{R}} t$ implies
  $\mathsf{aliens}(t) \subseteq NF(\mathcal{R})$ and $\mathsf{clean}_{z_a}(s) \xrightarrow{i}_{\mathcal{R}} \mathsf{clean}_{z_a}(t)$
- to obtain non-terminating term with $\mathsf{aliens}(s) \subseteq NF(\mathcal{R})$ start from first root reduction of a minimal non-terminating term

# Overview

# Signature Extensions for Dependency Pairs

| | $\mathsf{clean}_z$ | $\mathsf{clean}_{z_a}$ | $\mathsf{clean}_{coll}$ |
|---|---|---|---|
| simulation of root step | 1 | 1 | 1 |
| simulation of arbitrary step | $0-1$ | $1(inn.)$ | 1 |
| reverse simulation | *no* | *yes* | *no* |

- without reverse simulation, $SN(t)$ does not imply $SN(\mathsf{clean}(t))$

$\Rightarrow$ $\mathsf{clean}_z$ and $\mathsf{clean}_{coll}$ are problematic when dealing with minimal infinite chains

- $\mathsf{clean}_z$ has the reverse simulation property if $\mathcal{R}$ is left-linear
- known results:
  - signature extensions are sound for minimal chains and left-linear $\mathcal{R}$ ($\mathsf{clean}_z$)
  - signature extensions are sound for innermost minimal chains ($\mathsf{clean}_{z_a}$)
  - signature extensions are unsound for minimal chains

# Overview

# Related Work

- modularity results of Middeldorp entail signature extension result for termination (not for relative termination)
- sometimes similar reasoning (usage of multiset-extensions, aliens, ...)
- however, Middeldorp uses alternating layers of aliens
- with minimality-trick, for our purpose two layers suffice

# Summary

- Signature extensions are sound for
  - termination ($\mathsf{clean}_z$)
  - relative termination ($\mathsf{clean}_z, \mathsf{clean}_{coll}$)
  - innermost termination ($\mathsf{clean}_{z_a}$)
  - innermost minimal chains ($\mathsf{clean}_{z_a}$)
  - minimal chains and left-linear $\mathcal{R}$ ($\mathsf{clean}_z$)
- Signature extensions are unsound for
  - relative termination without variable condition
  - minimal chains
- using minimal non-terminating terms, no alternating layers of aliens are required
- everything has been formalized in IsaFoR