Resources in λ-calculus

(an introduction to the differential approach to computation)

Stéphane Gimenez

University of Innsbruck
Computational Logic Group

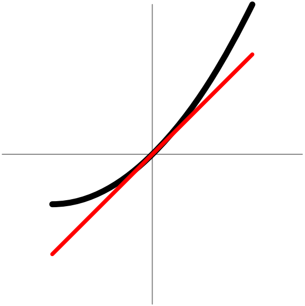
Master Seminar 3 – May, 16 2012

1. Introduction (first slide contains maths!).
2. Resource calculus.
3. Differential λ-calculus.
4. Promoted resources.
5. Conclusion.

Introduction

Differentials:

$f : \mathbb{R} \to \mathbb{R}$, for example $f(x) = x^2$:

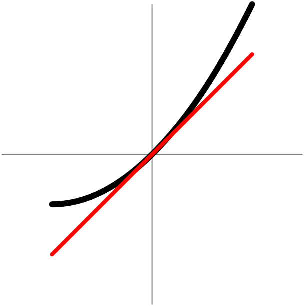


$Df : \mathbb{R} \to \mathcal{L}(\mathbb{R}, \mathbb{R})$, here $Df(x)(u) = 2x \cdot u = \frac{\partial}{\partial x} f(x) \cdot u$.

Introduction

Differentials:

$f : \mathbb{R} \to \mathbb{R}$, for example $f(x) = x^2$:



$Df : \mathbb{R} \to \mathcal{L}(\mathbb{R}, \mathbb{R})$, here $Df(x)(u) = 2x \cdot u = \frac{\partial}{\partial x} f(x) \cdot u$.

- λ-terms are functions, can we get linear approximations for them?

Ideas

What are resources?

$$t \quad ::= \quad x \;\big|\; \lambda x\, t \;\big|\; \langle t \rangle\, T \qquad\qquad T \quad ::= \quad 1 \;\big|\; T T \;\big|\; t$$

Ideas

What are resources?

$$t \quad ::= \quad x \;\big|\; \lambda x\, t \;\big|\; \langle t \rangle\, T \qquad\qquad T \quad ::= \quad 1 \;\big|\; T T \;\big|\; t$$

Example

$$\langle \lambda x \, \langle x \rangle \, x x \rangle \; abc$$

Ideas

What are resources?

$$t \quad ::= \quad x \;\big|\; \lambda x\, t \;\big|\; \langle t\rangle\, T \qquad\qquad T \quad ::= \quad 1 \;\big|\; TT \;\big|\; t$$

Example

$$\langle \lambda x\, \langle x\rangle\, xx\rangle\, abc$$

$$\longrightarrow \langle a\rangle\, bc + \langle a\rangle\, cb + \langle b\rangle\, ac + \langle b\rangle\, ca + \langle c\rangle\, ab + \langle c\rangle\, ba$$

Ideas

What are resources?

$$t \quad ::= \quad x \;\mid\; \lambda x\, t \;\mid\; \langle t \rangle\, T \qquad\qquad T \quad ::= \quad 1 \;\mid\; TT \;\mid\; t$$

Example

$$\langle \lambda x \langle x \rangle\, xx \rangle\, abc$$

$$\longrightarrow 2\ \langle a \rangle\, bc + 2\ \langle b \rangle\, ac + 2\ \langle c \rangle\, ab$$

Ideas

What are resources?

$$t \quad ::= \quad x \;\big|\; \lambda x\, t \;\big|\; \langle t\rangle\, T \qquad\qquad T \quad ::= \quad 1 \;\big|\; TT \;\big|\; t$$

Example

$$\langle \lambda x \langle x\rangle\, xx\rangle\, abc$$
$$\longrightarrow 2\, \langle a\rangle\, bc + 2\, \langle b\rangle\, ac + 2\, \langle c\rangle\, ab$$

Instantiating with $a = \lambda x x$, $b = \lambda x x$, and $c = \lambda x \langle x\rangle\, x$ ...

Ideas

What are resources?

$$t \quad ::= \quad x \;\Big|\; \lambda x\, t \;\Big|\; \langle t \rangle\, T \qquad\qquad T \quad ::= \quad 1 \;\Big|\; T T \;\Big|\; t$$

Example

$$\langle \lambda x \langle x \rangle\, x x \rangle\; abc$$

$$\longrightarrow 2\; \langle a \rangle\; bc + 2\; \langle b \rangle\; ac + 2\; \langle c \rangle\; ab$$

Instantiating with $a = \lambda x x$, $b = \lambda x x$, and $c = \lambda x \langle x \rangle\, x$ ...

$$\longrightarrow 0 + 0 + 4\; \langle \lambda x x \rangle\; \lambda x x$$

Ideas

What are resources?

$$t \quad ::= \quad x \;\Big|\; \lambda x\, t \;\Big|\; \langle t\rangle\, T \qquad\qquad T \quad ::= \quad 1 \;\Big|\; TT \;\Big|\; t$$

Example

$$\langle \lambda x\, \langle x\rangle\, xx\rangle\, abc$$

$$\longrightarrow 2\, \langle a\rangle\, bc + 2\, \langle b\rangle\, ac + 2\, \langle c\rangle\, ab$$

Instantiating with $a = \lambda x x$, $b = \lambda x x$, and $c = \lambda x\, \langle x\rangle\, x$ ...

$$\longrightarrow 0 + 0 + 4\, \langle \lambda x x\rangle\, \lambda x x$$

$$\longrightarrow 4\, \lambda x x$$

1. Resource calculus.

Syntax of Resource λ-terms

Definition

Resource calculus:

$$t ::= x \mid \lambda x\, t \mid \langle t \rangle\, T \qquad T ::= 1 \mid TT \mid t$$
$$\quad 0 \mid t + t \qquad\qquad\qquad 0 \mid T + T$$

Syntax of Resource λ-terms

Definition

Resource calculus:

$$t \quad ::= \quad \begin{array}{c} x \;\big|\; \lambda x\, t \;\big|\; \langle t \rangle\, T \\ 0 \;\big|\; t + t \end{array} \qquad\qquad T \quad ::= \quad \begin{array}{c} 1 \;\big|\; TT \;\big|\; t \\ 0 \;\big|\; T + T \end{array}$$

- Bags are quotiented by associativity, commutativity, neutrality:

$$t_1 \cdots t_n$$

Syntax of Resource λ-terms

Definition

Resource calculus:

$$t \quad ::= \quad x \;\mid\; \lambda x\, t \;\mid\; \langle t\rangle\, T \qquad\qquad T \quad ::= \quad 1 \;\mid\; TT \;\mid\; t$$
$$\qquad\qquad 0 \;\mid\; t + t \qquad\qquad\qquad\qquad\qquad\qquad 0 \;\mid\; T + T$$

- Bags are quotiented by associativity, commutativity, neutrality:

$$t_1 \cdots t_n$$

- Quotient by (bi-)linearity of all constructions:

$$\lambda x 0 \;=\; 0 \qquad\qquad \lambda x\,(t_1 + t_2) \;=\; \lambda x\, t_1 + \lambda x\, t_2$$

Linear substitution

$$t \;=\; \langle x \rangle\, xx \qquad\qquad \frac{\partial}{\partial x}\, t \cdot u \;=\; \langle u \rangle\, xx + \langle x \rangle\, ux + \langle x \rangle\, xu$$

Linear substitution

$$t \;=\; \langle x\rangle\, xx \qquad\qquad \frac{\partial}{\partial x}\, t\cdot u \;=\; \langle u\rangle\, xx + \langle x\rangle\, ux + \langle x\rangle\, xu$$

Definition

We use $\frac{\partial}{\partial z}\,\square\cdot u$ to denote linear substitution of z by u in a term:

$$\frac{\partial}{\partial z}\, x\cdot u \;=\; 0 \qquad\qquad \frac{\partial}{\partial z}\, z\cdot u \;=\; u$$

$$\frac{\partial}{\partial z}\left(\lambda x\, s\right)\cdot u \;=\; \lambda x\, \frac{\partial}{\partial z}\, s\cdot u$$

$$\frac{\partial}{\partial z}\left(\langle r\rangle\; R\right)\cdot u \;=\; \left\langle \frac{\partial}{\partial z}\, r\cdot u \right\rangle R \;+\; \langle r\rangle\, \frac{\partial}{\partial z}\, R\cdot u$$

Linear substitution

$$t = \langle x \rangle\, xx \qquad \frac{\partial}{\partial x} t \cdot u = \langle u \rangle\, xx + \langle x \rangle\, ux + \langle x \rangle\, xu$$

Definition

We use $\frac{\partial}{\partial z} \Box \cdot u$ to denote linear substitution of z by u in a term:

$$\frac{\partial}{\partial z} x \cdot u = 0 \qquad \frac{\partial}{\partial z} z \cdot u = u$$

$$\frac{\partial}{\partial z} (\lambda x\, s) \cdot u = \lambda x \frac{\partial}{\partial z} s \cdot u$$

$$\frac{\partial}{\partial z} (\langle r \rangle\, R) \cdot u = \left\langle \frac{\partial}{\partial z} r \cdot u \right\rangle R + \langle r \rangle \frac{\partial}{\partial z} R \cdot u$$

or in bags:

$$\frac{\partial}{\partial z} 1 \cdot u = 0 \qquad \frac{\partial}{\partial z} ST \cdot u = \left(\frac{\partial}{\partial z} S \cdot u \right) T + S \left(\frac{\partial}{\partial z} T \cdot u \right)$$

Properties

Property

Operators $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ commute, *Schwarz/Clairaut* property:

$$\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x} t \cdot u\right) \cdot v \;=\; \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y} t \cdot v\right) \cdot u$$

Properties

Property

Operators $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ commute, *Schwarz/Clairaut* property:

$$\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\, t \cdot u\right) \cdot v + \frac{\partial}{\partial y}\, t \cdot \left(\frac{\partial}{\partial x}\, v \cdot u\right)$$
$$=$$
$$\frac{\partial}{\partial x}\, t \cdot \left(\frac{\partial}{\partial y}\, u \cdot v\right) + \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\, t \cdot v\right) \cdot u$$

In the usual case when $x \notin v$ and $y \notin u$:

$$\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\, t \cdot u\right) \cdot v \;=\; \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\, t \cdot v\right) \cdot u$$

Reduction

Redexes are sub-terms of the form $\langle \lambda x\, s\rangle\, R$.

Definition

Reduction is defined using linear substitution:

$$\langle \lambda x\, s\rangle\, rR \;\longrightarrow\; \left\langle \lambda x\, \frac{\partial}{\partial x} s \cdot r \right\rangle R$$

$$\langle \lambda x\, s\rangle\, 1 \;\longrightarrow\; s[0/x]$$

Reduction

Redexes are sub-terms of the form $\langle \lambda x\, s \rangle\, R$.

Definition

Reduction is defined using linear substitution:

$$\langle \lambda x\, s \rangle\, rR \;\longrightarrow\; \left\langle \lambda x \frac{\partial}{\partial x} s \cdot r \right\rangle R$$

$$\langle \lambda x\, s \rangle\, 1 \;\longrightarrow\; s\left[{}^{0}/_{x}\right]$$

- A single redex can be reduced in several ways.
- A sum is reduced by reducing any term it contains.
- *Schwarz* property ensures confluence.
- Normalization holds as long as sums are finite.

Reduction

Redexes are sub-terms of the form $\langle \lambda x\, s \rangle\, R$.

Definition

Reduction is defined using linear substitution:

$$\langle \lambda x\, s \rangle\, rR \;\longrightarrow\; \left\langle \lambda x \frac{\partial}{\partial x} s \cdot r \right\rangle R$$

$$\langle \lambda x\, s \rangle\, 1 \;\longrightarrow\; s\left[{}^{0}/_{x}\right]$$

- A single redex can be reduced in several ways.
- A sum is reduced by reducing any term it contains.
- *Schwarz* property ensures confluence.
- Normalization holds as long as sums are finite.
- [ER08] *Uniformity and the taylor expansion of ordinary λ-terms*.

Decomposition of standard λ-terms

Decomposition is defined inductively on standard λ-terms:

$$
\begin{aligned}
(x)^* &= x \\
(\lambda x\, s)^* &= \lambda x\, s^* \\
((t)\, s)^* &= \sum_{n=0}^{+\infty} \langle t^* \rangle\, {s^*}^n \\
&= \langle t^* \rangle\, 1 + \langle t^* \rangle\, s^* + \langle t^* \rangle\, s^* s^* + \cdots
\end{aligned}
$$

Example.

Decomposition of standard λ-terms

Decomposition is defined inductively on standard λ-terms:

$$
\begin{aligned}
(x)^* &= x \\
(\lambda x\, s)^* &= \lambda x\, s^* \\
((t)\, s)^* &= \sum_{n=0}^{+\infty} \frac{1}{n!} \langle t^* \rangle\, {s^*}^n \\
&= \langle t^* \rangle\, 1 + \langle t^* \rangle\, s^* + \frac{1}{2} \langle t^* \rangle\, s^* s^* + \frac{1}{6} \cdots
\end{aligned}
$$

Example.

2. Differential λ-calculus.

A differential operator inside the syntax

The differential of a function $f : A \to B$ has type:

$$Df : A \to \mathcal{L}(A, B)$$

This suggests that we could add the following construct to the standard λ-calculus:

$$Df \cdot u : A \to B$$

We obtain Differential λ-calculus ([Vau07]):

$$t \quad ::= \quad \begin{array}{c} x \;\mid\; \lambda x\, t \;\mid\; (t)\, t \;\mid\; Dt \cdot u \\ 0 \;\mid\; t + t \end{array}$$

Warning: application is only left-linear.

A differential operator inside the syntax

The differential of a function $f : A \to B$ has type:

$$Df : A \to A \to B$$

This suggests that we could add the following construct to the standard λ-calculus:

$$Df \cdot u : A \to B$$

We obtain Differential λ-calculus ([Vau07]):

$$t \quad ::= \quad \begin{array}{c} x \;\mid\; \lambda x\, t \;\mid\; (t)\, t \;\mid\; Dt \cdot u \\ 0 \;\mid\; t + t \end{array}$$

Warning: application is only left-linear.

A differential operator inside the syntax

The differential of a function $f : A \to B$ has type:

$$Df : A \to A \multimap B$$

This suggests that we could add the following construct to the standard λ-calculus:

$$Df \cdot u : A \to B$$

We obtain Differential λ-calculus ([Vau07]):

$$t \quad ::= \quad \begin{array}{c} x \;\mid\; \lambda x\, t \;\mid\; (t)\, t \;\mid\; Dt \cdot u \\ 0 \;\mid\; t + t \end{array}$$

Warning: application is only left-linear.

Example

$$(\lambda y\, Dy \cdot x)\ \lambda z\, (z)\ z$$

Example

$$(\lambda y\, Dy \cdot x)\ \lambda z\,(z)\ z$$

$$\downarrow_\beta$$

$$D(\lambda z\,(z)\ z) \cdot x$$

Example

$$(\lambda y\, Dy \cdot x)\ \lambda z\,(z)\ z$$

$$\downarrow_\beta$$

$$D(\lambda z\,(z)\ z) \cdot x$$

$$\downarrow_\partial$$

$$\lambda z\, \frac{\partial}{\partial z} \big((z)\ z\big) \cdot x$$

Example

$$(\lambda y\, Dy \cdot x)\ \lambda z\,(z)\ z$$

$$\downarrow_\beta$$

$$D(\lambda z\,(z)\ z) \cdot x$$

$$\downarrow_\partial$$

$$\lambda z\, \frac{\partial}{\partial z}\big((z)\ z\big) \cdot x$$

$$=$$

$$\lambda z\,(x)\ z + \lambda z\,(Dz \cdot x)\ z$$

Taylor expansion

Nice analogy between:

$$(t)\ s \quad \approx \quad \sum_{n=0}^{+\infty} \frac{1}{n!} (D^n t \cdot (\underbrace{s, \ldots, s}_{n}))0$$

and:

$$f(x) = \sum_{n=0}^{+\infty} \frac{1}{n!} f^{(n)}(0) x^n$$

or:

$$f(x) = \sum_{n=0}^{+\infty} \frac{1}{n!} D^n f(\underbrace{x, \ldots, x}_{n})(0)$$

3. Promoted resources.

Syntax of λ-calculus with resources and promotion

Definition

$$t ::= x \mid \lambda x\, t \mid \langle t \rangle\, T \mid 0 \mid t + t \qquad T ::= 1 \mid TT \mid t \mid t^{!} \mid 0 \mid T + T$$

Syntax of λ-calculus with resources and promotion

Definition

$$t ::= \begin{matrix} x \mid \lambda x\, t \mid \langle t\rangle\, T \\ 0 \mid t + t \end{matrix} \qquad T ::= \begin{matrix} 1 \mid TT \mid t \mid t^! \\ 0 \mid T + T \end{matrix}$$

- $(s)\, t$ becomes $\langle s\rangle\, t^!$.
- $(D(D(Df \cdot u) \cdot v) \cdot w)\, t$ becomes $\langle f\rangle\, uvwt^!$

Syntax of λ-calculus with resources and promotion

Definition

$$t \quad ::= \quad \begin{array}{c} x \;\mid\; \lambda x\, t \;\mid\; \langle t\rangle\, T \\ 0 \;\mid\; t+t \end{array} \qquad T \quad ::= \quad \begin{array}{c} 1 \;\mid\; TT \;\mid\; t \;\mid\; t^! \\ 0 \;\mid\; T+T \end{array}$$

- $(s)\, t$ becomes $\langle s\rangle\, t^!$.
- $(D(D(Df \cdot u) \cdot v) \cdot w)\, t$ becomes $\langle f\rangle\, uvwt^!$

Here *promotion* is not linear:

$$(u+v)^! \neq u^! + v^!$$

Syntax of λ-calculus with resources and promotion

Definition

$$t \quad ::= \quad \begin{array}{c} x \;\mid\; \lambda x\, t \;\mid\; \langle t\rangle\, T \\ 0 \;\mid\; t+t \end{array} \qquad T \quad ::= \quad \begin{array}{c} 1 \;\mid\; TT \;\mid\; t \;\mid\; t^{!} \\ 0 \;\mid\; T+T \end{array}$$

- $(s)\, t$ becomes $\langle s\rangle\, t^{!}$.
- $(D(D(Df \cdot\, u) \cdot\, v) \cdot\, w)\, t$ becomes $\langle f\rangle\, uvwt^{!}$

Here *promotion* is not linear:

$$(u+v)^{!} = u^{!} v^{!}$$

Syntax of λ-calculus with resources and promotion

Definition

$$t \quad ::= \quad x \mid \lambda x\, t \mid \langle t \rangle\, T \mid 0 \mid t + t \qquad T \quad ::= \quad 1 \mid TT \mid t \mid t^! \mid 0 \mid T + T$$

- $(s)\, t$ becomes $\langle s \rangle\, t^!$.
- $(D(D(Df \cdot u) \cdot v) \cdot w)\, t$ becomes $\langle f \rangle\, uvwt^!$

Here *promotion* is not linear:

$$(u + v)^! = u^! v^!$$

$$0^! = 1$$

Atomic reduction

Linear substitution is extended to promoted resources:

$$\frac{\partial}{\partial z} t^! \cdot u \;=\; \left(\frac{\partial}{\partial z} t \cdot u\right) t^!$$

A new reduction rule:

$$\langle \lambda x\, s \rangle\, 1 \;\xrightarrow{a}\; s[0/x] \qquad \langle \lambda x\, s \rangle\, rR \;\xrightarrow{a}\; \left\langle \lambda x \frac{\partial}{\partial x} s \cdot r \right\rangle R$$

$$\langle \lambda x\, s \rangle\, r^! R \;\xrightarrow{a}\; \langle \lambda x\, s \rangle\, R + \left\langle \lambda x \frac{\partial}{\partial x} s \cdot r \right\rangle r^! R$$

Conclusion

We introduced:

- A notion of (non-duplicable and non-erasable) resource.
- A reduction based on linear substitution.
- A notion of “vector space” over terms to control non-determinism.
- Analogies with differential analysis (in math).

What we have not talked about:

- Differential Linear Logic.
- Links with Process algebras.

Open perspectives:

- Links with Implicit Complexity?
- Links with Quantum computation?
- Find a better syntax!